THEORIE DER KONDENSIERTEN MATERIE:
Quantum mechanics on macroscopic scales

Blatt 1 SS 2012, Prof. J. Ankerhold and Prof. M. Blencowe

Exercise 1: Tricks for fermions and bosons

Show for bosonic destruction and creation operators, b, $b^\dagger$, and a complex number α:

a) $b(b^\dagger)^n - (b^\dagger)^n b = \frac{\partial (b^\dagger)^n}{\partial b^\dagger}$ and $b^\dagger (b)^n - (b)^n b^\dagger = -\frac{\partial (b)^n}{\partial b}$

Hint: Use complete induction (vollständige Induktion). **(1 point)**

b) The exponential of an operator is defined as

$$\exp(\alpha b) = \sum_{n=0}^{\infty} \frac{\alpha^n b^n}{n!} .$$

Use the results from a) to show,

$$e^{-\alpha b^\dagger} b e^{\alpha b^\dagger} = b + \alpha \quad \text{and} \quad e^{-\alpha b} b^\dagger e^{\alpha b} = b^\dagger - \alpha ,$$

where expressions of the type $b + \alpha$ should be understood as $b + \alpha \cdot id$ with identity operator id.

(2 points)

c) Find corresponding results as in b) for fermions? **(1 point)**

Exercise 2: Shifted Oscillator

A harmonic oscillator in a constant (electric) field is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 - \sqrt{2m\hbar\omega}\gamma x .$$

a) Write this Hamiltonian expressed in terms of the operators a and $a^\dagger$ defined on the last exercise sheet. Show that

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) - \hbar\gamma (a^\dagger + a) . \tag{1}$$

(1 point)

b) Find shifted operators $\tilde{a}$ and $\tilde{a}^\dagger$, so that the Hamiltonian is given as

$$\tilde{H} = \hbar\omega \left(\tilde{a}^\dagger \tilde{a} + \frac{1}{2} \right) + \epsilon .$$

Find the constant energy contribution ϵ. Show that $\tilde{a}$ and $\tilde{a}^\dagger$ fulfill bosonic commutation relations.

(2 points)

c) The ground state of $\tilde{H}$ can be determined by $\tilde{a}|\tilde{0}\rangle = 0$. Write this equation in terms of the original annihilation operator a and show that the ground state is given by

$$|\tilde{0}\rangle = e^{-\gamma^2/2\omega^2} \exp\left(\frac{\gamma}{\omega} a^\dagger \right) |0\rangle .$$

(2 points)