A new proof of the Voronoï summation formula

Sebastian Endres

Institut für Theoretische Physik, Universität Ulm
Albert-Einstein-Allee 11, 89081 Ulm, Germany

E-mail: `sebastian.endres@uni-ulm.de`

Frank Steiner

Institut für Theoretische Physik, Universität Ulm
Albert-Einstein-Allee 11, 89081 Ulm, Germany

E-mail: `frank.steiner@uni-ulm.de`

Abstract. We present a short alternative proof of the Voronoï summation formula.

PACS numbers: 03.65.Ca, 03.65.Db

1. The Voronoï summation formula

The Voronoï summation formula provides a connection between a sum involving the divisor function $d(n)$ smeared with some function $f(n)$ and a sum involving again the divisor function smeared with a function $F(n)$ which arises from a certain Bessel transformation of the function f. The divisor function is defined as the number of divisors of n, unity and n itself included, i.e.

$$d(k) := \#\left\{(n,m); \quad nm = k\right\}, \quad n, m, k \in \mathbb{N}. \tag{1}$$

Note that the divisor function $d(n)$, $d(1) = 1$, $d(2) = 2$, $d(3) = 2$, $d(4) = 3$, $d(5) = 2$, $d(6) = 4$, …, with $d(p) = 2$ for p prime, is a very irregular function with asymptotic behaviour

$$d(n) = \mathrm{O}\left(n^{\epsilon}\right), \quad n \to \infty \quad \text{for all} \quad \epsilon > 0. \tag{2}$$

In 1849 Dirichlet proved [1] the following asymptotic formula

$$D(x) := \sum_{n=1}^{x} d(n) = x \ln x + (2\gamma - 1)x + \Delta(x), \quad x \to \infty \tag{3}$$

with

$$\Delta(x) = \mathrm{O}\left(\sqrt{x}\right), \quad x \to \infty, \tag{4}$$

where γ is Euler's constant. The famous Dirichlet's divisor problem is that of determining as precisely as possible the maximum order of the error term $\Delta(x)$. In 1903 Voronoï [2] was able to improve Dirichlet's result by proving

$$\Delta(x) = \mathrm{O}\left(x^{\frac{1}{3}} \ln x\right), \quad x \to \infty. \tag{5}$$

Voronoï based his prove on a new summation formula carrying now its name, see [3]. The estimate on $\Delta(x)$ was later improved, see e.g. [4, 5]. In [6] and [7] it was proved that

$$\Delta(x) = \mathrm{O}\left(x^{\alpha}\right), \quad x \to \infty \quad \text{with} \quad \alpha \geq \frac{1}{4}, \tag{6}$$

but the exact order of $\Delta(x)$ is still unknown. In various articles (e.g. [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]) the authors investigate the Voronoï summation formula and specify proper function spaces for which the Voronoï summation formula is valid. The result of Hejhal [17] states:

Theorem 1.1 (Hejahl:1979, [17]). *If f is two times continuously differentiable and possesses compact support ($f \in C_0^2(\mathbb{R})$) then the following formula is valid*

$$\begin{aligned}\sum_{n=1}^{\infty} d(n) f(n) = &\int_0^{\infty} \left(\ln k + 2\gamma\right) f(k) \mathrm{d}k + \frac{f(0)}{4} \\ &+ 2\pi \sum_{n=1}^{\infty} d(n) \int_0^{\infty} \left[\frac{2}{\pi} K_0\left(4\pi\sqrt{nk}\right) - Y_0\left(4\pi\sqrt{nk}\right)\right] f(k) \mathrm{d}k,\end{aligned} \tag{7}$$

where the functions $K_0(x)$ *and* $Y_0(x)$ *are modified Bessel functions, i.e. the McDonald function respectively the Neumann function (see e.g.* [*18*, p. 65, 66]*) and* γ *denotes the Euler constant.*

The Voronoï summation formula (7) can be interpreted as a trace formula for a certain noncompact quantum graph with pure Dirichlet boundary conditions (see [19]) whose eigenvalues are given by n^2, $n \in \mathbb{N}$, with multiplicities precisely given by the divisor function $d(n)$. It is the purpose of this note to prove the Voronoï summation formula (7) for a function space different from that in theorem 1.1 and e.g. in [12, 13, 14, 17]).

2. An alternative proof of the Voronoï summation formula

In order to present an alternative derivation of the Voronoï summation formula, we introduce the function (a particular Lambert series)

$$\Theta(t) := \sum_{n=1}^{\infty} d(n)\mathrm{e}^{-nt} = \sum_{n=1}^{\infty} \frac{1}{\mathrm{e}^{nt}-1}, \quad t \in (0,\infty). \tag{8}$$

We remark that $\Theta(t)$ coincides with the function $\Theta_{\Delta_D^{\frac{1}{2}}}(t)$ defined in (3.6) of [19] which is the trace of the wave group of the Dirichlet Laplacian considered in [19]. In [19] (see (3.12) therein) we have derived the following asymptotic relations

$$\begin{aligned} &\Theta(t) = -\frac{\ln t}{t} + \frac{\gamma}{t} + \frac{1}{4} + \mathrm{O}(t), \quad t \to 0^+, \\ &\Theta(t) = \mathrm{O}\left(\mathrm{e}^{-t}\right), \quad t \to \infty. \end{aligned} \tag{9}$$

For further convenience, we "regularise" $\Theta(t)$ in (8) at $t = 0$:

$$\begin{aligned} \Theta(t) &= \sum_{n=1}^{\infty} \frac{1}{\mathrm{e}^{nt}-1} \\ &= \sum_{n=1}^{\infty} \left[\frac{1}{\mathrm{e}^{nt}-1} - \frac{\mathrm{e}^{-nt}}{nt}\right] + \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-nt}}{nt} \\ &= \sum_{n=1}^{\infty} g(nt) - \frac{\ln\left(1-\mathrm{e}^{-t}\right)}{t}, \quad t \in (0,\infty) \end{aligned} \tag{10}$$

with

$$g(x) := \frac{1}{\mathrm{e}^{x}-1} - \frac{\mathrm{e}^{-x}}{x} = \mathrm{e}^{-x}\left[\frac{1}{1-\mathrm{e}^{-x}} - \frac{1}{x}\right], \quad x \in (0,\infty). \tag{11}$$

We notice that

$$g(x) = \frac{1}{2} + \mathrm{O}(x), \quad x \to 0^+, \quad \text{and} \quad g(x) = \mathrm{O}\left(\mathrm{e}^{-x}\right), \quad x \to \infty. \tag{12}$$

Now, we use theorem 5 of [14, p. 73] (in a slightly modified form) which can be considered as a special version of the classical Poisson summation formula.

Theorem 2.1 (Dixon/Ferrar: 1937, [14]). *If, for any finite $x_0 > 0$, g(x) is a real function of bounded variation in the interval $(0, x_0)$, then $(t \in (0,\infty))$*

$$\sum_{n=1}^{\infty} g(nt) = -\frac{1}{2} g\left(0^+\right) + \frac{1}{t}\int_0^{\infty} g(x)\mathrm{d}x + \sum_{n=1}^{\infty} \frac{2}{t}\int_0^{\infty} g(x)\cos\left(\frac{2\pi n}{t}x\right)\mathrm{d}x \quad (13)$$

provided that

- *the sum on the l.h.s and the first integral on the r.h.s in* (13) *exist,*
- $g(x) \to 0$ *as* $x \to \infty$,
- $g(x)$ *is the integral of* $g'(x)$ *in* $x \geq x_0$,
- $|g'(x)|$ *is integrable over* (x_0, ∞).

With the identity [18, p. 16] for the digamma function

$$\psi(z) = \ln z + \int_0^{\infty} \left[\frac{1}{x} - \frac{1}{1-\mathrm{e}^{-x}}\right] \mathrm{e}^{-zx}\mathrm{d}x, \quad \operatorname{Re} z > 0, \quad (14)$$

we obtain the relation $(n \in \mathbb{N}_0,\ t \in (0,\infty))$:

$$\int_0^{\infty} g(x)\cos\left(\frac{2\pi n}{t}x\right)\mathrm{d}x = \operatorname{Re}\left\{\ln\left(1+\mathrm{i}\frac{2\pi n}{t}\right) - \psi\left(1+\mathrm{i}\frac{2\pi n}{t}\right)\right\}. \quad (15)$$

Furthermore, with $(n \in \mathbb{N},\ t \in (0,\infty))$

$$\operatorname{Re}\ln\left(1+\mathrm{i}\frac{2\pi n}{t}\right) = \ln\left(\frac{2\pi n}{t}\right) + \frac{1}{2}\ln\left(1+\left(\frac{t}{2\pi n}\right)^2\right) \quad (16)$$

and the identity [20, p. 85]

$$\sinh z = z\prod_{n=1}^{\infty}\left(1+\left(\frac{z}{n\pi}\right)^2\right), \quad z \in \mathbb{C}, \quad (17)$$

we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \ln\left[1+\left(\frac{t}{2\pi n}\right)^2\right] &= \ln\left[\frac{\sinh\left(\frac{t}{2}\right)}{\left(\frac{t}{2}\right)}\right] \\ &= \frac{t}{2} + \ln\left(1-\mathrm{e}^{-t}\right) - \ln t, \quad t \in (0,\infty).\end{aligned} \quad (18)$$

Using

$$g\left(0^+\right) = \frac{1}{2} \quad (19)$$

and (see (15))

$$\int_0^{\infty} g(x)\mathrm{d}x = -\psi(1) = \gamma \quad (20)$$

and combining the results of (10), (13), (15), (16), (18), (19) and (20), we altogether obtain the relation ($t \in (0, \infty)$):

$$\Theta(t) = -\frac{\ln t}{t} + \frac{\gamma}{t} + \frac{1}{4} - \frac{2}{t} \sum_{n=1}^{\infty} \left[\operatorname{Re} \psi \left(1 + \mathrm{i} \frac{2\pi n}{t} \right) - \ln \left(\frac{2\pi n}{t} \right) \right]. \tag{21}$$

We remark that Wigert [21, p, 203] derived the same result, however, without introducing the digamma function $\psi(z)$, by means of the Euler-MacLaurin summation formula.

We recall that in [19, p. 191] we have derived (using the Voronoï summation formula valid for the test function space in [14]) the decomposition of $\Theta(t)$ in a "Weyl" term and an "oscillatory" term

$$\Theta(t) =: \Theta^{W}(t) + \Theta^{\mathrm{Osc.}}(t), \quad t \in (0, \infty) \tag{22}$$

with

$$\begin{aligned} \Theta^{W}(t) &:= \int_0^\infty (\ln k + 2\gamma) \mathrm{e}^{-tk} \mathrm{d}k + \frac{1}{4} \\ &= -\frac{\ln t}{t} + \frac{\gamma}{t} + \frac{1}{4}, \quad t \in (0, \infty) \end{aligned} \tag{23}$$

and

$$\begin{aligned} \Theta^{\mathrm{Osc.}}(t) &:= 2\pi \sum_{n=1}^{\infty} d(n) \int_0^\infty \left[\frac{2}{\pi} K_0 \left(4\pi \sqrt{nk} \right) - Y_0 \left(4\pi \sqrt{nk} \right) \right] \mathrm{e}^{-tk} \mathrm{d}k \\ &= -\frac{2}{t} \sum_{n=1}^{\infty} d(n) \left[\exp \left(\frac{4\pi^2 n}{t} \right) \mathrm{Ei} \left(-\frac{4\pi^2 n}{t} \right) \right. \\ &\qquad \left. + \exp \left(-\frac{4\pi^2 n}{t} \right) \mathrm{Ei} \left(\frac{4\pi^2 n}{t} \right) \right], \quad t \in (0, \infty). \end{aligned} \tag{24}$$

Here $\mathrm{Ei}(x)$ is the exponential integral function defined for $x > 0$ by the Cauchy principal value (see [18, p. 342]), where $\mathrm{Ei}(x) \equiv \mathrm{E}^*(x)$ and $\mathrm{Ei}(-x) \equiv -\mathrm{E}_1(x)$ for $x > 0$. From their asymptotics for $x \to \infty$ [18, pp. 346, 347] we obtain (after replacing e^{-x} by e^{x} on p. 347 l.c.)

$$e^{x} \, \mathrm{Ei}(-x) + e^{-x} \, \mathrm{Ei}(x) = \sum_{m=0}^{M} \frac{2(2m+1)!}{x^{2m+2}} + \mathrm{O} \left(\frac{1}{x^{2M+4}} \right), \quad x \to \infty. \tag{25}$$

Therefore, by (2) and (25), the sum on the r.h.s of (24) converges absolutely. Note that in (23) and (24) the test function ($h_t(k) := \mathrm{e}^{-tk}$, $t > 0$ in [19, p. 191]) is chosen in such a way that the integrals in (23) and (24) correspond to well defined Laplace transformations. Combining (21) with (22), (23) and (24) we infer

$$
\begin{aligned}
\Theta^{\mathrm{Osc.}}(t) &= -\frac{2}{t}\sum_{n=1}^{\infty}\left[\operatorname{Re}\psi\left(1+\mathrm{i}\frac{2\pi n}{t}\right)-\ln\left(\frac{2\pi n}{t}\right)\right] \\
&= -\frac{2}{t}\sum_{n=1}^{\infty} d(n)\left[\exp\left(\frac{4\pi^2 n}{t}\right)\mathrm{Ei}\left(-\frac{4\pi^2 n}{t}\right)\right. \\
&\qquad\qquad \left. +\exp\left(-\frac{4\pi^2 n}{t}\right)\mathrm{Ei}\left(\frac{4\pi^2 n}{t}\right)\right], \quad t\in(0,\infty).
\end{aligned} \tag{26}
$$

In order to prove (26) in a direct, alternative way, in particular without using the Voronoï summation formula, we need the following modified Poisson summation formula, theorem 6 in [14, p. 74] (in fact there is a typographic mistake therein and we have replaced $\gamma-\ln 2$ by $\ln(2\pi)$ in the first summand on the r.h.s of (6.21) in [14, p. 74]).

Theorem 2.2 (Dixon/Ferrar: 1937, [14])**.** *If $h(x)$ is a function which satisfies the conditions of theorem* 2.1 *except in so far as they concern the neighbourhood of the origin, and if, for a certain constant b,*

$$
H(x) := h(x) - b\ln x, \quad x\in(0,\infty), \tag{27}
$$

is of bounded variation in the neighbourhood of $x=0$, then the following formula holds ($\alpha\in(0,\infty)$):

$$
\begin{aligned}
\sum_{n=1}^{\infty} h(\alpha n) &= \frac{1}{2}b\ln(2\pi) - \frac{1}{2}\lim_{x\to 0^+} H(\alpha x) + \frac{1}{\alpha}\int_0^{\infty} h(x)\mathrm{d}x \\
&\quad + 2\sum_{n=1}^{\infty}\left[\frac{1}{\alpha}\int_0^{\infty} h(x)\cos\left(\frac{2\pi n}{\alpha}x\right)\mathrm{d}x + \frac{b}{4n}\right].
\end{aligned} \tag{28}
$$

(Note that in [14] the relation (28) has been stated for the special value $\alpha\equiv 1$). First, due to (25), we can write the last sum in (24) as a double sum

$$
\sum_{n=1}^{\infty} d(n) h\left(\frac{4\pi^2 n}{t}\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} h\left(\frac{4\pi^2 m}{t}n\right), \quad t\in(0,\infty), \tag{29}
$$

where we have defined

$$
h(x) := \exp(x)\,\mathrm{Ei}(-x) + \exp(-x)\,\mathrm{Ei}(x), \quad x\in(0,\infty). \tag{30}
$$

Notice that, due to [18, p. 343, 346, 347] and (25), the function $h(x)$ satisfies the required conditions of theorem 2.2. Furthermore, it holds [18, p. 343]

$$
h(\alpha x) = 2\gamma + 2\ln\alpha + 2\ln x + \mathrm{O}(x^2\ln x), \quad x\to 0^+, \quad \alpha\in(0,\infty) \tag{31}
$$

and thus we identify in our case $b=2$. By Fourier's inversion formula we obtain with [22, p. 8]

$$
\int_0^{\infty} h(x)\cos\left(\frac{2\pi n}{\alpha}x\right)\mathrm{d}x = -\frac{\frac{2\pi^2 n}{\alpha}}{\left(\frac{2\pi n}{\alpha}\right)^2+1}, \quad \alpha\in(0,\infty), \quad n\in\mathbb{N}_0. \tag{32}
$$

Therefore, we obtain by (28) and (32) setting $\alpha := \frac{4\pi^2 m}{t}$ ($m \in \mathbb{N}$, $t \in (0,\infty)$)

$$\begin{aligned}\sum_{n=1}^{\infty} h\left(\frac{4\pi^2 m}{t} n\right) &= \ln(2\pi) - \gamma - \ln\left(\frac{4\pi^2 m}{t}\right) + \sum_{n=1}^{\infty}\left[\frac{-n}{n^2 + \left(\frac{2\pi m}{t}\right)^2} + \frac{1}{n}\right] \\ &= -\gamma - \ln\left(\frac{2\pi m}{t}\right) + \sum_{n=1}^{\infty} \frac{\left(\frac{2\pi m}{t}\right)^2}{\left(n^2 + \left(\frac{2\pi m}{t}\right)^2\right) n}.\end{aligned} \tag{33}$$

With the identity [23, p. 106] ($x, y \in (0,\infty)$)

$$\sum_{k=1}^{\infty} \frac{1}{((kx)^2 + y^2)\, k} = \frac{1}{2y^2}\left[\psi\left(1 + \mathrm{i}\frac{y}{x}\right) + \psi\left(1 - \mathrm{i}\frac{y}{x}\right) + 2\gamma\right] \tag{34}$$

we get for the last sum on the r.h.s. in (33) ($m \in \mathbb{N}$, $t \in (0,\infty)$)

$$\sum_{n=1}^{\infty} \frac{\left(\frac{2\pi m}{t}\right)^2}{\left(n^2 + \left(\frac{2\pi m}{t}\right)^2\right) n} = \operatorname{Re}\psi\left(1 + \mathrm{i}\frac{2\pi m}{t}\right) + \gamma. \tag{35}$$

Putting (28), (30), (33) and (35) together we gain the expected first line in (26).

For the next steps we need the notion of the Laplace transform in the sense of [24]. Therefore, we define the L-function space [24, p. 13].

Definition 2.3. *A real or complex function f is an L-function iff*

- *f is defined at least for $t > 0$,*
- *in each finite interval $0 < T_1 \le t \le T_2$ the function f is Riemann integrable,*
- *the improper Riemann integral*

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{T} |f(t)| \mathrm{d}t, \quad T \in (0,\infty) \tag{36}$$

  exists,
- *there exists a real or complex s_0 such that for some fixed $T > 0$ the following improper Riemann integral*

$$\lim_{\omega \to 0} \int_{T}^{\omega} |f(t)| \mathrm{e}^{-s_0 t} \mathrm{d}t \tag{37}$$

  exists.

Now, let f be an L-function. We consider expressions of the form

$$\int_{0}^{\infty} \Theta(t) f(t) \mathrm{d}t, \tag{38}$$

where $\Theta(t)$ is defined in (8). Due to (9) the improper integral (38) exists if

$$f(t) \sim t^{\beta} L(t), \quad t \to 0, \quad \beta > 0, \quad k \in \mathbb{N}_0 \quad \text{arbitrary}, \tag{39}$$

where $L(t)$ is a slowly increasing function at $t = 0$ (for instance the absolute value of the logarithmic function) which is continuous and positive possessing the property [24, p. 201]

$$\frac{L(ut)}{L(t)} \to 1, \quad t \to 0, \quad u > 0 \quad \text{arbitrary.} \tag{40}$$

For such a function one can show [24, p. 202], [25, p. 45] (replacing t by $\frac{1}{t}$) that

$$t^{\epsilon} L(t) \to 0, \quad t \to 0, \quad \epsilon > 0 \quad \text{arbitrary.} \tag{41}$$

Furthermore, we assume for further convenience that the function $f(t)$ is a so-called regulated function (i.e. for every t in the domain of definition both the left and right limits $f(t-)$ and $f(t+)$ exist [but must not be equal]) and possesses compact support. Since

$$\Theta_N(t) := \sum_{n=1}^{N} d(n) \mathrm{e}^{-nt} \leq \Theta(t) \quad \text{for all} \quad N \in \mathbb{N}, \quad t \in (0, \infty) \tag{42}$$

the improper Riemann integral (38) converges uniformly and absolutely with respect to the summation in $\Theta(t)$. Furthermore, $\Theta_N(t)$ converges locally uniformly on $(0, \infty)$ to $\Theta(t)$. Thus, integration and summation can be interchanged in (38) and we obtain

$$\int_0^{\infty} \Theta(t) f(t) \mathrm{d}t = \sum_{n=1}^{\infty} d(n) \tilde{f}(n), \tag{43}$$

where the tilde denotes the Laplace transform of $f(t)$. In order to check that the sum on the r.h.s. in (43) converges, we can use a theorem of [24, p. 202]:

Theorem 2.4 (Doetsch, [24])**.** *Let f be an L-function possessing the asymptotics*

$$f(t) \sim B t^{\beta} L(t), \quad t \to 0, \quad \beta > -1, \quad B \quad \textit{arbitrary}, \tag{44}$$

where $L(t)$ is a regulated increasing function at $t = 0$ (and fulfills therefore (40) *and* (41)*). Then for the Laplace transform $\tilde{f}$ of f the following asymptotic formula holds:*

$$\tilde{f}(s) \sim B \frac{\Gamma(\beta + 1)}{s^{\beta+1}} L\left(\frac{1}{s}\right), \quad s \to \infty, \quad s \in \mathbb{R}^+. \tag{45}$$

Therefore, the sum on the r.h.s. of (43) converges absolutely due to (2). Now, we use (21) and (26) and attain with (43)

$$\begin{aligned} \sum_{n=1}^{\infty} d(n) \tilde{f}(n) &= \int_0^{\infty} \left[\Theta^{W}(t) + \Theta^{\mathrm{Osc.}}(t)\right] f(t) \mathrm{d}t \\ &= -\int_0^{\infty} \frac{\ln t}{t} f(t) \mathrm{d}t + \gamma \int_0^{\infty} \frac{f(t)}{t} \mathrm{d}t + \frac{1}{4} \int_0^{\infty} f(t) \mathrm{d}t \\ &\quad + \int_0^{\infty} f(t) \Theta^{\mathrm{Osc.}}(t) \mathrm{d}t. \end{aligned} \tag{46}$$

Using the identities [22, p. 149] ($t \in (0,\infty)$)

$$-\int\limits_0^\infty (\gamma + \ln k)\,\mathrm{e}^{-kt}\mathrm{d}k = \frac{\ln t}{t}, \qquad \int\limits_0^\infty \mathrm{e}^{-kt}\mathrm{d}k = \frac{1}{t} \tag{47}$$

we obtain, by Fubini's theorem for improper Riemann integrals and with our previous assumption on the function f, the identity

$$-\int\limits_0^\infty \frac{\ln t}{t} f(t)\mathrm{d}t + \gamma \int\limits_0^\infty \frac{f(t)}{t}\mathrm{d}t + \frac{1}{4}\int\limits_0^\infty f(t)\mathrm{d}t = \int\limits_0^\infty (\ln k + 2\gamma)\tilde{f}(k)\mathrm{d}k + \frac{\tilde{f}(0)}{4}. \tag{48}$$

Now, we use the required condition of the compact support of the function f. Thus, the last integral on the r.h.s in (46) is in fact an integral over a finite interval and on this interval, due to (25), the sum ($t \in (0,\infty)$)

$$\begin{aligned}\Theta^{\mathrm{Osc.}}{}_N(t) := \sum_{n=1}^{N} d(n)\Bigg[&\exp\left(\frac{4\pi^2 n}{t}\right)\mathrm{Ei}\left(-\frac{4\pi^2 n}{t}\right)\\ &+\exp\left(-\frac{4\pi^2 n}{t}\right)\mathrm{Ei}\left(\frac{4\pi^2 n}{t}\right)\Bigg]\end{aligned} \tag{49}$$

converges uniformly to $\Theta^{\mathrm{Osc.}}(t)$ for $N \to \infty$. Therefore, we can interchange integration and summation in the last summand on the r.h.s of (46). Finally, we can again use Fubini's theorem for improper Riemann integrals and obtain with relation (24) respectively [26, p. 352] and [26, p. 266] the following theorem:

Theorem 2.5. *Let f be a regulated L-function possessing compact support and the asymptotics* (39)*. Then the following formula holds:*

$$\begin{aligned}\sum_{n=1}^\infty d(n)\tilde{f}(n) = &\int\limits_0^\infty (\ln k + 2\gamma)\tilde{f}(k)\mathrm{d}k + \frac{\tilde{f}(0)}{4}\\ &+ 2\pi\sum_{n=1}^\infty d(n)\int\limits_0^\infty \left[\frac{2}{\pi}K_0\left(4\pi\sqrt{nk}\right) - Y_0\left(4\pi\sqrt{nk}\right)\right]\tilde{f}(k)\mathrm{d}k,\end{aligned} \tag{50}$$

where $\tilde{f}$ denotes the Laplace transform of f

Note that (50) is identical to the Voronoï summation formula (7) if $\tilde{f}(k)$ is renamed $f(k)$.

Acknowledgements

S. E. would like to thank the graduate school "Analysis of complexity, information and evolution" of the Land Baden-Württemberg for the stipend which has enabled this paper.

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