Reynolds transport theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega(t)} f(x,t)\,\mathrm{d}\Omega = \int\limits_{\Omega(t)} \left\{\frac{\partial f}{\partial t}(x,t) + \nabla\cdot(f\,\vec{u}\,)\right\}\,\mathrm{d}\Omega$$

Preliminaries Let $\Omega_0 \subset \mathbb{R}^n$ and let $\phi : \Omega_0 \times [0,\infty) \to \Omega(t) \subset \mathbb{R}^n$ be the transformation defined by

$$(x,t) \mapsto y := \phi(x,t).$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}y =: u(y,t) = u(\phi(x,t),t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(x,t) =: \phi_t(x,t)$$

Calcuation of the Jacobian-matrix:

$$D_x\phi_t(x,t) =: \phi_{xt}(x,t) \overset{S.v.S}{=} \phi_{tx}(x,t) = u_x(\phi(x,t),t)\cdot\phi_x(x,t)$$

Proof: For a more detailed proof we refer to the lecture notes of Prof. Dr. J. Lorenz, "Die Naviers-Stokes Gleichung ..", RWTH-Aachen, 1994.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega(t)} f(y,t)\,\mathrm{d}\Omega &\overset{1}{=} \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_0} f(\phi(x,t),t)\ \left|\det\left(\phi_x(x,t)\right)\right|\mathrm{d}\Omega_0 \\
&= \int\limits_{\Omega_0}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[f(\phi(x,t),t)\right]\ \left|\det\left(\phi_x(x,t)\right)\right| + f(\phi(x,t),t)\ \frac{\mathrm{d}}{\mathrm{d}t}\left|\det\left(\phi_x(x,t)\right)\right|\right\}\mathrm{d}\Omega_0 \\
&\overset{2}{=} \int\limits_{\Omega_0}\left\{\left[\frac{\partial f}{\partial t}(\phi(x,t),t) + \nabla f(\phi(x,t),t)\cdot u(\phi(x,t),t)\right]\ \left|\det\left(\phi_x(x,t)\right)\right|\right. \\
&\left. +\ f(\phi(x,t),t)\ (\nabla\cdot u)(\phi(x,t),t)\left|\det\left(\phi_x(x,t)\right)\right|\right\}\mathrm{d}\Omega_0 \\
&= \int\limits_{\Omega_0}\left\{\left[\frac{\partial f}{\partial t}(\phi(x,t),t) + \nabla f(\phi(x,t),t)\cdot u(\phi(x,t),t)\right]\ \left|\det\left(\phi_x(x,t)\right)\right|\right. \\
&\left. +\ f(\phi(x,t),t)\ (\nabla\cdot u)(\phi(x,t),t)\right\}\left|\det\left(\phi_x(x,t)\right)\right|\mathrm{d}\Omega_0 \\
&\overset{1}{=} \int\limits_{\Omega(t)}\left\{\frac{\partial f}{\partial t}(y,t) + \nabla\cdot\left[f(y,t)\ u(y,t)\right]\right\}\mathrm{d}\Omega
\end{aligned}$$

1. Transformation theorem with respect to $\phi(t,\cdot) : \Omega_0 \to \Omega(t)$
2. Derivation rule for Wronski determinants: (würde hier das ganze Lemma hinschreiben sonst ist nicht klar was $Y(t)$ bzw. $A(t)$ überhaupt ist.) $\frac{\mathrm{d}}{\mathrm{d}t}\det Y(t) = \mathrm{tr}A(t)\cdot\det Y(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\det\left(\phi_x(x,t)\right)\right| = \mathrm{tr}u_x(\phi(x,t),t)\ \left|\det\left(\phi_x(x,t)\right)\right|$$

and

$$\mathrm{tr}u_x(\phi(x,t),t) = (\nabla\cdot u)(\phi(x,t),t)$$