Computational Biomechanics 2017

Lecture 2:

Basic Mechanics 2

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</table>
Mechanical Basics
1.3 Variables, Dimensions and Units

Standard: ISO 31, DIN 1313

Variable = Number \cdot Unit
Length L = 2 \cdot m = 2 m

\{Variable\} = Number
[Variable] = Unit

Three mechanical SI-Units:
\begin{itemize}
  \item m (Meter)
  \item kg (Kilogram)
  \item s (Seconds)
\end{itemize}
2.1 Force

- We all believe to know what a force is.
- But, force is an invention not a discovery!
- ... it can not be measured directly.

**Newton’s 2nd Law [Axiom]:**

\[ \text{Force} = \text{Mass times Acceleration} \quad \text{or} \quad F = m \cdot a \]

**Note to Remember:**

„A force is the cause of acceleration or deformation of a body“
Representation of Forces

... with arrows

Forces are Vectors with
- Magnitude
- Direction
- Sense of Direction

5 N
Line of action
Screw
Units of Force

Newton

\[ N = \text{kg} \cdot \text{m/s}^2 \]

\[ F_G = m \cdot g = 0,1 \text{ kg} \cdot 9,81 \text{ m/s}^2 \]

\[ = 0,981 \text{ kg m/s}^2 \]

\[ \approx 1 \text{ N} \]

Note to Remember:

1 Newton \approx \text{Weight of a bar of chocolate (100 g)}
2.2 Method of Sections (Euler) [Schnittprinzip]

Free-Body Diagramm (FBD) [Freikörper-Bild]

Note to Remember:
First, cut the system, then include forces and moments.
Free-body diagram = completely isolated part.
2.2 Method of Sections

Cut through joint (2D)

Cut through bone (2D)
2.2 Method of Sections

Cut through bone (2D)  $\equiv$  Cut through joint (2D)
2.3 Combining and Decomposing Forces

Summation of Magnitudes

\[ 5 \text{ N} + 3 \text{ N} = 8 \text{ N} \]

Subtraction of Magnitudes

\[ 5 \text{ N} - 3 \text{ N} = 2 \text{ N} \]

Vector Addition

\[ F_1 = 5 \text{ N}, \quad F_2 = 3 \text{ N} \]

\[ F_R \approx 7 \text{ N} \]

Decomposition into Components

\[ F_{\text{axial}} \]

\[ F_{\text{transverse}} \]
2.4 The Moment [Das Moment]

Note to remember:
The moment $M = F \cdot a$ is equivalent to a force couple $(F, a)$.
A moment is the cause for angular acceleration or angular deformation (Torsion, Bending) of a body.
Units for Moment

Newton-Meter

\[ N \cdot m = \text{kg} \cdot \text{m}^2/\text{s}^2 \]

Representation of Moments

... with rotation arrows or double arrows

Moments are Vectors with ...

- Magnitude
- Direction
- Sense of Direction

Rechte-Hand-Regel:
2.5 Moment of a Force about a Point

[\text{Versetzungsmoment}]

Note to Remember:

\text{Moment} = \text{Force times lever-arm}
2.7 Static Equilibrium

Important:
Free-body diagram (FBD) first, then equilibrium!

For 2D Problems max. 3 equations for each FBD:

The sum of all forces in x-direction equals zero: \( F_{1,x} + F_{2,x} + \ldots = 0 \)

The sum of all forces in y-direction equals zero: \( F_{1,y} + F_{2,y} + \ldots = 0 \)

The sum of Moments with respect to P equals zero: \( M_{1,z}^P + M_{2,z}^P + \ldots = 0 \)

(For 3D Problems max. 6 equations for each FBD)
2.7 Static Equilibrium

Important:
Free-body diagram (FBD) first, then equilibrium!

3 equations of equilibrium for each FBD in 2D:

- Sum of all forces in x-direction: \( F_{1,x} + F_{2,x} + \ldots = 0 \),
- Sum of all forces in y-direction: \( F_{1,y} + F_{2,y} + \ldots = 0 \),
- Sum of all moments w.r.t. to P: \( M_{1,z}^P + M_{2,z}^P + \ldots = 0 \).

- Force EEs can be substituted by moment EEs
- 3 moment reference points should not lie on one line
6 equilibrium equations for one FBD in 3D:

Summe aller Kräfte in x - Richtung : \( \sum F_{ix} = 0, \)

Summe aller Kräfte in y - Richtung : \( \sum F_{iy} = 0, \)

Summe aller Kräfte in z - Richtung : \( \sum F_{iz} = 0, \)

Summe aller Momenteum x - Achse bezüglich Punkt P : \( \sum M_{ix}^P = 0. \)

Summe aller Momenteum y - Achse bezüglich Punkt Q : \( \sum M_{iy}^Q = 0. \)

Summe aller Momenteum z - Achse bezüglich Punkt R : \( \sum M_{iz}^R = 0. \)

- Force EEs can be substituted by moment EEs
- Max. 2 moment axis parallel to each other
- Determinant of coef. matrix not zero
2.8 Recipe for Solving Problems in Statics

Step 1: Model building. Generate a simplified replacement model (diagram with geometry, forces, constraints).

Step 2: Cutting, Free-body diagram. Cut system and develop free-body diagrams. Include forces and moments at cut, as well as weight.

Step 3: Equilibrium equations. Write the force- and moment equilibrium equations (only for free-body diagrams).

Step 4: Solve the equations. One can only solve for as many unknowns as equations, at most.

Step 5: Display results, explain, confirm with experimental comparisons. Are the results reasonable?
2.9 Classical Example: „Biceps Force“

From:
„De Motu Animalium“
G.A. BORELLI
(1608-1679)
Step 1: Model building

Rope (fixed length) 10 kg Beam (rigid, massless) Joint (frictionless)
Schritt 2: Schneiden und Freikörperbilder

10 kg

100 N
### More to Step 2: Cutting and Free-Body Diagrams

![Free-Body Diagrams](image)

### Step 3 and 4: Equilibrium and Solving the Equations

<table>
<thead>
<tr>
<th>Sum of all forces in vertical direction = 0</th>
<th>Sum of all forces in “rope” direction = 0</th>
<th>Sum of all moments with respect to Point G = 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 N + (−S₁) = 0</td>
<td>S₂ + (−S₃) = 0</td>
<td>−S₁ \cdot h₁ + S₂ \cdot h₂ = 0</td>
</tr>
<tr>
<td>⇒ S₁ = 100 N</td>
<td>⇒ S₃ = S₂</td>
<td>−100 N \cdot 35 \text{cm} + S₂ \cdot 5 \text{cm} = 0</td>
</tr>
</tbody>
</table>

⇒ S₂ = 100 N \cdot \frac{35 \text{cm}}{5 \text{cm}} = 700 N
3.1 Stresses

... to account for the loading of the material!

500 N

Fotos: Lutz Dürselen
Note to Remember:
Stress = „smeared“ force
Stress = Force per Area or $\sigma = \frac{F}{A}$
(Analogy: „Nutella bread teast“)
Units of Stress

Pascal: \( 1 \text{ Pa} = 1 \text{ N/m}^2 \)
Mega-Pascal: \( 1 \text{ MPa} = 1 \text{ N/mm}^2 \)

3.2 Example:

"Tensile stress in Muscle:

\[
\sigma_1 = \frac{F}{A_1} = \frac{700 \text{ N}}{7000 \text{ mm}^2} = 0,1 \frac{\text{N}}{\text{mm}^2} = 0,1 \text{ MPa}
\]
\[
\sigma_2 = \frac{F}{A_2} = \frac{700 \text{ N}}{70 \text{ mm}^2} = 10 \frac{\text{N}}{\text{mm}^2} = 10 \text{ MPa}
\]
3.3 Normal and Shear Stresses

Tensile bar

Cut 1:
- Normal stress $\sigma_1$

Cut 2:
- Normal stress $\sigma_2$
- Shear stress $\tau_2$
Note to Remember:

First, you must choose a point and a cut through the point, then you can specify (type of) stresses at this point in the body.

Normal stresses (tensile and compressive stress) are oriented perpendicular to the cut-surface.

Shear stresses lie tangential to the cut-surface.
General (3D) Stress State: Stress Tensor

... in one point of the body:  How much numbers do we need?

- 3 stress components in one cut (normal str., 2x shear str.)
  
  times

- 3 cuts (e.g. frontal, sagittal, transversal)
  
  results in

- 9 stress components for the full stress state in the point.

- But **only 6** components are linear independent („equality of shear stresses“)

\[
\begin{pmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{pmatrix}
\]

„Stress Tensor“
Symmetry of the Stress Tensor

Boltzmann Continua: Only volume forces ($f_x$ und $f_y$), no volume moments assumed → “Equality of corresponding shear stresses”

\[
\sigma = \begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\cdot & \sigma_{yy} & \tau_{yz} \\
sym & \cdot & \sigma_{zz}
\end{bmatrix}
\]

\[
\sum M^{(c)} = 2 \cdot \tau_{xy} \Delta y \Delta z \cdot \frac{1}{2} \Delta x - 2 \cdot \tau_{yx} \Delta x \Delta z \cdot \frac{1}{2} \Delta y = 0.
\]
General 3D Stress State

6 Components → 6 Pictures
Problem:
- How to produce nice Pictures?
- Which component should I use?
- Do I need 6 pictures at the same time?

So called „Invariants“ are „smart mixtures“ of the components

\[
\sigma_{\text{Mises}} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - \sigma_{xx} \sigma_{yy} - \sigma_{xx} \sigma_{zz} - \sigma_{yy} \sigma_{zz} + 3 \tau_{xy}^2 + 3 \tau_{xz}^2 + 3 \tau_{yz}^2}
\]
3.4 Strains

• Global, (external) strains

\[ \varepsilon := \frac{\text{Change in length}}{\text{Original length}} = \frac{\Delta L}{L_0} \]

• Local, (internal) strains

Units of Strain

without a unit

1

1/100 = %

1/1,000,000 = \( \mu \varepsilon \) (micro strain)

= 0,1 %
3D Local Strain State: Strain Tensor

\[ \mathbf{\Delta}x^+ + \delta \phi_{xy} \]

Displacements [Verschiebungen]
3D Local Strain State: Strain Tensor

Definition:

\[ \varepsilon_{xx} = \lim_{x_0 \to 0} \frac{\Delta x}{x_0}, \quad \varepsilon_{yy} = \lim_{y_0 \to 0} \frac{\Delta y}{y_0}, \quad \varepsilon_{zz} = \lim_{z_0 \to 0} \frac{\Delta z}{z_0} \]

\[ \varepsilon_{xy} = \frac{1}{2} \cdot \Delta \gamma, \quad \varepsilon_{xz} = \frac{1}{2} \cdot \Delta \beta, \quad \varepsilon_{yz} = \frac{1}{2} \cdot \Delta \alpha \]

Universal Strain Definition:

\[ \varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right), \quad i, j = \{x, y, z\} \]

\[ \varepsilon \equiv \begin{pmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz}
\end{pmatrix} \]

Note to Remember:
Strain is relative change in length (and shape)
Displacement vs. Strain

Displacement $u_x$

Strain, $\epsilon_{xx}$
Material Laws for Biological Tissues
Characterizing Mechanical Properties

Structural Properties
(Organ Properties)

3-Point Bending
of a bone specimen

Material Properties
(Tissue Properties)

Tensile Test
of a standardized Specimen
# Mechanical Properties

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<tr>
<th></th>
<th>Organ Properties</th>
<th>Tissue Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Elastic</strong></td>
<td>Different stiffnesses: load/deformation</td>
<td>Young's Modulus</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Poisson's Ratio</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Shear Modulus</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Compressive Modulus</td>
</tr>
<tr>
<td><strong>Ultimate</strong></td>
<td>Ultimate forces, moments, displacements: tension, compression, bending, torsion</td>
<td>Ultimate stresses, strains: tensile, compressive, shear at fracture</td>
</tr>
<tr>
<td></td>
<td></td>
<td>at end of elastic region, at start of yielding</td>
</tr>
</tbody>
</table>
Example: Fracture Callus


- Haematoma
- Granulation tissue
- Fibrous tissue
- Cartilage
- Woven bone
- Cortical bone

0.01 ... 3 MPa

0.4 ... 500 MPa

400 ... 6000 MPa

10 ... 20 GPa
Overview

**Simplest Material Law:**
- Linear-elastic, isotropic (Hook’s Law)

**More complex Laws:**
- Nonlinear
- Hyperelastic (nonlinear elastic + large deformations)
- Viscoelastic
- Non-elastic, plastic, yielding, hardening
- Anisotropic
- Fatigue, recovery
- Remodeling, healing
In principal:

Biological Tissues know all of these bad things.
They often combine the bad things
Stress-strain curve

- $\sigma_B$: Ultimate stress
- $\sigma_S$: Yield point
- $\varepsilon_B$: Ultimate strain
- $\varepsilon_S$: Strain at yield point
- $E$: Young’s modulus
Linear Elastic Law

Linear stress-strain relation

\[ \sigma = E \cdot \varepsilon \]

\[ \sigma = \frac{E}{\varepsilon} \]

\[ \sigma_{ij} = E_{ijkl} \cdot \varepsilon_{kl} \quad (81) \]

- Equality of shear stresses (Boltzmann Continua) and strains \( (36) \)
- Reciprocity Theorem from Maxwell* \( \rightarrow \) fully anisotropic \( (21) \)
- Orthotropic \( (9) \)
- Transverse Isotropic \( (5) \)
- Isotropy \( (2) \)

*) also known as Betti’s theorem or Maxwell-Betti reciprocal work theorem
Linear Elastic Law

\[ \sigma = \frac{E}{1 + \nu} \cdot \varepsilon \]

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix}
= \frac{E}{(1 + \nu) \cdot (1 - 2\nu)}
\begin{bmatrix}
(1 - \nu) & \nu & \nu & 0 & 0 & 0 \\
(1 - \nu) & \nu & 0 & 0 & 0 & 0 \\
(1 - \nu) & 0 & 0 & 0 & 0 & 0 \\
(1 - 2\nu) & 0 & 0 & 0 & 0 & 0 \\
(1 - 2\nu) & 0 & 0 & 0 & 0 & 0 \\
(1 - 2\nu) & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix}
\]

\[ E \] - Young's modulus
\[ \nu \] - Poisson's ratio \( (0 \ldots 0.5) \)
\[ G \] - Shear modulus
\[ K \] - Bulk modulus
\[ \mu, \lambda \] - Lame Constants

\[ \leftarrow 2 \text{ of these} \]
Bone tissue

### Anisotropic Properties

<table>
<thead>
<tr>
<th>Bone Type</th>
<th>Anisotropic Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corticale bone</td>
<td>Transverse isotropic (5)</td>
<td>Like Wood</td>
</tr>
<tr>
<td>Trabecular bone</td>
<td>Orthotropic (9)</td>
<td>3D Grid</td>
</tr>
</tbody>
</table>
## Anisotropic Properties

<table>
<thead>
<tr>
<th>Material</th>
<th>E Moduli in MPa</th>
<th>Strength in MPa</th>
<th>Fracture strain in %</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Spongy bone</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vertebra</td>
<td>60 (male)</td>
<td>4,6 (male)</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>35 (female)</td>
<td>2,7 (female)</td>
<td></td>
</tr>
<tr>
<td>prox. Femur</td>
<td>240</td>
<td>2.7</td>
<td>2.8</td>
</tr>
<tr>
<td>Tibia</td>
<td>450</td>
<td>5...10</td>
<td>2</td>
</tr>
<tr>
<td>Bovine</td>
<td>200...2000</td>
<td>10</td>
<td>1,7...3,8</td>
</tr>
<tr>
<td>Ovine</td>
<td>400...1500</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td><strong>Cortical bone</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Longitudinal</td>
<td>17000</td>
<td>200</td>
<td>2,5</td>
</tr>
<tr>
<td>Transversal</td>
<td>11500</td>
<td>130</td>
<td></td>
</tr>
</tbody>
</table>
Nonlinear Elastic and Hyperelastic Law

→ soft, fibrous tissues

Hyperelastic Laws: Nonlinear + large deformations
SED-stress relation, (Mooney-Rivlin, Neo-Hookean, ...)

Stress $\sigma$

progressive

linear

degressive

Strain $\varepsilon$
# Properties for Ligaments

<table>
<thead>
<tr>
<th>Properties</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile strength</td>
<td>20 – 50 MPa</td>
</tr>
<tr>
<td>Shear strength</td>
<td>1 kPa (fast loading 0.04mm/h)</td>
</tr>
<tr>
<td></td>
<td>740 kPa (slow loading 420mm/h)</td>
</tr>
<tr>
<td>Rupture strain</td>
<td>2 m/m (Shear)</td>
</tr>
<tr>
<td></td>
<td>0,1 – 0,5 m/m (Tension)</td>
</tr>
<tr>
<td>Rupture force</td>
<td>1500 N (Finger)</td>
</tr>
<tr>
<td></td>
<td>100 - 400 N (Ankle)</td>
</tr>
<tr>
<td></td>
<td>1200 N (ACL)</td>
</tr>
<tr>
<td>Stiffness</td>
<td>100 - 700 N/mm (ACL)</td>
</tr>
<tr>
<td>E Modulus, tension</td>
<td>100 – 1000 MPa</td>
</tr>
</tbody>
</table>
Non-elastic = plastic

→ Bone strength

ISO bone screw

Local Damage

Stress $\sigma$

loading

unloading

Strain $\varepsilon$
Viscoelastic Laws (Time Dependent)

→ Cartilage, Fibrous Tissue, Bone
Effects of Viscoelastic Materials

- Retardation (Creep):
  - Applied: Stress steps
  - Measured: Strain history

- Relaxation:
  - Applied: Strain steps
  - Measured: Stress history
Viscoelastic Laws (Time Dependent)

Typ: Inner damping

Typ: Memory effect
Poroelastic

→ Solid structure with pores: Cartilage, Fibrous Tissue, Bone

Hyaloron
## Properties for Articular Cartilage

<table>
<thead>
<tr>
<th>Properties</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>1300 kg/m³</td>
</tr>
<tr>
<td>Water content</td>
<td>75%</td>
</tr>
<tr>
<td>Compressive strength</td>
<td>5 MPa</td>
</tr>
<tr>
<td>Tensile strength</td>
<td>20 MPa</td>
</tr>
<tr>
<td>E Modulus, eq.</td>
<td>0.8 MPa</td>
</tr>
<tr>
<td>E Modulus, initial</td>
<td>3 MPa</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>3.5 MPa</td>
</tr>
<tr>
<td>Friction</td>
<td>0.005</td>
</tr>
<tr>
<td>Rupture strain</td>
<td>6...8%</td>
</tr>
<tr>
<td>Permeability</td>
<td>0.8 $10^{-14}$ m⁴/Ns</td>
</tr>
</tbody>
</table>
When is simplification allowed?

- Non linear: Small strains $\rightarrow$ “linear"
- Plastic: Approximation: „Ideal elastic plastic“
- Viscoelastic: Quasi-static Deformations $\rightarrow$ „elastic“
- Anisotropic: $\rightarrow$ „Worst-case scenario“
4 Simple Load Cases
1 Tension, Compression

Tensile bar (tensional rigidity $EA$)

Device stiffness

$$F = \frac{EA}{L_0} \Delta L, \quad k = \frac{EA}{L_0}$$

Axial Stress

$$\sigma = \frac{F}{A}$$

Strain

$$\varepsilon = \frac{\Delta L}{L_0}$$

$$\varepsilon_q = \frac{\Delta d}{d_0} = -\nu \varepsilon$$
2 Shear

Shear bolt (shear rigidity $GA$)

Device stiffness

\[ F = \frac{GA}{L} w, \quad k = \frac{GA}{L} \]

Shear stress

\[ \tau = \frac{F}{A} \]

Shear strain

\[ \varepsilon_{xy} = \frac{1}{2} \gamma \]
3 Bending (Cantilever Beam)

Beam (bending rigidity $EI_a$, Length $L$)

Cut

Cantilever formula:

$$w = \frac{L^3}{EI_a} F + \frac{L^2}{EI_a} M,$$
$$\varphi = \frac{L^2}{EI_a} F + \frac{L}{EI_a} M.$$
3 Torsion

Torsional rod (torsional rigidity $G_I$)

Length $L$, radius $r$

Cut

$M = \frac{G_I}{L} \phi, \quad c = \frac{G_I}{L}$

Note to Remember:
A hollow bone reaches a high stiffness and strength against bending and torsion with relatively little material.
**Second Moment of Area** \( I \) (SMA)  
[Flächenmoment zweiten Grades]

**Axial** moment of area (bending)  
\[
I_a = \frac{b \cdot h^3}{12} \\
I_a = \frac{\pi}{64} D^4 \\
I_a = \frac{\pi}{64} (D^4 - d^4)
\]

**Polar** (torsional) moment of area (torsion)  
\[
I_T = I_p = \frac{\pi}{32} D^4 \\
I_T = I_p = \frac{\pi}{32} (D^4 - d^4)
\]