Lévy Finance

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Contents

Exercises by Thomas Liebmann, first exercise class 28.10.08, 16-18h.

Lecture notes will be available on the website.

Outline

(1) Basic properties of Levy processes (LP)
(2) Stochastic calculus for Levy processes
(3) Financial market models based on LP

Motivation

Standard financial market model is a Black-Scholes (or Black-Scholes-Merton, BSM) model with a risky asset $dS_t = S_t\left(rdt + \sigma dW_t\right), \; S_0 = p_0$ and a bank account $dB_t = rB_tdt;\; B_0 = 1$. This implies $S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$. A European call option on S_t has the time T payoff $C_T = \left(S_T - K\right)^+$, where K denotes the strike and T the expiry time. Risk neutral valuation implies that the price at time 0 is given by $C_0 = E\left(e^{-rT}\left(S_T - K\right)^+\right) = S_0\Phi\left(d_1\right) - Ke^{-rT}\Phi\left(d_2\right)$, where Φ is the standard normal cumulative distribution function.

However, market data on European call options gives different σ for different K and T, the volatility surface. This shows that the model is inconsistent.

Different attempts have been made to correct for the volatility smile, such as time dependent volatility, volatility depending on S_T, or stochastic volatility models. However, these approaches cannot cope with the problem that markets can exhibit extreme valuation moves which are incompatible with the Black-Scholes model.

CHAPTER 1

Lévy Processes

1.1. Basic Definitions and Notations

DEFINITION 1.1.1. *(Stochastic basis, stochastic process, adapted, RCLL)*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration, i.e. an increasing family of σ-algebras $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$, $s \leq t$. A *stochastic basis* $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ satisfies the usual conditions:

(1) $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$.
(2) $\mathbb{F}$ is right-continuous, i.e. $F_t = F_{t+} := \bigcap_{s>t} F_s \ \forall t$.

A *stochastic process* $X = (X_t)_{t\geq 0}$ is a family of random variables on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$:

$$X(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R} \quad \text{on } \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$$

which is *adapted,* meaning that X_t is F_t measurable for every t, in an abuse of notation we will write $X_t \in F_t$. X is called *right-continuous with left limits (RCLL)* if it is continuous to the right *a.s.*.

Now we will consider different concepts for the "sameness" of two stochastic processes X and Y:

DEFINITION 1.1.2. *(Sameness of stochastic processes)*

(1) X and Y have the *same finite-dimensional distributions*, if for all n and $\{t_1, \ldots, t_n\}$ we have
$$(X(t_1), \ldots, X(t_n)) \stackrel{d}{=} (Y(t_1), \ldots, Y(t_n))$$
(2) Y is a *modification* of X, if $\mathbb{P}(X_t = Y_t) = 1$ for every t.[1]
(3) X and Y are *indistinguishable* if almost all their sample paths agree, i.e. $\mathbb{P}(X_t = Y_t;\ \forall\, 0 \leq t \leq \infty) = 1$.

REMARK 1.1.3. For RCLL processes, 2 and 3 are equivalent.

DEFINITION 1.1.4. *(Stopping time, optional time)*
A random variable $\tau : \Omega \to [0, \infty)$ is a *stopping time* if the set $\{\tau \leq t\} \in \mathcal{F}_t$, $\forall t$. It is an *optional time* if $\{\tau < t\} \in \mathcal{F}_t \ \forall t$.

REMARK 1.1.5.

(1) For a right-continuous $\mathbb{F}$, every optional time is a stopping time.
(2) A hitting time $\tau_A := \inf\{t > 0 :\ X_t \in A\}$, (where A is a Borel set) is a stopping time.

DEFINITION 1.1.6. *(Stopped σ-Algebra, Martingale)*

[1] This does not imply equality for almost all ω for all t.

(1) For RCLL-processes, we define the <u>*stopped σ-Algebra*</u> $\mathcal{F}_\tau$ as

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t, \ \forall t \ge 0\}.$$

(2) X is a <u>*(sub-/super-) martingale*</u> (with respect to $\mathbb{F}$ and $\mathbb{P}$) if
  (a) X is adapted, $E(|X(t)|) < \infty \quad \forall t$ and
  (b) $$E(X(t)|\mathcal{F}_s) \begin{cases} \le X(s) & \text{(super-martingale)} \\ = X(s) & \text{(martingale)} \\ \ge X(s) & \text{(sub-martingale)} \end{cases} \quad a.s. \text{ for all } 0 \le s \le t.$$

LEMMA 1.1.7. *Let X be a (sub-) martingale and ϕ a convex function with $E(|\phi(X_t)|) < \infty$. Then $\phi(X_t)$ is a sub-martingale.*

PROOF. $E(\phi(X_t)|\mathcal{F}_t) \ge \phi(E(X_t|\mathcal{F}_s)) \ge \phi(X_s)$, the first inequality by Jensens inequality. □

EXERCISE 1.1.8. ξ a random variable with $E(|\xi|) < \infty$ then $E(\xi|\mathcal{F}_t) = X_t$ is a martingale.

DEFINITION 1.1.9. *(Brownian Motion)*
$X = (X_t)_{t \ge 0}$ is a <u>*standard Brownian motion*</u> (BM) if

(1) $X(0) = 0$ *a.s.*
(2) X has independent increments: $X(t+u) - X(t)$ is independent of $\sigma(X(s);\ s \le t)$ for any $u \ge 0$.
(3) X has stationary increments: the law of $X(t+u) - X(t)$ depends only on u.
(4) X has Gaussian increments: $X(t+u) - X(t) \sim N(0, u)$.
(5) $X_t(\omega)$ has continuous paths for all ω.

THEOREM 1.1.10. *(Wiener) Brownian motion exists.*

NOTATION. We will use W as a symbol for Brownian motion.

FACT. *(Properties of Brownian motion)*

(1) $Cov(W_s, W_t) = \min(s, t)$.
(2) *$(W(t_1), \dots, W(t_n))$ is multivariate Gaussian.*
(3) *BM can be identified as Gaussian process with continuous paths.*
(4) *W is a martingale with respect to its own filtration $\mathcal{F}_t = \sigma(W_s,\ s \le t)$:*

$$E(W_t|\mathcal{F}_s) = E(W_t - W_s|\mathcal{F}_s) + E(W_s|\mathcal{F}_s) = W_s$$

Lectures: 4.11., 11.11, 25.11, 2.12, 16.12.

Exercises 28.10., 18.11., 9.12.

1.2. Characteristic Functions

DEFINITION 1.2.1. *(Characteristic function)*
If X is a random variable with cumulative distribution function F, then its <u>*characteristic function*</u> (cf) ϕ_X (or ϕ if we do not need to emphasize X) is defined as

$$\phi_X(t) = E\left(e^{itX}\right) = \int_{-\infty}^{\infty} e^{itx} F(dx), \quad t \in \mathbb{R}.$$

NOTE 1.2.2. Here $i = \sqrt{-1}$, the imaginary unit. The characteristic function always exists.

FACT 1.2.3. *(Some properties of the characteristic function)*

(1) *If X and Y are independent, then*

$$\begin{aligned}\phi_{X+Y}(t) &= E\left(e^{it(X+Y)}\right) = E\left(e^{itX}e^{itY}\right)\\ &\overset{(*)}{=} E\left(e^{itX}\right)E\left(e^{itY}\right) = \phi_X(t)\phi_Y(t),\end{aligned}$$

where $(*)$ follows from independence. So characteristic functions take convolution into multiplication.

(2) $\phi(0) = 1$.

(3) $|\phi(t)| = \left|\int_{-\infty}^{\infty} e^{itx}F(dx)\right| \leq \int_{-\infty}^{\infty}\left|e^{itx}\right|F(dx) \leq 1$

(4) *ϕ is continuous:*

$$\begin{aligned}|\phi(t+u)-\phi(t)| &= \left|\int_{-\infty}^{\infty}\left(e^{i(t+u)x} - e^{itx}\right)F(dx)\right|\\ &\leq \int_{-\infty}^{\infty}\underbrace{\left|e^{itx}\right|}_{\leq 1}\underbrace{\left|e^{iux}-1\right|}_{\leq 2}F(dx) \overset{(*)}{\to} 0\end{aligned}$$

For $u \to 0$ we have $\left|e^{iux}-1\right| \to 0$, so by Lebesgue's dominated convergence theorem, the last term tends to 0 $(*)$.[2] *Since the whole argument does not depend on t, we have in fact uniform continuity.*

(5) *Uniqueness theorem: ϕ determines the distribution function F uniquely.*

(6) *Continuity theorem: If $(X_n)_{n=0}^{\infty}$ and X are random variables with corresponding cumulative distribution functions $(\phi_n)_{n=0}^{\infty}$ and ϕ, then convergence of (ϕ_n) to ϕ, i.e. $\phi_n(t) \overset{(n\to\infty)}{\longrightarrow} \phi(t)\ \forall t$, is equivalent to convergence of F_n to F.*

EXAMPLE 1.2.4. *(Characteristic function of normally distributed random variables)*

(1) $\mathcal{N}(0,1)$, the normal density $f(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)$:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{tx}f(x)dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(tx - \frac{1}{2}x^2\right)dx\\ &\overset{(*)}{=} \frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}t^2}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}(x-t)^2\right)dx = e^{\frac{1}{2}t^2}.\end{aligned}$$

Thus substituting it for t we have $\phi_{\mathcal{N}(0,1)}(t) = \exp\left(-\frac{1}{2}t^2\right)$.

(2) $\mathcal{N}(\mu,\sigma^2):\ X \sim \mathcal{N}(0,1)$

$$E\left(e^{it(\mu+\sigma X)}\right) = e^{it\mu}E\left(e^{i\sigma tX}\right) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$$

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[2] The last term is dominated by $\int_{-\infty}^{\infty} 2F(dx) = 2 < \infty$. By the DCT, integrals and limits $(u \to 0)$ can then be interchanged.

1.3. Point Processes

1.3.1. Exponential Distribution.

DEFINITION 1.3.1. *(Exponential distribution)*
We say that the random variable T has an *exponential distribution* with parameter λ, $T \sim \text{exponential}(\lambda)$, if $\mathbb{P}(T \leq t) = 1 - e^{-\lambda t}$ for $t \geq 0$.

FACT 1.3.2. *Recall that* $E(T) = \frac{1}{\lambda}$ *and* $Var(T) = \frac{1}{\lambda^2}$.

PROPOSITION 1.3.3. (Properties of the exponential distribution)

(1) *"Lack of memory":* $\mathbb{P}(T > s + t | T > t) = \mathbb{P}(T > s)$.
(2) *Let* $T_1, T_2, \ldots, T_n$ *be independent exponentially distributed random variables with parameters* $\lambda_1, \lambda_2, \ldots, \lambda_n$. *Then* $\min\{T_1, \ldots, T_n\} \sim \textit{exponential}\,(\lambda_1 + \ldots + \lambda_n)$.
(3) $T_1, T_2, \ldots, T_n$ *i.i.d. exponential(λ) random variables. Then* $G = T_1 + T_2 + \ldots + T_n \sim \textit{Gamma}(n, \lambda)$ *with density* $\lambda e^{-\lambda} \frac{(\lambda t)^{n-1}}{(n-1)!}$ *for* $t \geq 0$.

1.3.2. Poisson Process.

DEFINITION 1.3.4. *(Poisson process)*
Let $(t_i)_{i=1}^{\infty}$ be independent, exponentially distributed random variables with parameter λ. Let $T_n = t_1 + \ldots + t_n$ for $n \geq 1$, $T_0 = 0$, then define

$$N(s) = \max\{n : \; T_n \leq s\}.$$

$N(s)$ is called a *Poisson process*.

LEMMA 1.3.5. *$N(s)$ has a Poisson distribution.*

THEOREM 1.3.6. *(Properties of the Poisson process)*
If $\{N(s), s \geq 0\}$ *is a Poisson process, then*

(1) $N(0) = 0$,
(2) $N(t + s) - N(t) \sim \textit{Poisson}(\lambda s)$,
(3) $N(t)$ *has independent increments.*

Conversely if (1), (2), and (3) hold, then $\{N(s)\}$ *is a Poisson process.*

DEFINITION 1.3.7. *(Non-homogeneous Poisson process)*
We say that $\{N(s), s \geq 0\}$ is a Poisson process with rate $\lambda(r)$ if

(1) $N(0) = 0$,
(2) $N(t+s) - N(s) \sim \text{Poisson}\left(\int_s^{t+s} \lambda(r)dr\right)$, $\quad \lambda : \mathbb{R}_+ \to \mathbb{R}_+$, a deterministic process.
(3) $N(t)$ has independent increments.

NOTE 1.3.8. The Poisson distribution with parameter λ has probability mass function $\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{N}_0$. Its characteristic function is

$$\begin{aligned} \phi(t) = E\left(e^{itX}\right) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} e^{itn} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\left(\lambda e^{it}\right)^n}{n!} \\ &= e^{-\lambda} \exp\left\{\lambda e^{it}\right\} = \exp\left\{-\lambda\left(1 - e^{it}\right)\right\}. \end{aligned}$$

DEFINITION 1.3.9. *(Compound Poisson process)*
The process $S(t) = Y_1 + \ldots + Y_{N(t)}$, $S(t) = 0$ if $N(t) = 0$, is called a *compound Poisson process*, where N is a Poisson process and Y_i are i.i.d. random variables.

THEOREM 1.3.10. *Let (Y_i) be i.i.d., N an independent non-negative integer-valued random variable and S as above, then*

1. $E(N) < \infty, \quad E(|Y_i|) < \infty$, *then* $E(S) = E(N) E(Y_1)$.
2. $E(N^2) < \infty, E\left(|Y_i|^2\right) < \infty$, *then* $Var(S) = E(N) Var(Y_1) + Var(N) E(Y_1^2)$.
3. *If* $N = N(t)$ *is Poisson*(λt), *then* $Var(S) = t\lambda \left(E(Y_1)\right)^2$.

1.4. Infinitely Divisible Distributions and the Lévy-Khintchine Formula

1.4.1. Lévy processes.

DEFINITION 1.4.1. *(Lévy process)*
A process $X = (X_t,\ t \geq 0)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *Lévy process* (LP), if it possesses the following properties:

1. The paths of X are $\mathbb{P}$-almost surely right-continuous with left limits (RCLL).
2. $X(0) = 0 \quad a.s.$
3. X has independent increments: $X(t+u) - X(t)$ is independent of $\sigma\left(\{X(s), s \leq t\}\right)$ for any $u \geq 0$.
4. X has stationary increments, i.e. the law of $X(t+u) - X(t)$ depends only on u.

Prime examples are Brownian motion and the Poisson process.
Say we have $X \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow$ cf: $\phi(t) = \phi_X(t) = \exp\left\{i\mu t - \frac{1}{2}\sigma^2 t^2\right\}$. For each n we have $\phi(t) = (\phi_n(t))^n = \exp\left\{i\mu \frac{t}{n} - \frac{1}{2}\frac{\sigma^2 t^2}{n}\right\}^n$. So $X = X_1^{(n)} + \cdots + X_n^{(n)}$ with $X_i^{(n)} \sim \mathcal{N}\left(\frac{\mu}{n}, \frac{\sigma^2}{n}\right)$, i.i.d.
Also $Y \sim \text{Poi}(\lambda)$, then $\phi_Y(t) = \exp\left\{-\lambda\left(1 - e^{it}\right)\right\} = \exp\left\{-\frac{\lambda}{n}\left(1 - e^{it}\right)\right\}^n$ so the product of the characteristic function of n Poi$\left(\frac{\lambda}{n}\right)$ random variables $Y = Y_1^{(n)} + \cdots + Y_n^{(n)}$ with $Y_i^{(n)} \sim \text{Poi}\left(\frac{\lambda}{n}\right)$, i.i.d.

DEFINITION 1.4.2. *(Infinitely divisible)*
A random variable X (or its distribution function F) is *infinitely divisible* if for each $n = 1, 2, \ldots$ there exist independent indentically distributed $X_{n,i}$ $i = 1, \ldots, n$ with $X_{n,i} \sim F_n$ such that $X = X_{n,1} + \ldots + X_{n,n}$ or equivalently

$$F = \underbrace{F_n * \cdots * F_n}_{\text{n times}} = *^n F_n.$$

FACT 1.4.3. *Recall that* $\psi(u) := -\log E\left(e^{iuX}\right)$ *is the* characteristic exponent *of a random variable* X.

THEOREM 1.4.4. *(Levy-Khintchine formula)*
A probability law μ of a real-valued random variable is infinitely divisible if and only if there exists a triple (a, σ, π), where $a \in \mathbb{R}$, $\sigma \geq 0$, and π is a measure concentrated on $\mathbb{R}\setminus\{0\}$ satisfying $\int_{\mathbb{R}\setminus\{0\}} (1 \wedge x^2)\pi(dx) < \infty$ such that the characteristic exponent of μ (resp. $X \sim \mu$) is given by

$$\psi(\theta) = i\alpha\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x \mathbf{1}_{\{|x|<1\}}\right) \pi(dx)$$

for every $\theta \in \mathbb{R}$*.*

Proof. (Parts)

(1) Observe that for a compound Poisson process $X(t) = \sum_{j=1}^{N(t)} \xi_j$ with ξ_i i.i.d. and independent of N and $\xi_i \sim F$ with no atoms at zero. Then

$$\begin{aligned} E\left(e^{i\theta X(t)}\right) &= \sum_{n\geq 0} E\left(e^{i\theta \sum_{j=1}^{n} \xi_j}\right) e^{-\lambda} \frac{\lambda^n}{n!} \\ &= \sum_{n\geq 0} \left(\int_{\mathbb{R}} e^{i\theta x} F(dx)\right)^n e^{-\lambda} \frac{\lambda^n}{n!} \\ &= \exp\left\{-\lambda \int_{\mathbb{R}} \left(1 - e^{i\theta x}\right) F(dx)\right\}. \end{aligned}$$

Thus we have the triple $a = \lambda \int_{\{|x|<1\}} xF(dx)$, $\sigma = 0$, $\pi(dx) = \lambda F(dx)$.

(2) Define $\psi_n(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{|x|>\frac{1}{n}} \left(1 - e^{i\theta x} + i\theta x \mathbf{1}_{\{|x|\leq 1\}}\right) \pi(dx)$ this is the convolution of a Gaussian and a compound Poisson and hence it is the characteristic exponent of an infinitely divisible distribution (because the sum of infinitely divisible distributions is infinitely divisible: $F * G = *^n F_n * *^n G_n = *^n (F_n * G_n)$).

(3) Property of characteristic functions: If a sequence of characteristic functions $\phi_n(t)$ converges to a function $\phi(t)$ for every t and $\phi(t)$ is continuous at $t = 0$, then $\phi(t)$ is the characteristic function of some distribution.

So we only need to show that $\psi(\theta)$is continuous in $\theta = 0$.

$$\begin{aligned} |\psi(\theta)| &= \left| \int_{\{|x|<1\}} \left(1 + i\theta x - e^{i\theta x}\right) \pi(dx) + \int_{\{|x|\geq 1\}} \left(1 - e^{-i\theta x}\right) \pi(dx) \right| \\ &\overset{\text{Taylor}}{\leq} \frac{1}{2} |\theta|^2 \int_{\{|x|<1\}} |x|^2 \pi(dx) + \int_{\{|x|\geq 1\}} \underbrace{\left|1 - e^{-i\theta x}\right|}_{\leq 2} \pi(dx) \end{aligned}$$

By dominated convergence we have $\psi(\theta) \to 0$ as $\theta \to 0$.

□

Let X be a Levy process, then for every t

$$X_t = X_{\frac{t}{n}} + \left(X_{\frac{2t}{n}} - X_{\frac{t}{n}}\right) + \cdots + \left(X_{\frac{nt}{n}} - X_{\frac{(n-1)t}{n}}\right)$$

so $X_t$is infinitely divisible (from the definition of a Levy process: stationary and independent increments). Define for $\theta \in \mathbb{R}$, $t \geq 0$

$$\psi_t(\theta) = -\log E\left(e^{i\theta X_t}\right).$$

For m, n positive integers

$$m \cdot \psi_1(\theta) = \psi_m(\theta) = n \cdot \psi_{\frac{m}{n}}(\theta)$$

so for any rational t: $\psi_t(\theta) = t \cdot \psi_1(\theta)$ $(*)$. For t irrational we can choose a decreasing sequence of rationals (t_n) such that $t_n \downarrow t$. Almost sure right continuity of X implies right-continuity of $\exp\{-\psi_t(\theta)\}$. By dominated convergence and so $(*)$ holds for every t.

For any Lévy process$E\left(e^{i\theta X_t}\right) = e^{-t\psi(\theta)}$ where $\psi(\theta) = \psi_1(\theta)$ is the charactersitic exponent of X_1.

DEFINITION 1.4.5. $\psi(\theta)$ is called the *characteristic exponent* of the Levy process X.

THEOREM 1.4.6. (The Levy-Khintchine formula for Levy processes)
Suppose that $a \in \mathbb{R}$, $\sigma \geq 0$, and π is a measure concentrated on $\mathbb{R}\setminus\{0\}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\pi(dx) < \infty$. From this triple define for each $\theta \in \mathbb{R}$

$$\psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{-i\theta x} + i\theta x \mathbf{1}_{\{|x|<1\}}\right)\pi(dx).$$

Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a Levy process is defined having the characteristic exponent ψ.

CHAPTER 2

The Levy-Ito decomposition and the path structure

2.1. The Levy-Ito decomposition

$$\psi(\theta) = \underbrace{\left\{ia\theta + \frac{1}{2}\sigma^2\theta^2\right\}}_{=:\psi^{(1)}} + \underbrace{\left\{\pi(\mathbb{R}\backslash(-1,1)) + \int_{\{|x|\geq 1\}} \left(1 - e^{i\theta x}\right) \frac{\pi(dx)}{\pi(\mathbb{R}\backslash(-1,1))}\right\}}_{=:\psi^{(2)}} + \underbrace{\left\{\int_{\{0<|x|<1\}} \left(1 - e^{i\theta x} + i\theta x\right)\pi(dx)\right\}}_{=:\psi^{(3)}}$$

for all $\theta \in \mathbb{R}$, $a \in \mathbb{R}$, $\sigma \geq 0$ and π as above.
$\psi^{(1)}$ corresponds to $X_t^{(1)} = \sigma W_t - at,\ t \geq 0$.
$\psi^{(2)}$corresponds to $X_t^{(2)} = \sum_{i=1}^{N_t} \xi_i,\ t \geq 0$ with $\{N_t, t \geq 0\}$ is a Poisson process with rate $\pi(\mathbb{R}\backslash(-1,1))$ and $\{\xi_i,\ i \geq 1\}$ are i.i.d. with distribution $\frac{\pi(dx)}{\pi(\mathbb{R}\backslash(-1,1))}$ concentrated on $\{|x| \geq 1\}$. (In case of $\pi(\mathbb{R}\backslash(-1,1)) = 0$, think of $\psi^{(2)}$ as being absent.
We need to indentify $\psi^{(3)}$ as the characteristic exponent of a Levy process $X^{(3)}$.

$$\int_{\{0<|x|<1\}} \left(1 - e^{i\theta x} + i\theta x\right)\pi(dx) = \sum_{n\geq 0}\left\{\lambda_n \int_{2^{-(n+1)}\leq|x|<2^{-n}} \left(1 - e^{i\theta x}\right) F_n(dx) + i\theta\lambda_n \int_{2^{-(n+1)}\leq|x|<2^{-n}} xF_n(dx)\right\},$$

where $\lambda_n = \pi\left(\left\{x : 2^{-(n+1)} \leq |x| \leq 2^{-n}\right\}\right)$ and $F_n(dx) = \frac{\pi(dx)}{\lambda_n}$.

2.2. Poisson Random Measures

$X = \{X_t : t \geq 0\}$ a compound Poisson process with drift $X_t = \mu t + \sum_{i=1}^{N_t} \xi_i,\ t \geq 0$, $\mu \in \mathbb{R}$, $\{\xi_i,\ i \geq 1\}$ are i.i.d., N_t is a Poisson process with intensity λ. Let $\{T_i,\ i \geq 1\}$ be the times of arrival of the Poisson process. Pick a set $A \in \mathcal{B}[0,\infty) \times \mathcal{B}(\mathbb{R}\backslash\{0\})$, define $N(A) := \#\{i \geq 0 : (T_i, \xi_i) \in A\}$. Since X experiences an almost surely finite number of jumps over a finite timeperiod it follows that $N(A) < \infty$ a.s. for any finite A.

LEMMA 2.2.1. *Choose $k \geq 1$. If $A_1, \ldots, A_k$ are disjoint sets in $\mathcal{B}[0,\infty)\times\mathcal{B}(\mathbb{R}\backslash\{0\})$, then $N(A_1), \ldots, N(A_k)$ are mutually independent and Poisson distributed with parameter $\lambda_i = \lambda \cdot \int_{A_i} dt \times F(dx)$.*

Furthermore for almost every realization of X the corresponding

$$N : \mathcal{B}[0,\infty) \times \mathcal{B}(\mathbb{R}\backslash\{0\}) \to \{0,1,2,\ldots\} \cup \{\infty\}$$

is a measure.

DEFINITION 2.2.2. (Poisson random measures)

Let $(S, \mathcal{A}, \eta)$ be an arbitrary σ-finite measure space. Let $N : \mathcal{A} \to \{0,1,2,\ldots\} \cup \{\infty\}$ be such that the family $\{N(A),\ A \in \mathcal{A}\}$ are random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and then N is called a ***Poisson random measure*** on $(S, \mathcal{A}, \eta)$ if

(1) For mutually disjoint $A_1, \ldots, A_k \in \mathcal{A}$ the random variables $N(A_1), \ldots, N(A_k)$ are independent.
(2) For each $A \in \mathcal{A}$, $N(A)$ is Poisson distributed with parameter $\eta(A)$.
(3) N is $\mathbb{P} - a.s.$ a measure.

(N is sometimes called a Poisson random measure on $\mathcal{A}$ with intensity η.)

THEOREM 2.2.3. *There exists a Poisson random measure.*

FACT. *For N a Poisson random measure $(S, \mathcal{A}, \eta)$*

(1) *$\forall A \in \mathcal{A}$, $N(.\cap A)$ is a Poisson random measure on $(S \cap A, \mathcal{A} \cap A, \eta(.\cap A))$. If $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$, then $N(.\cap A)$and $N(.\cap B)$ are independent.*
(2) *The support of N is $\mathbb{P} - a.s.$ countable. If in addition η is finite then the support of N is a.s. finite.*

As N is $\mathbb{P}$-a.s. a measure, we have

$$\int_S f(x) N(dx)$$

is a $[0,\infty)$-valued random variable for measurable functions $f : S \to \mathbb{R}$. (Define for $f^+ = f \vee 0$, and $f^- = (-f) \vee 0$ in the usual way.)

THEOREM 2.2.4. *Let N be a Poisson random measure on $(S, \mathcal{A}, \eta)$ and $f : S \to \mathbb{R}$ be a measurable function. Then*

(1)
$$X = \int f(x) N(dx)$$
is almost surely absolutely convergent if and only if

$$\int_S (1 \wedge |f(x)|) \eta(dx) < \infty. \tag{2.2.1}$$

(2) *When condition (2.2.1) holds, then*

$$E\left(e^{i\beta X}\right) = \exp\left\{-\int_S \left(1 - e^{i\beta f(x)}\right) \eta(dx)\right\} \quad \forall \beta \in \mathbb{R}.$$

(3) *Furthermore*

$$E(X) = \int_S f(x)\eta(dx)$$

when $\int |f(x)|\,\eta(dx) < \infty$ *and*

$$E(X^2) = \int_S f(x)^2\eta(dx) + \left(\int_S f(x)\eta(dx)\right)^2$$

if $\int f(x)^2\eta(dx) < \infty$.

2.3. Square Integrable Martingales

Consider $(\underbrace{[0,\infty)\times\mathbb{R}}_{S}, \underbrace{\mathcal{B}[0,\infty)\times\mathcal{B}(\mathbb{R})}_{\mathcal{A}}, \underbrace{dt\times\pi(dx)}_{\eta})$, where π is a measure concentrated on $\mathbb{R}\backslash\{0\}$.

Lemma 2.3.1. *Suppose N is a Poisson random measure where π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ and $B \in \mathcal{B}(\mathbb{R})$ such that $0 < \pi(B) < \infty$. Then*

$$X_t := \int_{[0,t]}\int_B xN(ds\times dx), \qquad t \geq 0$$

is a compound Poisson process with arrival rate $\pi(B)$ and jump distribution $\pi(B)^{-1}\pi(dx)|_B$.

Proof. X_t is RCLL by the properties of Poisson random measures as a counting measure. For $0 \leq s < t < \infty$ we have

$$X_t - X_s = \int_{(s,t]}\int_B xN\left(ds\times dx\right)$$

which is independent of $\sigma\left\{X_u : u \leq s\right\}$, because N gives independent counts on disjoint regions.
From Theorem 2.2.4

$$E\left(e^{i\theta X}\right) = \exp\left\{-t\int_B\left(1-e^{i\theta x}\right)\pi(dx)\right\}.$$

From independent increments we see that

$$\begin{aligned} E\left(e^{i\theta(X_t - X_s)}\right) &= \frac{E\left(e^{i\theta X_t}\right)}{E\left(e^{i\theta X_s}\right)} = \exp\left\{-(t-s)\int_B\left(1-e^{i\theta x}\right)\pi(dx)\right\} \\ &= E\left(e^{i\theta X_{t-s}}\right), \end{aligned}$$

which shows stationarity.
We introduce $\frac{\pi(B)}{\pi(B)}$

$$E\left(e^{i\theta X}\right) = \exp\left\{-t\pi(B)\int_B\left(1-e^{i\theta x}\right)\frac{\pi(dx)}{\pi(B)}\right\}$$

and obtain the characteristic function of a compound Poisson process. □

Lemma 2.3.2. *Let N and B be as in lemma 2.3.1 and assume that $\int_B |x|\,\pi(dx) < \infty$.*

(1) *The compound Poisson process with drift*

$$M_t = \int_{[0,t]} \int_B xN(ds \times dx) - t \int_B x\pi(dx) \quad t \geq 0$$

is a $\mathbb{P}$*-martingale w.r.t the filtration*

$$\mathcal{F}_t = \sigma\left(N(A) : A \in \mathcal{B}[0,t] \times \mathcal{B}(\mathbb{R})\right)$$

(2) *If furthermore* $\int_B X^2 \pi(dx) < \infty$*, then it is a square-integrable martingale.*

Proof.

(1) M_t is $\mathcal{F}_t$-measurable. Also for $t \geq 0$

$$E\left(|M_t|\right) \leq E\left(\int_{[0,t]} \int_B |x|\, N(ds \times dx)\right) + t \int_B |x|\, \pi(dx) < \infty$$

by theorem 2.2.4 (3).

$$\begin{aligned} E(M_t - M_s \mid \mathcal{F}_s) &\overset{(*)}{=} E(M_{t-s}) \\ &= E\left[\int_{[0,t-s]} \int_B xN(ds \times dx) - (t-s) \int_B x d\pi(dx)\right] \\ &= 0 \end{aligned}$$

where $(*)$ follows from the independence of the increments for X_t and stationarity and the last equation follows from theorem 2.2.4 (3).

(2) From $\int_B x^2 \pi(dx) < \infty$, then Theorem 2.2.2. (3) says $E\left(X_t^2\right) < \infty$ and

$$E\left(\left(M_t + t \int_B x\pi(dx)\right)^2\right) = t \int_B x^2 \pi(dx) + t^2 \left(\int_B x\pi(dx)\right)^2$$

but the left-hand side also gives

$$E\left(M_t^2 + 2t \int_B x\pi(dx) \underbrace{E\left(M_t\right)}_{=0} + t^2 \left(\int_B x\pi(dx)\right)^2\right)$$

so $E\left(M_t^2\right) = t \int_B x^2 \pi(dx) < \infty$, this shows that M_t is a square integrable martingale. In the following, we need to consider sets B_ε of the type $B_\varepsilon = (-1, -\varepsilon) \cup (\varepsilon, 1)$.

□

Theorem 2.3.3. *Assume that* N *is as in lemma 2.3.1 and* $\int_{(-1,1)} x^2 \pi\,(dx) < \infty$. *For each* $\varepsilon \in (0,1)$ *we define the martingale*

$$M_t^\varepsilon = \int_{[0,t]} \int_{B_\varepsilon} xN\left(ds \times dx\right) - t \int_{B_\varepsilon} x\pi\left(dx\right),\ t \geq 0$$

and let $\mathcal{F}_t^*$ *be equal to the completion of* $\bigcap_{s>t} \mathcal{F}_s$ *by the null sets of* $\mathbb{P}$ *where* $\mathcal{F}_t$ *is given as above. Then there exists a martingale* $M = \{M_t,\ t \geq 0\}$ *with the following properties:*

(1) *for each* $T > 0$*,there exists a deterministic subsequence* $\left\{\varepsilon_n^T,\ n = 1, 2, \ldots\right\}$ *with* $\varepsilon_n^T \downarrow 0$*along which* $\mathbb{P}\left(\lim_{n\to\infty} \sup_{0 \leq t \leq T} \left(M_s^{\varepsilon_n^T} - M_s\right)^2 = 0\right) = 1$.

(2) *It is adapted to the filtration* $\{\mathcal{F}_t^*,\ t \geq 0\}$.
(3) *It has right-continuous paths with left limits.*
(4) *It has at most a countable number of discontinuities in* $[0,T]$.
(5) *It has stationary and independent increments.*

In short, there exists a Lévy process, which is a martingale with a countable number of jumps in each interval $[0,T]$*in which for each* $T>0$ *the sequence of martingales* $\{M_T^\varepsilon,\ t \leq T\}$ *converges almost uniformly on* $[0,T]$ *and with probability 1 along a subsequence* ε *which may depend on* T.

We need some facts on square-integrable martingales. Assume that we have $(\Omega, \mathcal{F}, \mathcal{F}_t^*, \mathbb{P})$ is a stochastic basis satisfying the usual conditions.

DEFINITION 2.3.4. Fix $T>0$ and define $\mathcal{M}_T^2 = \mathcal{M}_T^2(\Omega, \mathcal{F}, \mathcal{F}_t^*, \mathbb{P})$ to be the space of real-valued, right-continuous, square integrable martingales with respect to the given filtration over the finite time period $[0,T]$.

So $\mathcal{M}_T^2$ is a vector sapce over $\mathbb{R}$ with zero element $M_t \equiv 0$. Indeed it is a Hilbert space with respect to the inner product $< M, N >= E(M_T N_T)$.
Note that if we have $< M, M >= 0$ then by Doob's inequality $E\left(\sup_{0 \leq t \leq T} M_t^2\right) \leq 4E\left(M_T^2\right)$, so $\sup_{0\leq t\leq T} M_t = 0$ *a.s.*. By right-continuity $M_t = 0\ \forall t \in [0,T]$.
Assume that $\left\{M^{(n)},\ n=1,2,\ldots\right\}$ is a Cauchy sequence. Then for any $\left\{M_T^{(n)},\ n=1,2,\ldots\right\}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Hence there exists a limiting variable M_T such that

$$E\left[\left(M_T^{(n)} - M_T\right)^2\right]^{1/2} \to 0 \quad (n\to\infty).$$

Define the martingale M to be the right-continuous version of $M_t := E(M_T \mid \mathcal{F}_t^*)$ for $t\in[0,T]$. By definition $\left\|M^{(n)} - M\right\| =< M^{(n)} - M, M^{(n)} - M >^{1/2} = E\left(\left(M_T^{(n)} - M_T\right)^2\right)^{1/2} \to 0\ (n\to\infty)$.
Clearly M_t is $\mathcal{F}_t^*$ adapted and by Jensen's inequality

$$E\left(M_t^2\right) = E\left(\left(E\left(M_T \mid \mathcal{F}_t^*\right)\right)^2\right) \leq E\left(E\left(M_T^2 \mid \mathcal{F}_t^*\right)\right) < \infty.$$

PROOF. (Theorem 2.3.3)

(1) Choose $0<\eta<\varepsilon<1$, fix $T>0$ and define $M^\varepsilon := \{M_t^\varepsilon : t\in[0,T]\}$. With the standard calculation (cf lemma 2.3.2)

$$E\left(\left(M_T^\varepsilon - M_T^\eta\right)^2\right) = E\left(\left\{\int_{[0,T]}\int_{\eta\leq|x|\leq\varepsilon} xN(ds\times dx)\right\}^2\right)$$
$$= T\int_{\eta\leq|x|<\varepsilon} x^2\pi(dx)$$

The left-hand side is $\|M^\varepsilon - M^\eta\|^2$. So $\lim_{\varepsilon\to0}\|M^\varepsilon - M^\eta\| = 0$, since $\int_{(-1,1)} x^2\pi(dx) < \infty$ and hence $\{M^\varepsilon : 0<\varepsilon<1\}$is a Cauchy family in $\mathcal{M}_T^2$.

As $\mathcal{M}_T^2$ is a Hilbert space, we know there exists a martingale $M = \{M_s : s\in[0,T]\} \in \mathcal{M}_T^2$ such that $\lim_{\varepsilon\to0}\|M - M^\varepsilon\|^2 = 0$.

By Doob's maximal inequality we find that

$$\lim_{\varepsilon\to 0} E\left(\sup_{0\le s\le T}(M_s - M_s^\varepsilon)^2\right) \le 4\lim_{\varepsilon\to 0}\|M - M^\varepsilon\| = 0.$$

So the limit does not depend on T.

Now L^2-convergence implies convergence in probability, which in turn implies a.s. convergence along a deterministic subsequence, thus (1) follows.

(2) Fix $0 \le t \le T$ then $M_t^{\varepsilon_n^T}$ is $\mathcal{F}_t^*$-measurable and the a.s. limit M_t is $\mathcal{F}_t^*$-measurable as well.
(3) The same argument as in (2) for RCLL.
(4) RCLL implies only countable many discontinuities
(5) Uniform convergence implies the convergence of the finite dimensional distributions. Then for $0 \le u \le v \le s \le t \le T < \infty$ and $\theta_1, \theta_2 \in \mathbb{R}$

$$\begin{aligned} E\left(e^{i\theta_1(M_v - M_u)}e^{i\theta_2(M_t - M_s)}\right) &\overset{\text{DCT}}{=} \lim_{\varepsilon\to 0} E\left(e^{i\theta_1\left(M_v^{\varepsilon_n^T} - M_u^{\varepsilon_n^T}\right)}e^{i\theta_2\left(M_t^{\varepsilon_n^T} - M_s^{\varepsilon_n^T}\right)}\right)\\ &= \lim_{n\to\infty} E\left(e^{i\theta_1 M_{v-u}^{\varepsilon_n^T}}\right)E\left(e^{i\theta_2 M_{t-s}^{\varepsilon_n^T}}\right)\\ &\overset{\text{DCT}}{=} E\left(e^{i\theta_1 M_{v-u}}\right)E\left(e^{i\theta_2 M_{t-s}}\right)\end{aligned}$$

□

2.4. The Levy-Ito Decomposition

THEOREM 2.4.1. *Given $a \in \mathbb{R}$, $\sigma \ge 0$, π a measure concentrated on $\mathbb{R}\setminus\{0\}$ satisfying $\int_{\mathbb{R}}\left(1\wedge x^2\right)\pi(dx) < \infty$, there exists a probability space on which independent Levy processes $X^{(1)}, X^{(2)}$ and $X^{(3)}$ exist, $X_t^{(1)} = \sigma B_t - at$, $t \ge 0$, a linear Brownian motion with drift, $X_t^{(2)} = \sum_{i=1}^{N_t}\xi_i$, $t \ge 0$ is a Poisson process with rate $\{N_t,\ t\ge 0\}$ is a Poisson process with rate $\pi(\mathbb{R}\setminus(-1,1))$ and $\{\xi_i, i = 1,2,\ldots\}$ are i.i.d with distribution ${\pi(dx)}/{\pi(\mathbb{R}\setminus(-1,1))}$ concentrated on $\{|x|\ge 1\}$ and $X^{(3)}$ is a square integrable martingale with an almost surely countable number of jumps on each finite time interval, which are of magnitude less than unity and charactersitic exponent given by $\psi^{(3)}$.*

REMARK. By taking $X = X^{(1)} + X^{(2)} + X^{(3)}$ we have the Levy-Khintchine formula (theorem 1.4.2) holds.

PROOF.

(1) $X^{(1)}$ is clear,
(2) large jumps in theorem 2.2.1
(3) According to theorem 2.3.1 we have $X^{(3)}$. Dependence of "small" and "large" jumps from PRM BM independent, use a different probability space. Combine on the product space.

□

CHAPTER 3

Financial Modelling with Jump-Diffusion Processes

3.1. Poisson Process

THEOREM 3.1.1. *Let $N(t)$be a Poisson process with intensity λ, then the compensated Poisson process $M(t) = N(t) - \lambda t$ is a martingale.*

PROOF. $E(M(t) \mid \mathcal{F}_s) = E(M(t) - M(s) \mid \mathcal{F}_s) + M(s) = E(N(t) - N(s)) - \lambda(t-s) + M(s) = M(s)$. □

Let $Y_1, Y_2, \ldots$ be a sequence of iid random variables with $E(Y_i) = \beta$ which are also independent of $N(t)$. Define the compound Poisson process $Q(t) = \sum_{i=1}^{N(t)} Y_i$. $E(Q(t)) = \beta\lambda t$.

THEOREM 3.1.2. *The compensated compound Poisson process $Q(t) - \beta\lambda t$ is a martingale.*

PROOF. $E(Q(t) - \beta\lambda t \mid \mathcal{F}_s) = E(Q(t) - Q(s) \mid \mathcal{F}_s) + Q(s) - \beta\lambda t = \beta\lambda(t-s) + Q(s) - \beta\lambda t = Q(s) - \beta\lambda s$. □

3.2. Jump Processes and Their Integrals

DEFINITION 3.2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathbb{F} = (\mathcal{F}_t)$ a filtration on the space, satisfying the usual conditions. We assume that W is a Brownian motion w.r.t. $(\mathbb{P}, \mathbb{F})$, N is a Poisson process, and Q is a compound Poisson process on this space.
We define

$$\int_0^t \Phi(s) dX(s)$$

where $X(0) = x_0$ is a non-random initial condition, $I(t) = \int_0^t \Gamma(s) dW(s)$ is an Ito-integral, called the Ito-integral part,

$$R(t) = \int_0^t \theta(s) ds$$

is a Riemann-integral, called the Riemann-integral part and $J(t)$ is an adapted right-continuous pure jump process with $J(0) = 0$ and $X(t) = x_0 + I(t) + R(t) + J(t)$.

The continuous part of X is $X^c = X(0) + I(t) + R(t)$ and the quadratic of this process is

$$[X^c, X^c](t) = \int_0^t \Gamma^2(s) ds$$

or $d[X^c](t) = \Gamma^2(t)dt$. $J(t)$ right-continuous means $J(t) = \lim_{s \downarrow t} J(s)$ and the left-continuous version is $J(t-)$, i.e. the value immediately before the jump. We assume that J has no jump at 0 and only finitely many jumps in each interval $(0,T]$ and is constant between the jumps ($\rightsquigarrow$pure jump process).

DEFINITION 3.2.2. $X(t)$ will be called a *jump process*. Observe that $X(t)$ is right-continuous and adapted. Its left continuous version is $X(t-) = x_0 + I(t) + R(t) + J(t-)$. The *jump size* of X and t is denoted by $\triangle X(t) = X(t) - X(t-) = \triangle J(t) = J(t) - J(t-)$.

DEFINITION 3.2.3. Let $X(t)$ be a jump process and $\Phi(t)$ an adapted process. The stochastic integral of Φ with respect to X is defined by

$$\int_0^t \Phi(s)dX(s) = \int_0^t \Phi(s)\Gamma(s)dW(s) + \int_0^t \Phi(s)\theta(s)ds + \sum_{0\leq s\leq t} \Phi(s)\triangle J(s).$$

In differential notation we write

$$\begin{aligned}\Phi(t)dX(t) &= \Phi(t)dI(t) + \Phi(t)dR(t) + \Phi(t)dJ(t)\\ &= \Phi(t)dX^c(t) + \Phi(t)dJ(t)\end{aligned}$$

EXAMPLE. $X(t) = M(t) = N(t) - \lambda t$, N a Poisson process with intensity λ. So $I(t) \equiv 0$, $R(t) = -\lambda t = X^c(t)$, $J(t) = N(t)$. Let $\Phi(s) = \triangle N(s) = \mathbb{1}_{\{\triangle N(s)\neq 0\}} = N(s) - N(s-)$. $\int_0^t \Phi(s)dX^c(s) = -\lambda \int_0^t \Phi(s)ds = 0$, since $\Phi(s) = 0$ except for finitely many points. $\int_0^t \Phi(s)dN(s) = \sum_{0\leq s\leq t} \Phi(s)\triangle N(s) = \sum_{0\leq s\leq t} (\triangle N(s))^2 = N(t)$

THEOREM 3.2.4. *Assume that the jump process $X(t)$ is a martingale, the integrand $\Phi(t)$ is left-continuous and adapted and $E\left[\int_0^t \Gamma^2(s)\Phi^2(s)ds\right] < \infty$ for all $t \geq 0$. Then the stochastic integral $\int_0^t \Phi(s)dX(s)$ is well-defined and also a martingale.*

PROOF. Sketch: Use the martingale transform lemma, properties of a Hilbert space and the Ito isometry. □

EXAMPLE. Let $M(t) = N(t) - \lambda t$ be as above and let $\Phi(s) = \mathbb{1}_{[0,S_1]}(s)$, that is Φ is 1 up to and including the time of the first jump of N ($S_1 \sim \mathrm{Exp}(\lambda)$) and 0 afterwards. Then

$$\int_0^t \Phi(s)dM(s) = \begin{cases} -\lambda t & ,0 \leq t \leq S_1 \\ 1 - \lambda S_1 & ,t \geq S_1 \end{cases}$$

is a martingale.

DEFINITION 3.2.5. Choose $0 = t_0 < t_1 < \ldots < t_n = T$, set $\pi = \{t_0, t_1, \ldots, t_n\}$ denote by $\|\pi\| = \max\{t_{j+1} - t_j\}$ the length of the larges subinterval of the partition π. Define

$$Q_\pi(X) = \sum_{j=1}^{n-1} (X(t_{j+1}) - X(t_j))^2 .$$

The quadratic variation of X on $[0,T]$ is defined to be $[X,X](T) = \lim_{\|\pi\|\to 0} Q_\pi(X)$.
We know $[W,W](T) = T$ for Ito-integrals $[I,I](T) = \int_0^T \Gamma^2(s)\,ds$.
We also need the cross variation of X_1 and X_2 which is defined $C_\pi(X_1,X_2) = \sum_{j=0}^{n-1} (X_1(t_{j+1}) - X_1(t_j))(X_2(t_{j+1}) - X_2(t_j))$ and $[X_1,X_2](T) = \lim_{\|\pi\|\to 0} C_\pi(X_1,X_2)$.

THEOREM 3.2.6. *Let $X_i(t) = X_i(0) + J_i(t) + R_i(t) + J_i(t)$, $i = 1,2$ be jump processes (with the usual conditions) Then*

$$[X_1, X_1](T) = [X_1^c, X_1^c](T) + [J_1, J_1](T) = \int_0^T \Gamma_1(s)^2 + \sum_{0 \le s \le T} (\triangle J_1(s))^2$$

and

$$\begin{aligned}[X_1, X_2](T) &= [X_1^c, X_2^c](T) + [J_1, J_2](T) \\ &= \int_0^T \Gamma_1(s)\Gamma_2(s)\,ds + \sum_{0 \le s \le T} (\triangle J_1(s))(\triangle J_2(s))\end{aligned}$$

PROOF.

$$\begin{aligned} C_\pi(X_1, X_2) = & \sum_{j=0}^{n-1} \left(X_1^c(t_{j+1}) - X_1^c(t_j) + J_1(t_{j+1}) - J_1(t_j)\right) \\ & \cdot \left(X_2^c(t_{j+1}) - X_2^c(t_j) + J_2(t_{j+1}) - J_2(t_j)\right) \\ = & \underbrace{\sum_{j=0}^{n-1} \left(X_1^c(t_{j+1}) - X_1^c(t_j)\right)\left(X_2^c(t_{j+1}) - X_2^c(t_j)\right)}_{\to [X_1^c, X_1^c](T) = \int_0^T \Gamma_1(t)\Gamma_2(t)dt \text{ for } |\pi| \to 0} \\ + & \underbrace{\sum_{j=0}^{n-1} \left(X_1^c(t_{j+1}) - X_1^c(t_j)\right)\left(J_2(t_{j+1}) - J_2(t_j)\right)}_{|.| \le \max_{0 \le j \le n-1} \left|X_1^c(t_{j+1}) - X_1^c(t_j)\right| \cdot \sum_{j=0}^{n-1} |J_2(t_{j+1}) - J_2(t_j)| \to 0 \text{ for } |\pi| \to 0} \\ + & \underbrace{\sum_{j=0}^{n-1} \left(X_2^c(t_{j+1}) - X_2^c(t_j)\right)\left(J_1(t_{j+1}) - J(t_j)\right)}_{|.| \le \max_{0 \le j \le n-1} \left|X_2^c(t_{j+1}) - X_2^c(t_j)\right| \cdot \sum_{j=0}^{n-1} |J_1(t_{j+1}) - J_1(t_j)| \to 0 \text{ for } |\pi| \to 0} \\ + & \sum_{j=0}^{n-1} \underbrace{\left(J_1(t_{j+1}) - J_1(t_j)\right)\left(J_2(t_{j+1}) - J_2(t_j)\right)}_{\text{only } \neq 0 \text{ when } J_1 \text{ and } J_2 \text{ jump together}} \end{aligned}$$

□

COROLLARY 3.2.7. *Let W be Brownian motion and $M(t) = N(t) - \lambda t$ a compensated Poisson process. Then $[W, M](t) = 0$ for $t = 0$.*

COROLLARY 3.2.8. *For $i = 1,2$ $\tilde{X}_i(t) = \tilde{X}_i(0) + \int_0^t \Phi_i(s)\,dX_i(s)$. Then*

$$\begin{aligned}\left[\tilde{X}_1, \tilde{X}_2\right](t) &= \int_0^t \Phi_1(s)\Phi_2(s)\,d[X_1, X_2](s) \\ &= \int_0^t \Phi_1(s)\Phi_2(s)\Gamma_1(s)\Gamma_2(s)\,ds + \sum_{0 \le s \le t} \Phi_1(s)\Phi_2(s)\triangle J_1(s)\triangle J_2(s).\end{aligned}$$

3.3. Stochastic Calculus for Jump Processes

THEOREM 3.3.1. (Itô-Doeblin formula for jump processes)
Let $X(t)$ be a jump process and $f(x)$ a function for which $f^{'}$ and f''exist and are continuous, i.e. $f \in C^2$. Then

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(t))\, dX'(s) + \frac{1}{2}\int_0^t f''(X(s)) d\left[X_1^c\right](s) + \sum\nolimits_{0\leq s\leq t} \left[f(X(s)) - f(X(s-))\right].$$

PROOF. Fix $\omega \in \Omega$ and let $0 < \tau_1 < \tau_2 < \ldots < \tau_{n-1} < t$ be the jump times in $[0,t]$. We set $\tau_n = 0$ if there is no jump and otherwise $\tau_n = t$. Whenever we have to points $u < v$ such that $u, v \in [\tau_j, \tau_{j+1}]$ for arbitrary $j = 1, \ldots, n-1$ there is no jump between u and v and Ito's formula for continuous processes applies.

$$\Rightarrow X(f(0)) - X(f(u)) = \int_u^v f'(X(s))\, dX^c(s) + \frac{1}{2}\int_u^v f''(X(s))\, d[X^c](s)$$

Letting $u \to \tau_j^+$and $v \to \tau_{j+1}^-$ then by the right continuity of X we obtain

$$f\left(X\left(\tau_{j+1}^-\right)\right) - f\left(X\left(\tau_j^+\right)\right) = \int_{\tau_j}^{\tau_{j+1}} f'(X(s))\, dX^c(s) + \frac{1}{2}\int_{\tau_j}^{\tau_{j+1}} f''(X(s))\, d[X^c](s).$$

Then by adding the jump at τ_{j+1}, i.e. $f(X(\tau_{j+1})) - f\left(X\left(\tau_{j+1}^-\right)\right)$, we get

$$f\left(X\left(\tau_{j+1}\right)\right) - f\left(X\left(\tau_j^+\right)\right) = \int_{\tau_j}^{\tau_{j+1}} f'(X(s))\, dX^c(s) + \frac{1}{2}\int_{\tau_j}^{\tau_{j+1}} f''(X(s))\, d[X^c](s) + f(X(\tau_{j+1})) - f\left(X\left(\tau_{j+1}^-\right)\right).$$

Since there is only a countable number of jumps, we obtain the claim by summing over all jumps. □

COROLLARY 3.3.2. *Let $W(t)$ be a Brownian Motion and $N(t)$ a Poisson process with intensity $\lambda > 0$, both defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and relative to the same filtration $(\mathcal{F}_t)_{t\geq 0}$.*
Then the processes $W(t)$ and $N(t)$ are independent.

PROOF. Let u_1 and $u_2$be fixed numbers, $t \geq 0$ fixed and define

$$Y(t) = \exp\left\{u_1 W(t) + u_2 N(t) - \frac{1}{2}u_1^2 t - \lambda\left(e^{u_2} - 1\right)t\right\}$$

To show: $\text{LT}(W+N) = \text{LT}(W)\text{LT}(N) \Leftrightarrow W$ and N are independent$\Leftrightarrow Y(t)$ is a martingale.
Define $X(s) = u_1 W(s) + u_2 N(s) - \frac{1}{2}u_1^2 s - \lambda\left(e^{u_2} - 1\right)s$ and $f(x) = e^x$.$\Rightarrow Y(t) = f(X(t))$. We have $dX^c(s) = u_1 dW(s) - \frac{1}{2}u_1^2 ds - \lambda\left(e^{u_2} - 1\right)ds \quad (*)$ and $d[X^c](s) = m^2 ds \ (**)$.

If Y has a jump at time s, then

$$\begin{aligned} Y(s) &= \exp\left[u_1 W(s) + u_2 (N(s-)+1) - \frac{1}{2}u_1^2 s - \lambda (e^{u_2}-1) s\right] \\ &= Y(s-)e^{u_2}. \end{aligned}$$

$\Rightarrow Y(s) - Y(s-) = (e^{u_2}-1) Y(s-) \underbrace{\triangle N(s)}_{=N(s)-N(s-)=1}$. According to the Ito-Doeblin formula we have

$$\begin{aligned} Y(t) = f(X(t)) &= f(X(0)) + \int_o^t f'(X(s))\, dX^c(s) \\ &\quad + \frac{1}{2}\int_0^t f''(X(s))\, d[X^c](s) + \sum_{0\le s\le t} (f(X(s)) - f(X(s-))) \\ &= 1 + u_1 \int_0^t Y(s) dW(s) - \frac{1}{2}u_1^2 \int_0^t Y(s) ds - \lambda (e^{u_2}-1) \int_0^t Y(s) ds \\ &\quad + \frac{1}{2}u_1^2 \int_0^t Y(s) ds + \sum_{0\le s\le t} (Y(s) - Y(s-)) \\ &= 1 + u_1 \int_0^t Y(s)\, dW(s) - \lambda (e^{u_2}-1) \int_0^t Y(s-)\, ds \\ &\quad + (e^{u_2}-1) \int_0^t Y(s-) dN(s) \\ &= 1 + u_1 \int_0^t Y(s) dW(s) + (e^{u_2}-1) \int_0 Y(s-) dM(s), \end{aligned}$$

where $M(s) = N(s) - \lambda s$ is a martingale, so the integral is also a martingale.
$\Rightarrow Y(t)$ is a martingale and $E(Y(t)) = E(Y(0)) = 1$. By taking expectations, we get

$$\begin{aligned} & E(\exp\{u_1 W(t) + u_2 N(t)\}) = \exp\left(\frac{1}{2}u_1^2 t\right) \exp(\lambda (e^{u_1}-1) t) \\ \Leftrightarrow \quad & \mathrm{LT}(W+N) = \mathrm{LT}(W)\, \mathrm{LT}(N). \end{aligned}$$

By the identity property of the moment generating function the factorizing yields the independence of $W(t)$ and $N(t)$. The same argument for $(W(t_1), \ldots, W(t_n))^T$ and $(N(t_1), \ldots, N(t_n))^T$ $\forall n \in \mathbb{N}$, $t_n > \ldots > t_1 \ge 0$ yields that the processes themselves are independent. □

THEOREM 3.3.3. *(Ito-Doeblin in higher dimensions)*
Let $X_1(t)$ and $X_2(t)$ be jump processes and the function $f \in C^{1,2,2}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$. Then

$$\begin{aligned} f(t, X_1(t), X_2(t) &= f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s))\, ds \\ &\quad + \int_0^t f_{X_1} dX_1^c + \int_0^t f_{X_2} dX_2^c + \frac{1}{2}\sum_{i,j=1}^{2} \int_0^t f_{X_i X_j} d\left[X_i^c, X_j^c\right] \\ &\quad + \sum_{0\le s\le t} (f(s, X_1(s), X_2(s)) - f(s, X_1(s-), X_2(s-)). \end{aligned}$$

COROLLARY 3.3.4. (Product Rule)
Let X_1, X_2 be jump processes. Then

$$X_1(t) \cdot X_2(t) = X_1(0) X_2(0) + \int_0^t X_2 dX_1^c + \int_0^t X_1 dX_2^c + [X_1^c, X_2^c](t) + \sum_{0 \le s \le t} (X_1(s) X_2(s) - X_1(s-) X_2(s-)).$$

PROOF. Theorem 3.3.3 with $f(t, x_1, x_2) = x_1 x_2$. □

$$X(t) = \underbrace{\underbrace{X(0)}_{=x_0} + I(t) + R(t)}_{X^c(t) \text{ continuous part}} + J(t)$$

$I(t) = \int_0^t \Gamma(s) dW(s)$ the Ito integral part
$R(t) = \int_0^t \theta(s) ds$ the Rieman integral part
$J(t)$ adapted, right-continuous pure jump process
We define $\int_0^t \Phi(s) dX(s)$ for a suitable class of processes Φ to be a martingale.

COROLLARY 3.3.5. *(Doleans-Dade exponent)*
Let $X(t)$ be a jump process. The D-D exponent of X is defined to be the process

$$Z^X(t) = \exp\left\{X^c(t) - \frac{1}{2}[X^c, X^c](t)\right\} \prod_{0 < s \le t} (1 + \triangle X(s)).$$

The process is the solution of the SDE

$$dZ^X(t) = Z^X(t-) dX(t)$$

or in integral from

$$Z^X(t) = 1 + \int_0^t Z^X(s-) dX(s).$$

PROOF. Define

$$Y(t) = \exp\left\{\int_0^t \Gamma(s) dW(s) + \int_0^t \theta(s) ds + \frac{1}{2}\int_0^t \Gamma^2(s) ds\right\} = \exp\left\{X^c(t) - \frac{1}{2}[X^c, X^c](t)\right\}$$

From the standard continuous-time Ito-formula we have that $dY(t) = Y(t) dX^c(t) = Y(t-) dX^c(t)$.
Define $K(t) = 1$ for $0 < t < \tau_1$, where τ_1 is the time of the first jump of X, and for $t \ge \tau_1$ we set $K(t) = \prod_{0 < s \le t} (1 + \triangle X(s))$. Then $K(t)$ is a pure jump process and $Z^X(t) = Y(t) \cdot K(t)$.
Also $\triangle K(t) = K(t) - K(t-) = K(t-) \triangle X(t)$ and $[Y, K](t) \equiv 0$ because Y is continuous and K is a pure jump process.

$$
\begin{aligned}
Z^X(t) = Y(t) \cdot K(t) &= 1 + \int_0^t K(s-)dY(s) + \int_0^t Y(s-)dK(s) \\
&= 1 + \int_0^t K(s-)Y(s-)dX^c(s) + \sum_{0<s\leq t} Y(s-)K(s-)\triangle X(s) \\
&= 1 + \int_0^t Y(s-)K(s-)dX(s) = 1 + \int_0^t Z^X(s-)dX(s)
\end{aligned}
$$

□

We now discuss how to change measure in a jump process framework. We start with a compound Poisson process. $Q(t) = \sum_{i=1}^{N(t)} Y_i$, where $N(t)$ is a Poisson process with intensity λ and $Y_1, Y_2, \ldots$ are iid random variables (independent of N) with density $f(y)$. Let $\tilde{\lambda} > 0$ and $\tilde{f}$ be another density with $\tilde{f}(y) = 0$ whenever $f(y) = 0$. Define

$$
Z(t) = e^{(\lambda - \tilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}.
$$

LEMMA 3.3.6. *The process Z is a martingale. In particular $E(Z(t)) = 1$ $\forall t$.*

PROOF. We define a pure jump process

$$
J(t) = \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}.
$$

At the jump times of the process J we have

$$
\begin{aligned}
J(t) &= J(t-)\frac{\tilde{\lambda}\tilde{f}(Y_{N(t)})}{\lambda f(Y_{N(t)})} = J(t-)\frac{\tilde{\lambda}\tilde{f}(\triangle Q(t))}{\lambda f(\triangle Q(t))} \\
\triangle J(t) = J(t) - J(t-) &= \left[\frac{\tilde{\lambda}\tilde{f}(\triangle Q(t))}{\lambda f(\triangle Q(t))} - 1\right] J(t-)
\end{aligned}
$$

Define the compound Poisson process $H(t) = \sum_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$ for which $\triangle H(t) = \frac{\tilde{\lambda}\tilde{f}(\triangle Q(t))}{\lambda f(\triangle Q(t))}$ and also

$$
E\left(\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}\right) = \frac{\tilde{\lambda}}{\lambda}\int_{-\infty}^{\infty} \frac{\tilde{f}(y)}{f(y)} f(y)dy = \frac{\tilde{\lambda}}{\lambda}
$$

so the compensated compound Poisson process $H(t) - \tilde{\lambda}t$ is a martingale.
Furthermore $\triangle J(t) = J(t-)[\triangle H(t) - \underbrace{\triangle N(t)}_{=1}]$. Because J, H, N are pure jump processes, this is

$$
dJ(t) = J(t-)(dH(t) - dN(t)).
$$

Using the product formula we now find that

$$\begin{aligned} Z(t) &= Z(0) + \int_0^t J(s-)(\lambda - \tilde{\lambda})e^{(\lambda-\tilde{\lambda})s}ds + \int_0^t e^{(\lambda-\tilde{\lambda})s}dJ(s) \\ &= 1 + \int_0^t J(s-)(\lambda - \tilde{\lambda})e^{(\lambda-\tilde{\lambda})s}ds + \int_0^t e^{(\lambda-\tilde{\lambda})s}J(s-)\left[dH(s) - dN(s)\right] \\ &= 1 + \int_0^t J(s-)e^{(\lambda-\tilde{\lambda})s}d\underbrace{\left[H(s) - \tilde{\lambda}s\right]}_{\text{a martingale}} - \int_0^t J(s-)e^{(\lambda-\tilde{\lambda})s}d\underbrace{\left[N(s) - \lambda s\right]}_{\text{a martingale}}. \end{aligned}$$

By Theorem 3.2.4 this implies that Z is a martingale since $Z(0) = 1$ we have $E(Z(t)) \equiv 1$. In different notation we have

$$dZ(t) = Z(t-)d[H(t) - \tilde{\lambda}(t)] - Z(t-)d\left[N(t) - \lambda t\right].$$

□

Fix a positive T and define $\tilde{\mathbb{P}}(A) = \int_A Z(T)d\mathbb{P}$, $A \in \mathcal{F}$.

THEOREM 3.3.7. (Change of measure for compound Poisson process)
Under the probability measure $\tilde{\mathbb{P}}$ the process $Q(t)$, $0 \le t \le T$ is a compound Poisson process with intensity $\tilde{\lambda}$. Furthermore, the jumps in $Q(t)$ are independent and identically distributed with density $\tilde{f}(y)$.

PROOF. We show that Q has under $\tilde{\mathbb{P}}$ the moment generating function

$$\tilde{E}\left(e^{uQ(t)}\right) = \exp\left\{\tilde{\lambda}t\left(\tilde{\varphi}_Y(u) - 1\right)\right\}$$

with $\tilde{\varphi}_Y(u) = \int_{-\infty}^{\infty} e^{uy}\tilde{f}(y)dy$ which is the moment generating function of a compound Poisson process with intensity $\tilde{\lambda}$ and jump size distribution $\tilde{f}$.
Define $X(t) = \exp\left\{uQ(t) - \tilde{\lambda}t(\tilde{\varphi}_Y(u) - 1)\right\}$ and show that $X(t)Z(t)$ is a $\mathbb{P}$-martingale. By the product rule

$$X(t)Z(t) = 1 + \underbrace{\int_0^t X(s-)dZ(s)}_{\text{a mg bc Z a mg, X left cont.}} + \underbrace{\int_0^t Z(s-)dX(s)}_{=:II} + [X, Z](t)$$

$$II = \int_0^t Z(s-)dX^c(s) + \sum_{0<s\le t} Z(s-)X(s-)(e^{u\triangle Q(s)} - 1) + \sum_{0<s\le t} \triangle X(s)\triangle Z(s)$$

Consider

$$\begin{aligned} \sum_{0<s\le t} \triangle X(s)\triangle Z(s) &= \sum_{0<s\le t} X(s-)Z(s-)(e^{u\triangle Q(s)} - 1)\triangle H(s) \\ &\quad - \sum_{0<s\le t} X(s-)Z(s-)(e^{u\triangle Q(s)} - 1)\triangle N(s) \\ &= \sum_{0<s\le t} X(s-)Z(s-)e^{u\triangle Q(s)}\triangle H(s) - \sum_{0<s\le t} X(s-)Z(s-)\triangle H(s) \\ &\quad - \sum_{0<s\le t} X(s-)Z(s-)(e^{u\triangle Q(s)} - 1) \end{aligned}$$

Observe that

$$V(t) = \sum_{i=1}^{N(t)} e^{uY_i} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$$

is a compound Poisson process with compensator $\tilde{\lambda}t(\tilde{\varphi}_Y(u) - 1)$.
From Ito's formula we know that

$$dX(t) = \underbrace{X(t-)d(-\tilde{\lambda}t(\tilde{\varphi}_Y(u) - 1))}_{dX^c(t)} + \underbrace{X(t-)(e^{u\triangle Q(t)} - 1)}_{dJ(t)}.$$

Thus

$$II = \int_0^t X(s-)Z(s-)d(V(s) - \tilde{\lambda}\varphi_\lambda(u)s) - \int_0^t X(s-)Z(s-)\left[H(s) - \lambda s\right]$$

□