Musterlösungen Blatt 1

1.

a) $$AB = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 2 & -1 \\ 0 & 4 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 0 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1+0-6 & 3+0+3 \\ 2+0+2 & -6+4-1 \\ 0+0-4 & 0+8+2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -7 & 6 \\ 4 & -3 \\ -8 & 10 \end{pmatrix}}}$$

b) $$B^T E = \begin{pmatrix} -1 & 0 & -2 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2+0-6 \\ -6+2+3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -4 \\ -1 \end{pmatrix}}}$$

c) $$CD = \begin{pmatrix} 5 & 0 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 10+0 & 10+0 \\ -6+6 & -6+8 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 10 & 10 \\ 0 & 2 \end{pmatrix}}}$$

d) $$E^T E = \begin{pmatrix} -2 & 1 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = (4+1+9) = (14)$$

$\neq$

e) $$EE^T = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}\begin{pmatrix} -2 & 1 & 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ -6 & 3 & 9 \end{pmatrix}}}$$

f) $$2C+D = 2\begin{pmatrix} 5 & 0 \\ -3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ -6 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 12 & 2 \\ -3 & 8 \end{pmatrix}}}$$

g) $$(C+D)^2 = \left[\begin{pmatrix} 5 & 0 \\ -3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}\right]^2 = \begin{pmatrix} 7 & 2 \\ 0 & 6 \end{pmatrix}^2 = \begin{pmatrix} 49+0 & 14+12 \\ 0+0 & 0+36 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 49 & 26 \\ 0 & 36 \end{pmatrix}}}$$

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$C^2+CD+DC+D^2$

$\neq$

h) $$C^2+2CD+D^2 = \begin{pmatrix} 5 & 0 \\ -3 & 2 \end{pmatrix}^2 + 2\begin{pmatrix} 5 & 0 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}^2$$

$$= \begin{pmatrix} 25 & 0 \\ -21 & 4 \end{pmatrix} + 2\begin{pmatrix} 10 & 10 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 10 & 12 \\ 18 & 22 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 55 & 32 \\ -3 & 30 \end{pmatrix}}}$$

2.

a) $A(B+C) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \left[\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right]$

$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{pmatrix}$

$= \begin{pmatrix} a_{11}(b_{11}+c_{11}) + a_{12}(b_{21}+c_{21}) & a_{11}(b_{12}+c_{12}) + a_{12}(b_{22}+c_{22}) \\ a_{21}(b_{11}+c_{11}) + a_{22}(b_{21}+c_{21}) & a_{21}(b_{12}+c_{12}) + a_{22}(b_{22}+c_{22}) \end{pmatrix}$

$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} + \begin{pmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{pmatrix}$

$= AB + AC$

oder allgemein: $A \in \mathbb{R}^{n \times m}$, $B, C \in \mathbb{R}^{m \times k}$

$[A(B+C)]_{ij} = \left[\sum_{l=1}^{m} a_{il}(b_{lj} + c_{lj}) \right] = \sum_{l=1}^{m} a_{il} b_{lj} + \sum_{l=1}^{m} a_{il} c_{lj}$

Eintrag i-te Zeile, j-te Spalte, $1 \leq i \leq n$, $1 \leq j \leq k$

$= [AB]_{ij} + [AC]_{ij}$

2.

b) Schreibe $A^T = \begin{pmatrix} \bar a_{11} & \bar a_{12} \\ \bar a_{21} & \bar a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$, $B^T = \begin{pmatrix} \bar b_{11} & \bar b_{12} \\ \bar b_{21} & \bar b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$.

$$(AB)^T = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}^T = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{21}b_{11} + a_{22}b_{21} \\ a_{11}b_{12} + a_{12}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \bar b_{11}\bar a_{11} + \bar b_{12}\bar a_{21} & \bar b_{11}\bar a_{12} + \bar b_{12}\bar a_{22} \\ \bar b_{21}\bar a_{11} + \bar b_{22}\bar a_{21} & \bar b_{21}\bar a_{12} + \bar b_{22}\bar a_{22} \end{pmatrix} = B^T A^T$$

c) oder allgemeiner: $A = (a_{ij})^{\overset{\text{Zeilen}}{i=1,\dots,m}}_{\underset{\text{Spalten}}{j=1,\dots,m}} \in \mathbb{R}^{m\times m}$, $B = (b_{ij})^{i=1,\dots,m}_{j=1,\dots,k} \in \mathbb{R}^{m\times k}$

$$A^T := (\bar a_{ij})^{i=1,\dots,m}_{j=1,\dots,m} = (a_{ji})^{i=1,\dots,m}_{j=1,\dots,m} \in \mathbb{R}^{m\times m}$$

$$B^T := (\bar b_{ij})^{i=1,\dots,k}_{j=1,\dots,m} = (b_{ji})^{i=1,\dots,k}_{j=1,\dots,m} \in \mathbb{R}^{k\times m}$$

$$[(AB)^T]_{ij} = [AB]_{ji} = \sum_{l=1}^{m} a_{jl} b_{li} = \sum_{l=1}^{m} \bar b_{il} \bar a_{lj} = [B^T A^T]_{ij}$$

Eintrag i-te Zeile, j-te Spalte

3. Schreibe $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$, dann bedeutet

$$M \begin{pmatrix} 2 & 1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 2m_{11} - 4m_{12} & m_{11} + 2m_{12} \\ 2m_{21} - 4m_{22} & m_{21} + 2m_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{einfach}$$

$$\left.\begin{aligned} 2m_{11} - 4m_{12} &= 1 \\ m_{11} + 2m_{12} &= 0 \end{aligned}\right\} \to \left.\begin{aligned} 2m_{11} - 4m_{12} &= 1 \\ 4m_{12} &= -\tfrac{1}{2} \end{aligned}\right\} \to \begin{aligned} m_{12} &= -\tfrac{1}{8} \\ m_{11} &= \tfrac{1}{4} \end{aligned}$$

$$\left.\begin{aligned} 2m_{21} - 4m_{22} &= 0 \\ m_{21} + 2m_{22} &= 1 \end{aligned}\right\} \to \left.\begin{aligned} 2m_{21} - 4m_{22} &= 0 \\ 4m_{22} &= 1 \end{aligned}\right\} \to \begin{aligned} m_{22} &= \tfrac{1}{4} \\ m_{21} &= \tfrac{1}{2} \end{aligned}$$

also $M = \begin{pmatrix} \frac{1}{4} & -\frac{1}{8} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$.

Man nennt M die inverse Matrix zu $\begin{pmatrix} 2 & 1 \\ -4 & 2 \end{pmatrix}$, und $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ heißt Einheitsmatrix (siehe Vorlesung).

4. Man stelle folgende Matrix, bzw. Vektor auf

$$\begin{array}{l} \\ \text{Maschinenlaufzeit} \\ \text{Rohstoffe} \\ \text{Arbeitszeit} \end{array}
\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 3 & 4 & 1 \\ 4 & 2 & 8 \\ 2 & 5 & 6 \end{pmatrix} \end{array}
\quad
\begin{array}{l} \text{Einheiten } A \\ \text{Einheiten } B \\ \text{Einheiten } C \end{array}
\begin{pmatrix} 20 \\ 15 \\ 40 \end{pmatrix}.$$

a)

$$\begin{pmatrix} 3 & 4 & 1 \\ 4 & 2 & 8 \\ 2 & 5 & 6 \end{pmatrix} \begin{pmatrix} 20 \\ 15 \\ 40 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 160 \\ 430 \\ 355 \end{pmatrix}}} \begin{array}{l} \text{Gesamtmaschinenlaufzeit} \\ \text{Gesamtrohstoffmenge} \\ \text{Gesamtarbeitszeit} \end{array}.$$

b) Man stelle folgenden Kostenvektor auf

$$\begin{array}{ccc} \text{Maschinenlaufzeit} & \text{Rohstoffe} & \text{Arbeitszeit} \\ (40+15 & 60+10 & 12) \end{array}.$$

Gesamtkosten: $(55 \quad 70 \quad 12)\begin{pmatrix} 160 \\ 430 \\ 355 \end{pmatrix} = 8800 + 30100 + 4260 = \underline{\underline{43160}}$

c) Änderung der Kosten:

$$(55, 70, 12)\begin{pmatrix} 3 & 4 & 1 \\ 4 & 2 & 8 \\ 2 & 5 & 6 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ -8 \end{pmatrix} = (55 \quad 70 \quad 12)\begin{pmatrix} 7 \\ -44 \\ -38 \end{pmatrix}$$

$$= 385 - 3080 - 456 = \underline{\underline{-3151}}$$