Qualitatives Risikomanagement in der Versicherung

Gerhard Stahl
Ulm, Juni 2015
1. Freitag, den 5. Juni 10:00 bis 17:00 Uhr

2. Samstag, den 6. Juni 10:00 bis 17:00 Uhr

3. Freitag, den 19. Juni 10:00 bis 17:00 Uhr

4. Samstag, den 20. Juni 10:00 bis 13:00 Uhr
Bibliography

1. ISO-Norm zu Risikomanagement
2. Luhmann, Soziologie des Risikos
3. Aven, Quantitative Risk Assessment
4. Aven et. al., Uncertainty in Risk Assessment
5. Hood et. al., The Government of Risk
6. Carrel, The Handbook of Risk
7. Diebold et. al., The Known, the Unknown and the Unknowable
Bibliography

8. Chapman, Simple tools and techniques for enterprise risk management
9. Rebonato, Plight of the Furtune Tellers
10. De Weert, Bank and insurance capital management
11. Dembo; Fremann, Die Revolution des finanziellen Risikomanagements
12. Lam, Enterprise Risk Management
13. Scandizzo, Risk and Governance
14. Tarantino, Essentials of Risk Management in Finance
15. Lyotard, Das postmoderne Wissen
Content

- SPAN
- Aven’s approach
- ISO Norm of risk management in Aven’s perspective
- Applications to Solvency II
■ SPAN
Variable of interest: \( \nu(\Pi(s_1) - \nu(\Pi_0)) = \nu(\Pi(s_1)) - \nu(\Pi_0) \)

Scenarios: \( S = s_1, \ldots, s_N \)

Weights: \( W = w_1, \ldots, w_n \)

Risk Measure: \( \rho_h(\Pi) = max\{w_1(\nu(\Pi(s_1) - \Pi_0)), \ldots, w_n(\nu(\Pi(s_n) - \nu(\Pi_0))\} \)
### Scenario matrix of SPAN

<table>
<thead>
<tr>
<th>Number</th>
<th>Scenario Weights</th>
<th>Changes in Underlying</th>
<th>Changes in Volatility of Underlying</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100%</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>100%</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>100%</td>
<td>+1/3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>100%</td>
<td>+1/3</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>100%</td>
<td>-1/3</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>100%</td>
<td>-1/3</td>
<td>-1</td>
</tr>
<tr>
<td>7</td>
<td>100%</td>
<td>+2/3</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>100%</td>
<td>+2/3</td>
<td>-1</td>
</tr>
<tr>
<td>9</td>
<td>100%</td>
<td>-2/3</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>100%</td>
<td>-2/3</td>
<td>-1</td>
</tr>
<tr>
<td>11</td>
<td>100%</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>100%</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>13</td>
<td>100%</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>100%</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>15</td>
<td>35%</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>35%</td>
<td>-2</td>
<td>0</td>
</tr>
</tbody>
</table>
Properties of SPAN

By means of SPAN a control circuit between the exchange and their participants is established, that uses the margin \( ?? \) as feedback that mirrors changes in the market by taking current prices \( \nu(\Pi_0) \) as well as changes over one day, \( \nu(\Pi(s_i)) \), into account, in order to control the system and keep it in an equilibrium. Here the margin serves as a control variable. All in all this is the example of setting standards (margins) gathering information (daily calculation) and modification of behavior (compare risk preferences with the market).
Properties of SPAN

- Uncertainty and time
  - the margin is determined ex-ante, expressing the uncertainty about the future
  - the marked-to-market evaluation is done ex-post
  - the time horizon is one trading day.

- Aspects of risk measurement of SPAN
  - it is portfolio based
  - a risk measure – the maximum loss is used to determine the margin by $\varrho_h(\Pi)$
  - portfolio-based stress tests are included by the means of $s_{15}$ and $s_{16}$

- Role of stakeholder = process owner of SPAN
Properties of SPAN

- the stakeholder – here the exchange – determines the methods
- the exchange organizes the communication process
- the exchange is responsible for the business processes.
Properties of SPAN

- SPAN establishes a feedback system between Stock Exchange and market actors, which uses the margin as feedback to obtain the system stability.

- It defines $\rho_h(\Pi)$ just for the portfolios, which are related to the certain underlying.

- The weights $w$ are time-invariant.

- SPAN is a deterministic model, the covariance structure of risk factors cannot be adequately captured by this model.

- SPAN is a closed model.
Systemic view on risk management

- CAPITAL
- Portfolio
- CONTROLLING SYSTEM

Systemic Protection
Process - a fundamental term

System theory is a method or language like mathematics, which it is universally applicable. Its basic term is that of a *process*. This flexible term allows to describe and analyze complex ensembles like a financial institution and its interactions. Typically a top-down approach is preferred when this method is applied and sub-processes are adjoint, if a finer granulation is needed. Compared to mathematics, where the focus is of *understanding* relationships, the focus of systemic models is to *control* or *change behavior of the system*. 
Process - a fundamental term
Process - a fundamental term

The fundamental term of cybernetics is that of a process. This flexible tool describes a system through its various components. In the context of an undertaking such a system may comprise: projects, stochastic models, external players and economic conditions. In order to describe and represent a process the following terms are useful: name of the process, process owner, description of triggers, input - items and sources, same for output, process units, their owners, business objective(s), business risks, key controls and measures of success. A process is called *under control* if its output is within prespecified limits. It is called *capable*, if the implemented process is able to achieve its prespecified goals.

Ulm, Juni 2015
Qualitatives Risikomanagement in der Versicherung
Page 16
Examples for the Process: Inputs

- Appointment
- Business objectives and plan
- Process map and organogram
- Value chain
- Audit committee
- Internal controls
- Risk management plan
- Financial reports
Examples for the Process: Inputs

- Marketing plan
- Ratio analysis
Examples for the Process: Constraints

- Business risk management culture
- Risk management resources
- Risk management study parameters
- Risk management plan
Examples for the Process: Mechanisms and Outputs

Mechanisms:

■ Finance analysis tools
■ Risk management process diagnostic
■ SWOT questions
■ PEST questions

Outputs:

■ Business analysis findings
Content

- Aven’s approach
Some definitions of risk

1. **Definition 1.** Risk is the measure of the probability and severity of adverse effects

2. **Definition 2.** Risk is the combination of probability and extent of consequences

3. **Definition 3.** Risk is equal to the triplet \((s_i, p_i, c_i)\), where \(s_i\) is the \(i\)-th scenario, \(p_i\) is the probability of that scenario, and \(c_i\) is the consequence of the \(i\)-th scenario, \(i = 1, \ldots, N\)
Some definitions of risk

4. \( \text{Risk} = (A; P; C) \), where \( A \) represents the events (initiating events, scenarios), \( C \) the consequences of \( A \) and \( P \) the associated probabilities.

Here a probability can be interpreted either as relative frequency \( P_f \) or as a subjective measure of uncertainty about future events and consequences, seen through the eyes of the assessor and based on some background information and knowledge (Bayesian perspective). The probability is referred to as a subjective or knowledge based probability.
1. **Definition 4.** Risk refers to uncertainty of outcome of actions and events

2. **Definition 5.** Risk is a situation or event where something of human value is at stake and where the outcome is uncertain

3. **Definition 6.** Risk is an uncertain consequence of an event or an activity with respect to something that humans value

4.
Risk is defined through uncertainties

**Definition 7.** Risk is equal to the two-dimensional combination of events / consequences and associated uncertainties

5.

**Definition 8.** Risk is uncertainty about an severity of a consequence (outcome) of an activity with respect to something that humans value

Note that these definitions emphasize the importance of uncertainty. Hence, in the light of these definitions risk is not characterized by \((A, C, P)\) but of \((A, C, U)\), where \(U\) - the uncertainties - replace the probabilities, i.e. \(P\).
Risk assessment based on the \((A, C, P_f)\) definition

Definition 1-3 are covered by the risk perspective if the probabilities are frequency interpreted. In this case the risk is unknown as \(P_f\) is unknown. Risk assessment is introduced to describe the risk. The description covers an estimate \(\hat{P}_f\) of \(P_f\) as well as the assessments of uncertainties about \(\hat{P}_f\) and \(P_f\). Thus, if this perspective to risk is the starting we alert to a risk description

Risk description in the \((A, C, P_f)\) case \(= (A, C, \hat{P}_f, U(\hat{P}_f), K)\), where \(U(\hat{P}_f)\) refers to an uncertainty description of \(\hat{P}_f\) relative to the true value \(P_f\) and \(K\) is the background knowledge that the estimate and uncertainty description is based on. We refer to \(U(\hat{P}_f)\) as a second order uncertainty description.

If we use subjective probabilities to express our uncertainties about \(P_f\) the risk description takes the following form:
Risk assessment based on the \((A, C, P_f)\) definition

Risk description according to the probability of frequency approach
\[ = (A, C, \hat{P}_{f}, P(P_f), K). \]

Risk description according to the pure traditional statistical approach
\[ = (A, C, \hat{P}_{f}, d(P_f), K), \text{ where } d \text{ is a traditional confidence interval for } P_f. \]
The concept of vulnerability and resilience

**Definition 9.** Vulnerability (antonym robustness) = \((C, P_f, | A)\), in other words the vulnerability is the two-dimensional of consequences and associated relative frequency interpreted probabilities given the occurrence of an initiating event \(A\). The vulnerability description in the \(= (C, P_f, | A)\) case = \((C, \hat{P}_f, U(\hat{P}_f), K \mid A)\).

**Definition 10.** We define resilience as: \((C, P_f, | A)\), including new types of \(A\) and the resilience description \((C, \hat{P}_f, U(\hat{P}_f), K \mid \text{any } A)\), including new types of \(A\).
Risk assessment based on the \((A, C, U)\) definition

The risk description covers the following components:

Risk description is \(=(A, C, U, P, K)\), where \(P\) denotes the knowledge-based probabilities, uncertainties \(U\) not captured by \(P\) and \(K\) the background knowledge that \(U\) and \(P\) are based on. This description covers probability distributions of \(A\) and \(C\) as well as prediction of \(A\) and \(C\).

Vulnerability (antonym robustness) \(=(C, U, | A)\). The description of vulnerability \((C, U, P, K | A)\)

Resilience: \((C, U, | any A)\), including new types of \(A\). Resilience description: \((C, U, P, K | any A)\), including new types of \(A\).
Reliability and Validity

Reliability is concerned with the consistency of the "measuring instrument" (analysts, experts, methods, procedures).

Validity is concerned with the success of "measuring" what ones sets out to "measure" in the analysis.

**Definition 11.** Reliability \( (R) \): The extent to which the risk analysis yields the same results when repeating the analysis.

**Definition 12.** Validity \( (V) \): The degree to which the risk analysis describes the specific concepts that one is attempting to describe.
Criteria for Reliability

1. R1: The degree to which the risk analysis, methods produce the same results at re-runs of these methods.

2. R2: The degree to which the risk analysis produces identical results when conducted by different analysis teams but using the same methods and data.

3. R3: The degree to which the risk analysis produces identical results when conducted by different analysis teams with the same analysis scope and objectives but no restrictions on methods and data.
Criteria for Validity

1. V1: The degree to which the produced risk numbers are accurate compared to the underlying true risk.

2. V2: The degree to which the assigned probabilities adequately describe the assessors’ uncertainties of the unknown quantities considered.

3. V3: The degree to which the epistemic uncertainty assessment are complete.

4. V4: The degree to which the analysis addresses the right quantities.
Aven’s Framework

Background knowledge $K$

- Measure of uncertainty $P(X \leq x)$
- Model input $X \to G(X)$
- Quantities of Interest $Z$
  - Uncertainty expressed by $P(Z \leq z)$
  - $EZ, VarZ$
  - Etc.

Uncertainty propagation

- Sensitivity analysis and importance ranking
- Uncertainty assessment of uncertainty factors

Managerial review and judgement

- Decision

Uncertainty evaluation

Figure 1: [Diagram of Aven’s Framework]
ISO Norm of risk management in Aven’s perspective
### Modeling Principles
- Market consistency
- Robustness
- Falsable
- Parsimonious
- Communicability
- Auditable
- Minimise systemic risk within a group

### Talanx Values
1. Entrepreneurial mindset and action within the Group context
2. Comprehensive customer orientation
3. Results and performance orientation
4. Mutual trust and open communication

### General principles of risk management
a) Creates value
b) Integral part of organizational processes
c) Part of decision making
d) Explicitly addresses uncertainty
e) Systematic, structured and timely
f) Based on the best available information
g) Tailored
h) Takes human and cultural factors into account
i) Transparent and inclusive
j) Dynamic, iterative and responsive to change
k) Facilitates continual improvement and enhancement of the organization

### Principles
- Mandate and commitment
- Design of framework for managing risk
- Continual improvement of the framework
- Implementing risk management
- Monitoring and review of the framework

### Framework
- Establishing the context
- Risk assessment
- Risk identification
- Risk analysis
- Risk evaluation
- Risk treatment

### Process
- Communication and consultation
- Monitoring and review
- Establishing the context
- Risk assessment
- Risk identification
- Risk analysis
- Risk evaluation
- Risk treatment
**Terms and definitions - the backbones of the ISO framework**

**Definition 13.** Risk is defined as an effect of uncertainty on objectives, where an effect is a deviation from the expected - positive or negative. Objectives can have different aspects and can apply at different levels (strategic, organization-wide, project, product, process). Uncertainty is the state, even partial, of deficiency of information related to understanding or knowledge of an event, its consequence or likelihood.

**Remarks:**

1. Luhmann uses in his definition of risk respectively danger an attribution process in order to differentiate the two terms. In the case of a self attribution by means of an action Luhmann speaks of risk. In the case an event happens externally, he speaks of danger. On the level of individuals, risk relates to deciders and danger to affected persons. Note that risk calculations are only but a forward moved ex-post judgement and hence they may enlarge risk in the sense of Luhmann. However, of course, they might limit the danger and hence optimize the faced
Terms and definitions - the backbones of the ISO framework

consequences.

2. Note that this definition is close to \((A, C, U)\) because uncertainty is used in the definition.

3. The ISO Norm applies the term likelihood in order to express uncertainty. Likelihood is much more general than probability and comprises classical probability approaches as well as those based on subjective probabilities.

4. Due to the fact that uncertainty is also related to the expressed likelihoods the risk description defined by Aven is the right structure to capture the definition of risk in a fully-fledged manner. This means that \((A, C, U, P, K)\) is the right framework.
Terms and definitions - the backbones of the ISO framework

**Definition 14.** Risk management is defined as coordinated activities to direct and control an organization with regard to risk.

Remarks:

1. Note that a direction presumes implicitly a target. Hence, a risk appetite must be defined.

2. The control loop is a feedback instrument already encountered in our process definition.

3. The coordinated activities refer to organizational aspects, especially to a risk governance.
From the remarks we see that the understanding of the term risk management is closely related to the principle building blocks of processes and hence can be understood as a steering tool that is based on a permanent update of information. Note that uncertainty is defined as the deficiency of information.

**Definition 15.** The risk management framework defines a set of components that provide the foundations and organizational arrangements for designing, implementing, monitoring, reviewing and continually improving risk management through the organization.

**Remarks:**

1. The foundations include the policy, objectives, mandate and commitment to manage risk.
   Essentially the **USE TEST** starts here. Furthermore, the objectives, mandate and commitment are important building blocks of a **RISK CULTURE**.
2. The organizational arrangements include plans, relationship, accountabilities, resources, processes and activities.

3. The risk management framework is embedded within the organization overall strategic and operational policies and practices. Again this sheds the light to the use test and enterprise risk management. Regulators interpret the word *embedded* in such a way that the risk strategy is derived from the business strategy. The operational policies mentioned in this remark relate also to the different levels of an organization. Hence, risk management is an organization-wide activity. For that reason, it is important to have an overall consistency and this is achieved by implementing a risk management framework. The consistency itself is a tool at least on an epistemic level to minimize logical discrepancies and hence contribute to minimize uncertainties.

**Definition 16.** *The risk management policy states the overall intentions and directions of an organization related to risk management.*
Terms and definitions - the backbones of the ISO framework

Remarks:

1. As this definition relates to intentions and directions, targets or values, i.e. some types of social norms must be defined in advance.

2. Given this it is evident that firm-specific targets and values and their implementation relate to the use test. In this respect it is important to have a sufficiently large understanding of the term risk management system and model. We come to this later on.

3. The risk management policy as well as the risk management framework are comprised in Aven’s $(A,C,U,P,K)$ framework in the component $K$ - the background knowledge. This means the background knowledge captures among others firm-specific knowledge including processes but also targets and values.

**Definition 17.** The risk attitude is an organization’s approach to assess and eventually pursue, retain, return or take-away from risk.
Definition 18. *The risk management plan is the scheme within the risk management framework specifying the approach the management components and resources to be applied to the management of risk.*

Remarks:

1. Management components include procedures, practices, assignment of responsibilities, sequence and timing of activities.

2. Risk management plan can be applied at different levels.

3. In Aven’s framework the risk management plan is captured by the background knowledge $K$.

4. Note that the risk management may also comprise persons which actively manage risks. By changing the risk profile they are in general not part of the independent
Terms and definitions - the backbones of the ISO framework

risk function. Hence, the risk management plan refers to broader scope than just the risk management function in a financial institution.

**Definition 19.** A risk owner is a person or entity with the accountability and authority to manage a risk.

Remarks:

1. First of all, the board of a financial institution is a risk owner in the sense of the given definition.

2. If we consider the cascading structure of an insurance holding, there must be a clear procedure of escalation which implies the necessity of materiality concept and a limit and threshold system.

3. Especially board members contribute very much to the risk culture by setting the tone from the top.
Terms and definitions - the backbones of the ISO framework

Definition 20. The risk management process is defined as a systemic application of management policies, procedures and practices to the activities of communicating, consulting, establishing the context and identifying, analyzing, evaluating, treating, monitoring and reviewing risk.

Remarks:

1. Again, the term process refers to the basic definition given before and consists of inputs, outputs, controls and mechanisms.

2. In the above definition

   (a) the application of management policies, procedures and practices refer to the input
   (b) the communication, consulting, context establishing is referred to an output
   (c) the identification, analysis, evaluation of risks is referred to a mechanism, i.e. some type of model - either internal or standardized methods
Terms and definitions - the backbones of the ISO framework

(d) the monitoring and reviewing of risk is related to the control loop

A financial undertaking is not independent from various sociological structures. This comprise markets, employees, legal issues and so on. All these conditioning aspects contribute to what is understood as the context. As laid down in the definition of risk and danger by Luhmann we have to differentiate between internal and external contexts. In general internal contexts may be actively influenced by decisions and hence relate to risk in terms of Luhmann whereas external context is treated often passively by an undertaking and is therefore very much related to danger.

Note interestingly that in the Holy Bible the term risk does not appear. However, the term danger is very well known - both in the old and new testament, see for example the Epistles of the Romans, Luke and Samuel. This is very much in line with Luhmann’s approach due to the fact that the individualization of a person as we understand it today, began with the middle ages and is hence beyond the scope of the Holy Bible for which its canon was finalized at the synod of Karthago,
Terms and definitions - the backbones of the ISO framework

approximately 400 A.C.

**Definition 21.** The external context is defined by the external environment in which the organization seeks to achieve its objectives.

Remarks: The external context includes

1. the cultural, social, political, legal, regulatory, financial, technological, economic, natural, competitive environment whether internal, national, regional or local.

(a) Note that for financial institutions the legal and regulatory framework is very important due to the fact that financial markets are regulated. The importance of these issues may be seen that financial institutions have a **Chief Compliance Officer**. This key function under Solvency II guarantees that the undertaking is compliant with existing laws. In the light of globalization, this might be quite complex, see for example embargo decisions by politicians. Hence, compliance risks intervene and are manifested in reputational risks.
Terms and definitions - the backbones of the ISO framework

Note that non-governmental bodies like rating agencies or self-organizations by the industry, e.g. the CFO- and CRO-Forum, may establish requirements in terms of best-practice-papers or in the case of rating agencies standards which undertakings have to fulfill.

In the term of Aven’s approach the legal and regulatory aspects are captured by the component $K$. Later on, we will split-up $K$ in our own definition of a model and we will use $R$, with $R \subset K$, to highlight the importance of the legal and regulatory frameworks - both under Basel III and Solvency II for financial institutions.

(b) The financial and economic data respectively information is either comprised in the components $A$ and $C$ in Aven’s framework, depending on whether they are considered as inputs ($A$) or as outputs ($C$). Both information are to a large extent objective in terms of values, prices or financial ratios. Compared to the component $R$, introduced above, which is to a large extent time-invariant or updated over decades, economic and financial data are updated frequently. Hence, we will introduce in our own framework the information set $I \sqcup$ which
Terms and definitions - the backbones of the ISO framework

comprises financial and economic data; \( I_t \subset A \). Examples might be balance sheet data or financial time series as interest rates.

(c) The competitive environment that is also to be taken into account is from an information theoretic point of view much more incomplete compared to financial and economic data. Given the importance of the judgmental component here, we introduce in our own model a component \( Z_t \) to capture the competitive situation. Of course, \( Z_t \subset K \) in the context of Solvency II, this component is also related to the forward-looking financial planning realized the controlling department. Hence, \( Z_t \) will play an important role within the ORSA context.

(d) The geographical components (internal, national etc.) refer to the applied level of the risk management framework. The level itself is mirrored in the organizational structure. In our own model we will capture this by a component \( O \).

2. Key drivers and trends having impact on the organization. Again, this is comprised in \( Z_t \subset K \).
Terms and definitions - the backbones of the ISO framework

3. Relationships with and perceptions with external stakeholders: this comprise clients (policy holders), investors and financial analysts. Note these bodies are captured by $R$. Given that the context and hence the danger is very much influenced by these stakeholders, the treatment of these stakeholders contributes to the risk culture and the use test. In a number of activities as: roadshows, quarterly financial statements and conferences, annual report, Regular Supervisory Report (RSR), the output of the risk management process is included. Note furthermore, that there is an interdependency between the use test and the establishing of the context of the risk management framework.
Terms and definitions - the backbones of the ISO framework

Regulatory Requirements, S&P, CRO Forum, Investors expectations

Controls: Validation, IKS, Compliance, Internal Audit, Committees, Policies

TERM

Risk Budget

LTS

NatCat
MR Holding
RD
LOBs Entities

Reporting

CRSA Limit Report

Risk Report
Definition 22. The internal context defines the internal environment in which the organization seeks to achieve its objectives.

Remarks: The internal context includes

1. governance, organizational structure, roles and accountabilities

   (a) the roles and accountabilities refer to \( R \subseteq K \), the organizational structure refers to \( O \). Even though documentation has to be reviewed on a yearly basis, this will realize only minor changes. For that reason, it is reasonable to assume time invariance to a large extent. Note that in the organizational structure, the concept of level is of highly importance. Furthermore, the organizational structure, especially for risk management or more general for key functions (risk management, compliance, internal audit and independent actuarial function) is influenced by regulatory prescriptions. Hence, the organization is

\[ O = T(R) \]
Terms and definitions - the backbones of the ISO framework

a function of $\mathbb{R}$. As a consequence, the regulatory framework influences also the exposure to operational risks.

2. policies, objectives and the strategies that are in place to achieve them.
   
   (a) Typically, objectives are financial ratios, limit- and threshold systems or risk parameters. By and large, these are captured at the level of the consequences, hence in $C$.
   
   (b) Policies can be seen as inputs in our process framework and strategies may be interpreted as a mechanism.

3. the capabilities understood in terms of resources and knowledge (time, people, processes, systems and technologies).

   (a) Most of the above mentioned capabilities refer to IT processes which are captured in Aven’s framework by $K$, in our framework we consider this as a part of $\emptyset \subset K$. Again it is evident that this component is very much related to
Terms and definitions - the backbones of the ISO framework

operational risk. Regulatory prescription which intend a certain redundancy with respect to organizational issues (four-eye-principle, a separation of development, test and production of IT systems) contribute to a reduction of operational risk due to introduce an independent control component that will dramatically decline the probability of an adverse event, however, this has a certain price. In principle, regulators prescribe here a mitigation technique in form of an operational hedge and increases the costs (to be paid for the hedge).

(b) In respect to knowledge it is important to understand risk management system as a learning process due to the fact that the regulatory environment is new and cannot be based on former experiences. Workshops that distribute knowledge are of high importance.

4. information systems, information flows and decision-making processes (both informal and informal)

(a) To tackle with information is in the heart of risk management. Understanding risk management systems a socio-technical system it is important to
Terms and definitions - the backbones of the ISO framework

differentiate between risk informed and risk based decisions. In any case the information is organized by an enterprise-wide reporting and a cascading system of committees where a clear structure of those participants of those who of voting or veto rights has to be clarified. Due to the fact that decision-making is about consequences, it is formally captured by $C$. Evidently, decision-making processes contribute very much to the use test. As explained before, targets have to be prespecified. Hence, the decisions are drawn within a disciplined framework.

5. relationships and perceptions and values of internal stakeholders

(a) Formally, these aspects are captured in $K$, in our framework it is mainly captured by $O$, however, for board members - a very important set of internal stakeholders - compliance is a high value. As we noticed before, compliance is related to $R$. The structure of internal stakeholders might be quite complex, depending on the complexity of the holding, say, furthermore the supervisory board and the employees are important examples.
6. the organization culture

(a) The term culture is not easy to explain. Given our systemic framework, we admire very much a definition given by Immanuel Kant, who defines culture as the autonomy of human beings. Hence, their ability to define independently from their actual preferences, the law of their will and action. This means, culture is a system of norms related to behavior and action. In the light of this definition, risk culture is very much defined by targets, i.e. the contribution of the risk management system to an enterprise risk management and to an adequate behavior of the persons in risk management departments. The latter is formalized in so-called fit and proper requirements. In our overview to risk management, the Talanx Values included as an example for such a normative setting.

7. standards, guidelines and models adopted by the organization

(a) With respect to the framework defined by the process the standards and
Terms and definitions - the backbones of the ISO framework

guidelines are to be understood as inputs whereas the models are understood as mechanisms.

(b) Under the regulatory regimes for financial institutions (banks, insurance companies, ucits) the undertakings may opt for a so-called internal model approach or model prescribed by regulators, so-called standardized methods as a tool to measure the risk to which they are exposed to. Note that even a regulatory model is a model and hence, related to some uncertainty. The latter is specific to the considered undertaking.

(c) For models it is important to differentiate between models in a narrow sense, which are only but a more or less complicated mathematical mechanism or to consider a model in a wide sense, where the whole risk management system is understood as a model.

8. Form and extent of contractual relationships

(a) Contractual relationships are referred either to internal deals or outsourcing activities within a holding between entities and centre of competencies say. In
Terms and definitions - the backbones of the ISO framework

the latter case, service level agreements have to be in place - internal deals have to follow the requirements as deals with external parties.

**Definition 23.** Communication and consultation is understood as a continual and iterative process that an organization conducts to provide, share or obtain information and to engage a dialogue with stakeholders regarding the management of risks.

Remarks:

1. The information can relate to the existence, nature, form, likelihood, significance, evaluation, acceptability and treatment of the management of risk.

   Given that the stakeholders contribute very much to the risk profile it does not take wonder that the before mentioned characteristics embrace all components of Aven’s approach, i.e. \((A, C, U, P, K)\). Given that the whole structure is touched, it is clear that communication to stakeholders is the key activity of a risk manager, especially the Chief Risk Officer. To most of these stakeholders...
Terms and definitions - the backbones of the ISO framework

the communication channels are institutionalized; internally by committees and reporting lines, externally by periodic meetings with rating agencies, regulators and investors.

Note that both rating agencies as well as financial analysts publish reports which include also judgments on risk management activities. Hence, these external stakeholders have a great influence - both on targets as well as the visible reputation of risk management departments. For that reason, a number of financial institution publish via internet white papers on risk management which formulate their specific philosophy.

2. Consultation is a two way process of informed communication between an organization and its stakeholders on an issue prior to making a decision or determining a direction on that issue. Consultation is

(a) a process with impacts on a decision influence rather than power and
(b) an input in a decision-making not joined decision-making.
From this we conclude that risk management should strive for influence. The consultation approach fits smoothly to independence requirements on risk management functions set out by regulators.

**Definition 24.** A stakeholder is a person or organization that can affect, be affected by or perceive themselves to be affected by a decision or activity.

Remarks:

1. From a process perspective stakeholders are in general affected by the output, i.e. the consequences. Hence, in general, stakeholders are affected by all components of a risk description, i.e. \((A, C, U, P, K)\).

2. Let us exemplify this: a hurricane, an element of \(A\) may impact investors. The climate change might influence the probability of future hurricanes, hence influence the likelihood and again impact the investors. The loss of a hurricane may be tremendous and the share price may fall. Regulatory changes may generate huge costs either in terms of regulatory capital or investments. Here
Terms and definitions - the backbones of the ISO framework

stakeholders are positively involved, because in general, these requirements will stabilize the financial system. At the same time they are negatively involved, because shareholders have a certain loss on performance.

**Definition 25.** Risk assessment is the overall process of risk identification, risk analysis and risk evaluation.

**Definition 26.** Risk identification is the process of finding, recognizing and describing risks.

Remarks:

1. Risk identification involves the identification of risk sources, events, their causes and their potential consequences.
   The definition of risk in Aven’s terminology

\[
Risk = (A, C, U)
\]

comprises the risk sources, events and the consequences. Mathematically the
Terms and definitions - the backbones of the ISO framework

Consequences $C$ are a function of $A$ and $U$:

$$C = T(U, A)$$

Here, $T$ stands for a forecast model or an evaluation model. Also a regulatory model as exemplified by SPAN fits in this framework.

2. Risk identification can involve historical data, theoretical analysis, informed and expert opinions and stakeholders needs. As already explained, historically data are captured by $\mathcal{I}_t \sqcup$, expert opinions are part of $Z_t$. Further expert judgments, especially those used through the modeling process are captured by $K \setminus \{\mathcal{I}_t \cup Z_t\}$. An important requirement from stakeholders is the capital requirements set out by rating agencies or supervisory institutions. For example under Solvency II, regulators require capital to survive a 200-year event. Furthermore between the set of all stakeholders, there might be conflicts of interest. This is especially true for shareholders and policy holders. Hence,
a financial institution has to clarify its preferences with respect to stakeholders which contribute very much to the level of risk or comfort a financial institution has to deliver to its stakeholders. 

Note that in this respect, companies in continental Europe and those with an anglo-saxon trial behave a little bit different, because on the European continent, the stewardship principle is applied whereas in anglo-saxon companies the agency principle dominates.

**Definition 27.** *A risk source is defined as an element which alone or in combination has the intrinsic potential to give rise to risk.*

Remark:
Note that the term risk source is more general than that of a risk factor. In general, a risk factor is understood as a source quantifiable risk that is an input in risk measurement. A risk source which gives rise to risk is also by definition related to uncertainty. Hence risk sources comprise also second order risks. Furthermore, a risk source may be related to an observable or unobservable quantity. An
correlation or volatility yield as examples for non-observables quantities. In Aven’s terminology a risk source is an element or a subset of $A$.

**Definition 28.** An event is defined as the occurrence or change of a particular set of circumstances.

Remarks:

1. An event with no consequences can also be referred to as a near miss or near hit.

2. In the terminology of stochastics, the term event is one of the building blocks of a probability or statistical space. A statistical space denoted by

$$(\Omega, \mathcal{A}, \mathbb{F}_\theta); \theta \in \Theta$$

where $\Omega$ denotes the samples space, $\mathcal{A}$ denotes the $\sigma$ - Algebra of events and $\mathbb{F}_\theta$ denotes the stochastic model where $\Theta$ denotes the parameter space. In the case of stochastic processes, the $\sigma$ - Algebra is replaced by a filtration $\mathcal{A}_t$. 
3. In the terminology of Aven a subset or an element of $A$. Recall, that Aven suppresses an indexation related to the time $t$.

**Definition 29.** A consequence is defined as an outcome of an event affecting objectives.

Remarks:

1. Evidently, the consequences are related to $C$ in Aven’s terminology. The consequences are in general related to an output of the risk management process.

2. Due to the fact that risk sources can be qualitatively or quantitatively the same is true for consequences. A good example for the first category is reputational risk.

3. From a practical point of view it is important to note that initial consequences can escalate through knock on effects. Again reputational risk may serve as an example but also a downgrade may be the starting point of a downturn spiral.
4. In the context of forecast models, the risk model is used at time $t$ for forecasting the state of affairs for $t + 1$, i.e. $V_{t+1|t}$, given the information available at $t$. Assume that the realized state of affairs at $t + 1$ is denoted by $x_{t+1}$. Then the function $d$

$$d(V_{t+1|t}, x_{t+1})$$

denotes the loss or consequences. The use test is exactly based on this term as far as the model in the narrow sense, i.e. the risk measurement, is concerned.

**Definition 30.** Likelihood is the chance of something happening.

Remark: The German’s translation of likelihood is Mutmaßlichkeit. It was introduced in the statistics by RA Fischer (likelihood ratio test, likelihood principle etc.). It is a synonym of probability, however, comprise concepts that are beyond the Kolmogorov approach to probability. Especially subjective and Bayesian approach.

**Definition 31.** Risk profile is a description of any set of risks.

Remarks:
1. Risk may be decomposed into a number of categories. This should be exhaustive and spotlight a cause and affect relationship between consequences and events. As earlier, the principle of leveling applies. Typically risk profiles are represented by so-called waterfall diagrams.
Terms and definitions - the backbones of the ISO framework

Risk components of Talanx Group¹
(as of 31 December 2013, €m)

```
Market risk:
- Non-life: 1,632
- Life: 433
- Pension: 390
- Diversification: 653

Total market risk: 3,908

Counterparty default risk: 1,802

Premium and investment risk (net): 2,196

Natural Cat. (net): 259

Diversification: 653

Further risk (life): 3,143

Operational risk: 781

Other risk: 389

Total risk before tax and before diversification: 4,857

Tax effect (non-life and retail): 16.1%

Total risk (net): 651

Net CAT (net): 1,850

Net total risk: 2,356

¹ Figures show risk categorisation of the Talanx Group after minorities, tax effects and diversification effects as of 2013. Solvency capital requirement determined according to 99.5% security level, economic view, after minorities.

High diversification between risk categories
2. As noted before the consequences $C$ are a function of $A$ and $U$:

$$C = T(A, U)$$

The decomposition of $A$ decomposes $C$ as seen in the waterfall diagram above. This decomposition is important in order to express the risk preferences, the exposure and a limit and threshold system. Furthermore, the decomposition is a building block of any performance measurement. The reason for that is that the decomposing categories are per category homogenous and in between categories heterogenous. By this dual approach risk return based comparison of different investment option is enabled. Hence, the risk profiling is the starting point of any quantitative or qualitative risk strategy. Hence, an important building block of the use test. Note that the so-called P&L Attribution defines a decomposition of the P&L related to risk categories.
Definition 32. Mit

\[ F(X_{t+h} | \mathcal{I}_t, Z_t, \mathbb{R}, \mathbb{O}) \]  \hspace{1cm} (1)

bezeichnen wir den durch ein Modelldefinierten relationalen Bezug. Hier ist dies eine Prognoseverteilung mit Prognosehorizont \( h \) für die interessierende Variable \( X \) - also der Ergebnisvariable - für die implizit zu Grunde liegenden erklärenden Variablen, die durch Informationsmengen \( \mathcal{I}_t, Z_t, \mathbb{R}, \mathbb{O} \) definiert sind. Dabei sind diesen Informationsmengen folgende Interpretationen zugeordnet:

1. \( \mathcal{I}_t \) bezeichnet klassische Datenmengen, die kontinuierlich erhoben werden bzw. anfallen und zu einem Update führen. Für einen ESG können dies Preiszeitreihen von Bonds oder Großschäden bei der Bestimmung von Schadensdreiecken sein. Neben diesen empirischen Daten können hier zusätzliche Expertenschätzungen z.B. bei operationellen Risiken verwendet werden. In jedem Fall handelt es sich um Inputdaten, die Inputmodellen (ESG, GES, ...) zu Grunde liegen.
Definition of the term Model

2. Bei der Informationsmenge $Z_t$ handelt es sich um Hintergrundwissen, das zum einen in die Modellierung einfließt, z.B. durch die Auswahl der Variablen, und zum anderen nach vorne gerichtete Expertenschätzungen darstellt, wie sie z.B. für strategische Entscheidungen im Rahmen des Controlings von Nöten sind. Im Gegensatz zu $I_t$ fallen diese Informationen nicht im Sinn eines produktiven Prozesses an, sondern sind Managementinformationen, also Experteneinschätzungen, die insbesondere Zukunfts- und keinen Vergangenheitsbezug haben.

**Definition of the term Model**

_Höhe der Kosten und des operationellen Risikos._

4. Ø bezeichnet die Aufbauorganisation eines Unternehmens. Mit der für Versicherungsgruppen typischen Spartentrennung geht eine komplexe Aufteilung von Rollen und Verantwortlichkeiten einher, die durch konzernweite arbeitsteilige Prozesse im Rahmen des internen Modells oder bei der Erstellung der IFRS-Bilanz nicht unwesentlich zum Profil des operationellen Risikos beiträgt.

**Example 1. Verwendungstest**

Am Beispiel des Verwendungstests lässt sich die Bedeutung des Modells im weiten Sinn bzw. der Bezug zu den unterschiedlichen Informationsmengen in (??) gut illustrieren.

1. Das Limit- und Schwellenwertsystem verknüpft zum einen die Aufbauorganisation über die Segmente, also Ø mit den Ergebnissen des internen Modells basierend auf \( \mathcal{J}_t \), also dem Ergebnis aus dem Rechenkern. Bei der Definition des
**Definition of the term Model**

*Risikobudgets kommen zusätzliche Erwägungen des Risikocontrollings zum Tragen, die idealtypisch in der Variable $Z_t$ abgebildet werden.*

2. Die Etablierung des Executive Risk Committees (ERC), also eine organisatorische Maßnahme, die der besonderen Bedeutung des internen Modells Rechnung trägt, und insoweit eine Folge des Verwendungstestes ist. Angemerkt sei hier, dass dies explizit auch als Modelländerung aufgefasst wurde, obgleich das ERC natürlich nicht Teil des Modells im engen Sinn, d.h. des Rechenkernes ist.

3. Die Verwendung der Modellergebnisse in Form des Economic Capital Report, der via Internet zur Verfügung steht, zur Kommunikation der Risikosituation gegenüber externen Stakeholdern, die in $\mathbb{R}$ abgebildet sind, bilden einen wesentlichen Teil des Use Testes, obwohl hier lediglich die Kommunikation zu Stakeholdern eine Rolle spielt.

Die Subsumtion der obigen Beispiele unter dem Begriff Verwendungstest...
Definition of the term Model

wäre ohne die Modellddefinition, wie sie in Definition (??) gegeben ist, nicht nachvollziehbar. Weiterhin lassen sich eigene Strategien zur Umsetzung des Verwendungstests nur aus einem abstrakten Ansatz deduzieren, da sonst lediglich eine Umsetzung von regulatorischen Vorgaben im Sinn von Level 1, 2 oder 3 Texten möglich wäre.

Nachdem die Grundbegriffe Modell, Risiko, Risikobeschreibung definitorisch gefasst sind, verbleibt noch wesentliche Risikoparameter zu fixieren. Dies geschieht mit folgender Definition:

**Definition 33. Risikomaßzahlen**

Im Rahmen von Solvency II ist es notwendig, für die Bestimmung des aufsichtlichen Solvenzzkapitals, eine Prognoseverteilung

\[ F(X_{t+h} \mid J_t, Z_t, R, O) \]

zu verwenden, wobei der Prognosehorizont \( h \) ein Jahr beträgt. Zur Beurteilung der durch den Prognosehorizont induzierten Unsicherheit wird ein \( \alpha \)-Quantil,
wobei \( \alpha = 99.5\% \) gesetzt ist, verwendet. Damit ergibt sich als Formel für das Solvenzkapital, kurz SCR,: 

\[
SCR := \mathbb{E}(X_{t+1}) - \text{VaR}(X_{t+1})
\]

(2)

Alternativ ließe Solvency II auch folgende Berechnung zu:

\[
SCR := SNA_0 - \text{VaR}(X_{t+1}),
\]

(3)

wobei \( SNA_0 \) die Eigenmittel zu \( t = 0 \) bezeichnet.
Definition of the term Model

- Required Capital
- Probability of Loss
- Probability of Default
- 0 VAR / TVAR
- SNA₀
- Expected Value
- SNA₁
In obiger Grafik wird die Dichte der zur Prognoseverteilung gehörenden Verteilungsfunktion und eine Reihe von Risikomaßzahlen dargestellt: der Erwartungswert, der Value-at-Risk, der Tail-Value-at-Risk sowie eine Ausfallwahrscheinlichkeit. Weiterhin werden die zum Zeitpunkt $h = 0$ vorhandenen Eigenmittel, hier als $SNA_0$ bezeichnet, dargestellt. Die interessierende Variable, also $X_t$ ist die Veränderung dieser Eigenmittel über ein Jahr hier als Variable $SNA_1$ auf der X-Achse abgetragen. Auf der Y-Achse werden die Werte der Dichtefunktion abgetragen.
Risikomaßzahlen

■ Erwartungswert.

\[ E_F(X) = \int x dF(x) = \int_0^1 F^{-1}(x) \, dx. \] (4)

■ Value at Risk.

\[ \text{VaR}_\alpha(X) = F^{-1}(\alpha) = \inf \{ x : F(x) \geq \alpha \} \] (5)

for \( 0 < \alpha < 1 \).

■ Varianz.

\[ \text{Var} = \int x^2 dF - \left( \int x dF \right)^2 = \int_0^1 (F^{-1}(x))^2 \, dx - \left( \int_0^1 F^{-1}(x) \, dx \right)^2. \] (6)
Risikomaßzahlen

- Tail VaR.

\[ TVaR_\alpha(X) = \mathbb{E}_F(X \mid X > VaR_\alpha) = \int_{VaR_\alpha}^{\infty} x\,dF = (1 - \alpha)^{-1} \int_{\alpha}^{1} F^{-1}(x)\,dx. \]  

(7)
Definition 34 (COSO). Definition: The Committee of Sponsoring Organizations (COSO) defines enterprise risk management as “a process, effected by an entity’s board of directors, management and other personnel, applied in strategy setting and across the enterprise, designed to identify potential events that may affect the entity, and manage risk to be within its risk appetite, to provide reasonable assurance of entity objectives.”
Model uncertainty in the context of derivative valuation

- $\Omega(\lambda, X, R)$
- $\lambda$ exposure vector
- $X$ risk factor
- $R$ behavior, e.g. management rules

$$D(X_{t+h}|I_t, R)$$

In an arbitrage-free market, the linearity of prices means there exist a risk-neutral probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the value $V_t(H)$ of an option with payoff $H$ is given by:

$$V_t(H) = B(t, T)E^{\mathbb{Q}}[H|\mathcal{F}_t]$$

where $B(t, T)$ is a discount factor.
From SPAN to an internal model

- Specification of the stochastic model \( X \sim F_\theta, \theta \in H \)
- Parameter estimation \( \hat{\theta} = T(x_1, \ldots, x_n) \)
- Evaluation of \( \nu(\pi) \)
- Determination of the right risk measure \( \rho \)
- Risk measurement procedure (Cont et al 1998)
Model Risk – or better Model Uncertainty – comprises both, the calculation of risk, say \textbf{VaR}, and the valuation of a portfolio (or instrument).

- Valuation
  - Marked-to-market vs. mark-to-model
  - Market consistent vs. actuarial
  - Actuarial vs accounting valuation (HGB vs IFRS)

Note risk is ex-ante

\[ \mathcal{D}(X_{t+h} | I_t) \]

where valuation (and hence performance measurement) is ex-post

\[ \nu(\Pi_t) \]
Interplay between the model and the various valuations

Reality

Insurance policy

Price/value?

\( R \)

Mapping

Model

\( X \)

\( \varphi \) valuation

\( A \)

Accounting

\( F \)
Definitions of Model Risk

- Definition 1: Every risk induced by the application of a statistical model is called model risk (Crouhge et al 1998)
- Definition 2: Every risk induced by the choice, specification and estimation of a statistical model is called model risk in the strict sense (Sibbertsen, Stahl, Lüdtke 2008)

\[ \prod D(x_t | X_{t-1}; \Theta) \]

Steps of the model specification:
1. Marginalization of the data generating process
2. Model specification with respect to the choice of variables
3. Model specification with respect to the functional form
4. Estimation of the parameters

Note according to current regulatory framework model risk has to be captured
Basic Structure of Risk Models

- $D(X_{t+h} | I_t)$ Riskmetrics (1994)
- $D(X_{t+h} | I_t, Z_t)$ MCEV, SiS (IFA, TX, DFA 2008)
- $D(X_{t+h} | I_t, Z_t, \mathbb{R}, \emptyset)$ Internal Model
- Variable of interest $X_{t+h} := \Delta \nu_{t+h}(\Pi) = \nu_{t+h}(\Pi) - \nu_t(\Pi)$
- Note that stakeholders impact the model specification and choice of risk measure as well as the considered level of significance.
- D denotes a data generating process in sense of Hendry
Definition: Evaluation

- The function $\nu$

\[ \nu : \mathbf{I} \times T \longrightarrow \mathbb{R} \]

\[ (I_i, t) \mapsto \nu_{it} = \nu_t(I_i), \]

denotes the price or value of instruments $I_i \in \mathbf{I}$ at times $t$. It is convenient to extend the function $\nu$ to $\mathbb{R}^{\mathbf{I} \times T \times T}$ by

\[ \nu_s(\lambda_{1t}, \cdots, \lambda_{ut}) := (\lambda_{1t}\nu_s(I_1), \cdots, \lambda_{ut}\nu_s(I_u)). \]

By ( ), $\nu$ is homogeneous (of degree one) w.r.t the $\lambda_i$. The market price at $s$ of a portfolio from epoch $t$ is determined by

\[ \nu_s(\Pi_t) = \Lambda_t^T \nu_s(\mathbf{I}) = \sum_{i=1}^{u} \lambda_{i,t} \nu_s(I_i). \]

- Note that this heterogeneity assumption is a rather strong assumption
Definition: Riskspace

- \( \Omega \) is called risk space w.r.t. \( I \), if there exists for every instrument \( I_i \in I \) a function \( \psi_{it} \) - called pricing function of instrument \( I_i \) - with:

\[
\psi_i : \Omega \times T \rightarrow \mathbb{R} \\
(\omega, t) \mapsto \psi_{it}(\omega) \equiv \psi_{it}(\omega_{i(1)}, \cdots, \omega_{i(s)}) = \nu_t(I_i),
\]

where \( i(1), \cdots, i(s) \) denote the relevant risk factors for instrument \( I_i \). If the function \( \psi_i \) is linear, the associated instrument \( I_i \) is called linear too. If all \( I_i \in \Pi_t \) are linear, the portfolio is called linear. For example portfolios in cash instruments (e.g. currencies and stocks) are linear ones.

- In practice \( |\Omega| \) is very large. A so-called standard set of risk factors is chosen. \( |R| \ll |\Omega| \)
Delta VaR Model

- \( R = (R^1, \ldots, R^n) \)

- \( \Delta V = V(R + \Delta R) - V(R) \approx \sum_{i=1}^{n} \frac{\delta V}{\delta R^i} \Delta R^i = D \Delta R \)

- \( D = (\delta V/\delta R^1, \ldots, \delta V/\delta R^n) \)
Coherent Risk Measure

- **Axiom A1.** Translation invariance and positiv homogeneity: 
  \[ \rho(aX + b) = a\rho(X) + b, \quad \forall a \geq 0, b \in \mathbb{R} \]

- **Axiom A2.** Monotonicity: \( \rho(X) \leq \rho(Y) \), if \( X \leq Y \) almost surely

- **Axiom A3.** Subadditivity: \( \rho(X + Y) \leq \rho(X) + \rho(Y) \), for any \( X, Y \in \mathcal{X} \)

- \( \rho(X) = \sup_{P \in \mathcal{P}} \{ E^P[X] \}, \quad \forall X \in \mathcal{X} \)
Insurance Risk Measure

■ Axiom C1. Conditional state independence: $\rho(X) = \rho(Y)$, if $X$ and $Y$ have the same distribution. This means that the risk of a position is determined only by the loss distribution.

■ Axiom C2. Monotonicity: $\rho(X) \leq \rho(Y)$, if $X \leq Y$ almost surely

■ Axiom C3. Comonotic additivity:
\[
\rho(X + Y) = \rho(X) + \rho(Y), \text{ if } X \text{ and } Y \text{ are comonotonic,}
\]
where random variables $X$ and $Y$ are comonotonic if and only if
\[
(X(w_1) - X(w_2))(Y(w_1) - Y(w_2)) \geq 0
\]
holds almost surely for $w_1$ and $w_2$ in $\Omega$

■ Axiom C4. Continuity:
\[
\lim_{d \to 0} \rho((X - d)^+) = \rho(X^+), \lim_{d \to -\infty} \rho(\min(X, d)) = \rho(X), \lim_{d \to -\infty} \rho(\max(X, d)) = \rho(X),
\]
where $(X - d)^+ = \max(X - d, 0)$
Natural Risk Statistic and its Representation

- \( \tilde{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) on the random variable \( X \)

- **Axiom D1.** Positiv homogeneity and translation invariance

\[
\hat{\rho}(a \tilde{x} + 1) = a \hat{\rho}(\tilde{x} + b), \quad \forall \tilde{x} \in \mathbb{R}^n, \quad a \geq 0, \quad b \in \mathbb{R}
\]

where \( \mathbf{1} = (1, 1, \ldots, 1)^T \in \mathbb{R}^n \)

- **Axiom D2.** Monotonicity

\[
\hat{\rho}(\tilde{x}) \leq \hat{\rho}(\tilde{y}), \quad \text{if} \quad \tilde{x} \leq \tilde{y}
\]

- **Axiom D3.** Comonotonic subadditivity:

\[
\hat{\rho}(\tilde{x} + \tilde{y}) \leq \hat{\rho}(\tilde{x}) + \hat{\rho}(\tilde{y}), \quad \text{if} \quad \tilde{x} \text{ and } \tilde{y} \text{ are comonotonic,}
\]

where \( \tilde{x} \) and \( \tilde{y} \) are comonotonic if and only if \( (x_i - x_j)(y_i - y_j) \geq 0, \) for any \( i \neq j \)

- **Axiom D4.** Permutation invariance:

\[
\hat{\rho}(x_1, \ldots, x_n) = \hat{\rho}(x_{i_1}, \ldots, x_{i_n}) \text{ for any permutation } (i_1, \ldots, i_n)
\]
Theorem 1: Let $x(1), \ldots, x(n)$ be the order statistics of the observation $\tilde{x}$ with $x(n)$ being the largest. Then $\hat{\rho}$ is a natural risk statistic if and only if there exists a set of weights $\mathcal{W} = \{\tilde{w} = (w_1, \ldots, w_n)\} \subset \mathbb{R}^n$ with each $\tilde{w} \in \mathcal{W}$ satisfying $\sum_{i=n}^{n} w_i = 1$ and $w_i \geq 0, \forall 1 \leq i \leq n$, such that

$$\hat{\rho}(\tilde{x}) = \sup_{\tilde{w} \in \mathcal{W}} \left\{ \sum_{i=1}^{n} w_i x(i) \right\}, \forall \tilde{x} \in \mathbb{R}^n$$
Natural Risk Statistic and its Representation

■ **Theorem 2**: Consider a fixed scenario set $\mathcal{W}$, where each $\tilde{w} \in \mathcal{W}$ satisfies $\sum_{i=1}^{n} w_i = 1$ and $w_i \geq 0, \forall 1 \leq i \leq n$. Let $\hat{\rho}$ be a natural risk statistic induced by $\mathcal{W}$:

$$\hat{\rho}(\tilde{x}) = \sup_{\tilde{w} \in \mathcal{W}} \left\{ \sum_{i=1}^{n} w_i x(i) \right\}, \forall \tilde{x} \in \mathbb{R}^n,$$

If every weight is monotonic, i.e.,

$$w_1 \leq w_2 \leq \ldots \leq w_n, \forall \tilde{w} \in \mathcal{W}$$

then $\hat{\rho}$ satisfies subadditivity and is therefore, a coherent risk statistic.

■ $TCE_\alpha(X) = E[X | X \geq VaR_\alpha(X)]$

■ $TCE_\alpha(X) = median[X | X \geq VaR_\alpha(X)]$