Periodicity and Nonsmoothness in Optimal Control: Applications to Legged Robotics

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Many mechatronic systems move inefficiently when their natural dynamics are rejected rather than exploited. Autonomous systems provide a useful reference for understanding what motions the mechanics of a system can generate on their own. Because these systems do not rely on active control, no energy has to be supplied through the input channels; their periodic motions can therefore be regarded as energetically optimal for a broad class of actuation costs. Passive walkers illustrate this idea particularly clearly, as their natural dynamics can generate globally optimal limit cycles without active actuation. During my doctoral work, I developed analytical and computational methods for finding, continuing, and optimizing such motions in nonsmooth/hybrid dynamical systems [1, 2]. The guiding principle is that natural dynamics are not merely a modeling difficulty, but a resource for understanding how mechanics, optimal control, and design shape the available motions of a system.

The central mathematical object is a periodic optimal control problem for a hybrid dynamical system. A trajectory is decomposed into N phases, such as flight, stance, or other contact configurations. Within each phase the state evolves smoothly; between phases, event conditions and reset maps describe the discrete transitions. A compact form of the problem is

$$
\begin{aligned}
\min_{x_i,u_i,T_i} \quad & \sum_{i=1}^{N} \int_0^{T_i} \ell_i\big(x_i(t), u_i(t)\big)\,\mathrm{d}t, & & \\
\text{s.t.} \quad & \dot{x}_i(t) = f_i\big(x_i(t), u_i(t)\big), \quad t \in [0, T_i] & & \text{(continuous dynamics)}, \\
& h_i\big(T_i, x_i(T_i), u_i(T_i)\big) = 0 & & \text{(switching and operational conditions)}, \\
& x_{i+1}(0) = g_i\big(x_i(T_i)\big), \quad i = 1, \dots, N & & \text{(discrete dynamics)}, \\
& x_{N+1}(0) = x_1(0) & & \text{(periodicity)}.
\end{aligned}
$$

In this formulation, nonsmoothness enters through the coexistence of continuous and discrete dynamics. The trajectory is assembled from smooth arcs, but the complete orbit contains switches and impacts; moreover, the feasible set may change when the sequence, timing, or activity of these events changes. Periodicity closes the hybrid cycle and turns the motion into a recurrent behavior rather than an isolated maneuver.

This perspective has led to methods for generating families of periodic orbits and energy-optimal gaits. Unactuated periodic motions provide meaningful seeds for trajectory optimization because they already solve the actuation problem in the limiting case of zero input. Building on this idea, passive motions can be continued into actuated gait families, connecting morphology, natural dynamics, and optimal control [3]. The resulting methods help explain how economical legged motions emerge from the interplay of mechanical structure, contact, and actuation.

Future work will extend this view from gait libraries toward a more systematic study of structure in mechanical optimization problems. One focus is model predictive control and turnpike behavior in hybrid and legged systems. In many long-horizon optimal control problems, solutions spend most of their time close to a characteristic optimal motion or set, with deviations mainly near the beginning and end of the horizon. For legged systems, an open question is whether such turnpike structures are periodic, how they interact with impacts and contact sequences, and how they can be used to classify optimal gaits or simplify predictive controllers.

Simpler mechanical systems can make these questions especially transparent. Consider, for example, a motorized sailplane that should travel between two points with minimal energy [4]. A naive solution might maintain nearly constant altitude and velocity, whereas an optimal solution may exploit the coupling between altitude, speed, gravity, and propulsion by alternating between climbing and descending arcs. Such a periodic or near-periodic pattern can act as a turnpike-like structure: the vehicle repeatedly returns to an energetically favorable cycle, while the boundary conditions determine how it enters and leaves this behavior. Although simpler than legged locomotion, the sailplane example exposes the same core questions: how do natural dynamics organize optimal motions, how does nonsmooth or constrained behavior shape the solution set, and how can these structures guide efficient mechatronic design and control?

References

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