# Properties of Codes in Sum-Rank Metric and Codes over Rings

Cornelia Ott

Ulm University

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# Outline

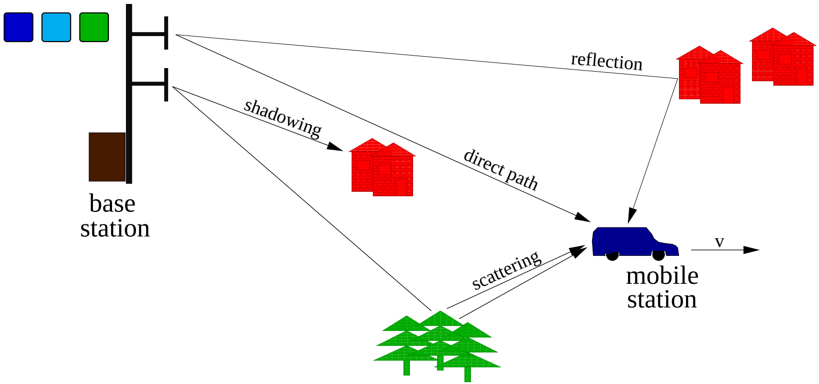
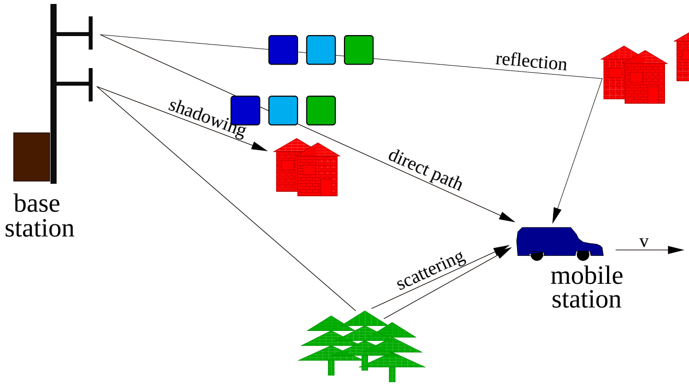



Figure: Multipath in mobile communications [BB10]

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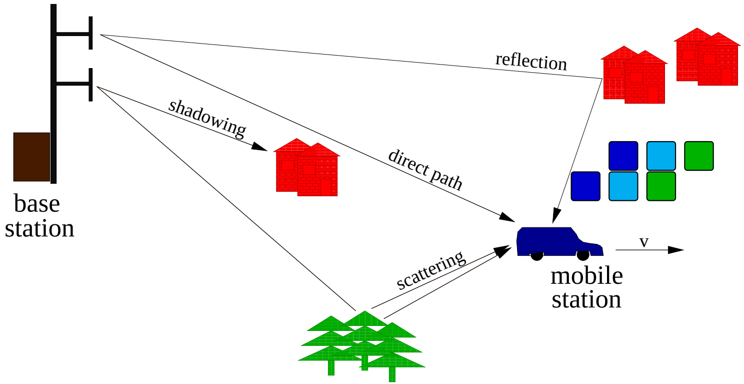
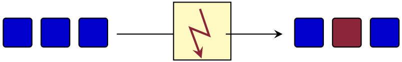



Figure: Multipath in mobile communications [BB10]

Problem: Transmission of $k$ symbols over an errornous channel



Channel Flannel

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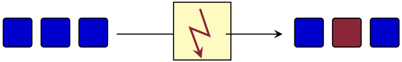

Channel Flannel not detectable

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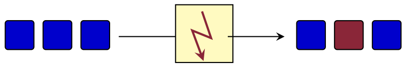

Channel
Book

Flannel
Boor

not detectable

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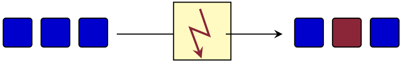

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Book?
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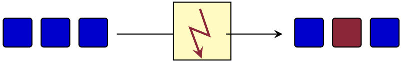

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| Channel | Flannel | not detectable | |
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# Preliminaries of Error Correcting Codes



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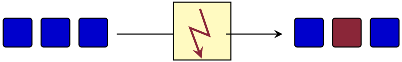

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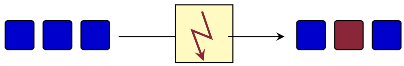

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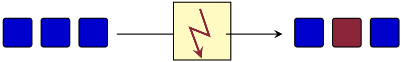

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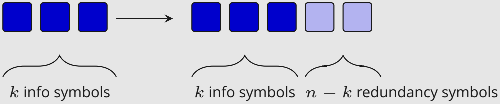
## Idea of Channel Coding

$k$ information symbols, $n-k$ redundancy symbols to get codewords of length $n$



## Blockcode $\mathcal{C}(n, M, d)$

Let $\Sigma$ be a set, $n, d \in \mathbb{Z}_+$ with $d \leq n$ and $\mathrm{d} : \Sigma^n \times \Sigma^n \to \mathbb{Z}_+$ a metric.
We define a blockcode $\mathcal{C}$ of length $n$, cardinality $M$ and minimum distance $d$ as

$$\mathcal{C} := \left\{ \boldsymbol{c}^{(1)}, \dots, \boldsymbol{c}^{(M)} \in \Sigma^n \mid \min_{\substack{\boldsymbol{c}^{(i)} \neq \boldsymbol{c}^{(j)} \\ i,j \in \{1,\dots,M\}}} \{\mathrm{d}(\boldsymbol{c}^{(i)}, \boldsymbol{c}^{(j)})\} = d \right\}.$$

Its elements $\boldsymbol{c} = (c_0, \dots, c_n) \in \mathcal{C}$ are called *codewords* and $\Sigma$ is called the *code alphabet*.

# Overview and Motivation

Commonly used setting:

- employ the Hamming metric $\mathrm{d}_H(\boldsymbol{c}, \boldsymbol{c}') = \sum_{i=0}^{n} \mathbb{1}_{c_i \neq c'_i}$ to measure the distance between two codewords $\boldsymbol{c}, \boldsymbol{c}'$.
- choose the code alphabet $\Sigma = \mathbb{F}_q$
  - define $\mathcal{C}$ as a $k$-dimensional subspace of $\mathbb{F}_q^n$

$\Rightarrow$ linear Blockcode $\mathcal{C}(n, k, d)$ over $\mathbb{F}_q$ in the Hamming metric with $k = \dim(\mathcal{C})$ and $|\mathcal{C}| = q^k$.

Settings in this work:

- use the socalled *sum-rank metric*
- choose a finite ring $\mathcal{R}$ as code alphabet



**Aim of Channel Coding**

large minimum distance $\Rightarrow$ good error correcting properties

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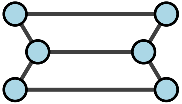

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# The Sum-Rank Metric

- $\mathbb{F}_{q^m}$ extension field of $\mathbb{F}_q$
- linear code $\mathcal{C} \subset \mathbb{F}_{q^m}^n$ subspace of dimension $k$
- split codelength $n = \eta \cdot \ell$ into $\ell$ blocks, each of size $\eta$

$$\boldsymbol{c} = [\ \underbrace{\boldsymbol{c}_1}_{\in \mathbb{F}_{q^m}^{\eta}} \mid \boldsymbol{c}_2 \mid \dots \mid \boldsymbol{c}_\ell\ ] \in \mathbb{F}_{q^m}^n$$

$$\boldsymbol{C} = [\ \underbrace{\boldsymbol{C}_1}_{\in \mathbb{F}_q^{m \times \eta}} \mid \boldsymbol{C}_2 \mid \dots \mid \boldsymbol{C}_\ell\ ] \in \mathbb{F}_q^{m \times n}$$

$\ell$-sum-rank weight/distance:

$$\mathrm{wt}_{SR,\ell}(\boldsymbol{c}) := \sum_{i=1}^{\ell} \mathrm{rk}_{\mathbb{F}_q}(\boldsymbol{C}_i) \leq \ell \cdot \underbrace{\mu}_{:=\min\{m,\eta\}}$$

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For $x \in \mathbb{F}_{q^m}$ it holds that $\mathrm{wt}_{SR,\ell}(x) \leq \mathrm{wt}_{SR,n}(x)$.

Proof: $\boldsymbol{x} = [\boldsymbol{x}_1 | \ldots | \boldsymbol{x}_\ell] \in \mathbb{F}_{q^m}^n$ with $\mathrm{wt}_{SR,n}(\boldsymbol{x}) = n - t = \eta - t_1 + \ldots + \eta - t_\ell$ where $\sum_{i=1}^{\ell} t_i = t$ and each $\boldsymbol{x}_i$ has $t_i$ zero entries.
$\Rightarrow \mathrm{wt}_{SR,\ell}(\boldsymbol{x}) = \sum_{i=1}^{\ell} \mathrm{rk}_q(\boldsymbol{x}_i) \leq \sum_{i=1}^{\ell} \min\{m, \eta - t_i\} \leq \sum_{i=1}^{\ell} (\eta - t_i) = n - t = \mathrm{wt}_{SR,n}(\boldsymbol{x})$.

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For $x \in \mathbb{F}_{q^m}$ it holds that $\mathrm{wt}_{SR,1}(x) \leq \mathrm{wt}_{SR,\ell}(x)$.

Proof: Assume w.l.o.g. $n \leq m$ and let $\mathrm{wt}_{SR,\ell}(\boldsymbol{x}) = t = t_1 + \ldots + t_\ell$ (i.e., each $\boldsymbol{x}_i$ has $t_i$ $\mathbb{F}_q$-linearly independent columns for $i \in \{1, \ldots \ell\}$)
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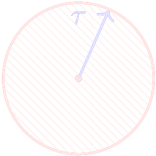
# Spheres and Balls in Sum-Rank Metric

Let $\tau \in \mathbb{Z}_{\geq 0}$ with $0 \leq \tau \leq \ell \cdot \mu$ and $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$.
A sphere in sum-rank metric with radius $\tau$ and center $\boldsymbol{x}$ is defined as

$$\mathcal{S}_\ell(\boldsymbol{x}, \tau) := \{\boldsymbol{y} \in \mathbb{F}_{q^m}^n \mid \mathrm{d}_{SR,\ell}(\boldsymbol{x}, \boldsymbol{y}) = \tau\}.$$

Analogously, we define a ball of sum-rank radius $\tau$ with center $x$ by

$$\mathcal{B}_\ell(\boldsymbol{x}, \tau) := \bigcup_{i=0}^{\tau} \mathcal{S}_\ell(\boldsymbol{x}, i).$$




We also define the following cardinalities:

$$\mathrm{Vol}_{\mathcal{S}_\ell}(\tau) := |\{\boldsymbol{y} \in \mathbb{F}_{q^m}^n \mid \mathrm{wt}_{SR,\ell}(\boldsymbol{y}) = \tau\}|,$$

$$\mathrm{Vol}_{\mathcal{B}_\ell}(\tau) := \sum_{i=0}^{\tau} \mathrm{Vol}_{\mathcal{S}_\ell}(i).$$

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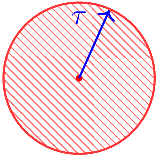


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$$\mathrm{Vol}_{\mathcal{S}_\ell}(\tau) := |\{\boldsymbol{y} \in \mathbb{F}_{q^m}^n \mid \mathrm{wt}_{SR,\ell}(\boldsymbol{y}) = \tau\}|,$$

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Let $\tau \in \mathbb{Z}_{\geq 0}$ with $0 \leq \tau \leq \ell \cdot \mu$ and $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$.
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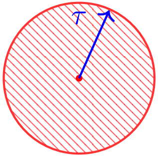


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## Volume of Spheres and Balls in Sum-Rank Metric

$$\mathrm{Vol}_{\mathcal{S}_\ell}(t) = \sum_{t=[t_1,\ldots,t_\ell]\in \underbrace{\mathcal{T}_{t,\ell,\mu}}_{\substack{\text{set of partitions of}\\ t=t_1+\ldots+t_\ell \text{ into terms } t_i\leq\mu}}} \prod_{i=1}^{\ell} \underbrace{\mathrm{NM}_q(\eta, m, t_i)}_{\substack{\text{number of } m\times\eta \text{ matrices over } \mathbb{F}_q\\ \text{for a given rank } t_i\leq\min\{m,\eta\}}}$$

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## Lower and Upper Bound on the Volume of a Single Sum-Rank Ball

$$q^{(m+\eta-\frac{t}{\ell})t-\frac{\ell}{4}} \cdot \gamma_q^{-\ell} \overset{\text{[OPB21, Lemma 2]}}{\leq} \mathrm{Vol}_{\mathcal{B}_\ell}(\rho) \overset{\text{if } \rho>1}{\leq} \rho \mathrm{Vol}_{\mathcal{S}_\ell}(\rho) \overset{\text{[PRR20, Theorem 5]}}{\leq} \rho \binom{\ell+\rho-1}{\ell-1} \gamma_q^{\ell} q^{\rho(m+\eta-\frac{\rho}{\ell})}.$$

Applications of these bounds:

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The volume of the intersection of two balls is $|\mathcal{B}_\ell(\boldsymbol{x}_1, \tau_1) \cap \mathcal{B}_\ell(\boldsymbol{x}_2, \tau_2)|$ independent of their centers, only dependent on their radii $\tau_1$, $\tau_2$ and the distance $\delta := \mathrm{d}_{SR,\ell}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ between their respective centers given as follows:

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where $\boldsymbol{d} \in \mathbb{F}_{q^m}^n$ arbitrary but fix with $\mathrm{wt}_{SR,\ell}(\boldsymbol{d}) = \delta$.
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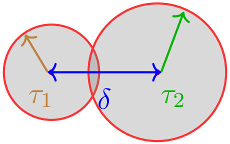

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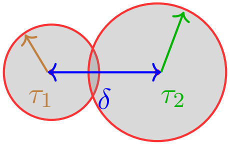


**New Result [OLWZ23]**

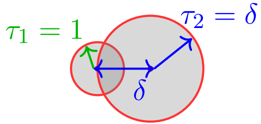
Let $u$, $s$, $t$ be positive integers such that $u + s \geq t$. The number of vectors $\boldsymbol{v} \in \mathbb{F}_{q^m}^n$ lying in the intersection of two balls with sum-rank radii $u$ and $s$ and sum-rank distance $t$ between their centers is

$$\mathrm{Vol}_{\mathcal{I}_\ell}(u,s,t) = \sum_{\substack{\boldsymbol{u}=[u_1,\ldots,u_\ell] \\ \in \mathcal{T}_{u,\ell,\mu}}} \sum_{\substack{\boldsymbol{s}=[s_1,\ldots,s_\ell] \\ \in \mathcal{T}_{s,\ell,\mu}}} \sum_{\substack{\boldsymbol{t}=[t_1,\ldots,t_\ell] \\ \in \mathcal{T}_{t,\ell,\mu}}} \prod_{i=1}^{\ell} \underbrace{\mathcal{I}(u_i, s_i, t_i, \eta, m)}_{\text{volume of the intersection of two \textbf{rank metric balls} of radii } u_i, s_i \text{ and distance } t_i \text{ between their centers over } \mathbb{F}_{q^m}^{\eta}}.$$

**New Result [OLWZ23]**

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_{q^m}^n$ such that $\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \boldsymbol{y}) = \delta$. Then

$$\begin{aligned}\mathrm{Vol}_{\mathcal{I}_\ell}(\delta, 1, \delta) =& |\mathcal{B}_\ell(\boldsymbol{x}, \delta) \cap \mathcal{B}_\ell(\boldsymbol{y}, 1)| \\ =& 1 + \frac{(q^m-1)(q^n-1)}{q-1} \\ & - \sum_{\substack{\boldsymbol{\delta}=[\delta_1,\ldots,\delta_\ell] \\ \in \tau_{\delta,\ell,\mu}}} \sum_{i=1}^{\ell} \frac{(q^\eta - q^{\delta_i}) \cdot (q^m - q^{\delta_i})}{q-1}.\end{aligned}$$



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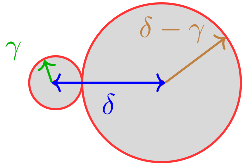
$$\begin{aligned}\mathrm{Vol}_{\mathcal{I}_\ell}(\gamma, \delta-\gamma, \delta) &= |\mathcal{B}_\ell(\boldsymbol{x}, \gamma) \cap \mathcal{B}_\ell(\boldsymbol{y}, \delta-\gamma)| \\ &= \sum_{\substack{\boldsymbol{\delta}=[\delta_1,\dots,\delta_\ell] \\ \in \mathcal{T}_{\delta,\ell,\mu}}} \sum_{\substack{\boldsymbol{\gamma}=[\gamma_1,\dots,\gamma_\ell] \\ \in \mathcal{T}_{\gamma,\ell,\boldsymbol{\delta}}}} \sum_{i=1}^{\ell} q^{\gamma_i \cdot (\delta_i - \gamma_i)} \cdot \begin{bmatrix} \delta_i \\ \gamma_i \end{bmatrix}_q\end{aligned}$$

for $0 \leq \gamma \leq \delta$.



Let $\mathcal{C}$ be a linear $[n, k, d]$ sum-rank metric code. Then it holds

$$d \leq \mu\ell - \frac{\mu}{\eta}k + 1$$

**MSRD codes**

$\mathcal{C}$ is called *maximum sum-rank-distance* (MSRD), if the Singleton like Bound is fulfilled with equality.

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# Bounds on codes in sum-rank metric Sphere-Packing Bound

## Sphere-Packing Bound [BGLR21, Theorem III.6]

For a linear $[n, k, d]$ sum-rank metric code $\mathcal{C}$, it holds that

$$q^{mk} \cdot \mathrm{Vol}_{\mathcal{B}_\ell}\left(\left\lfloor \frac{d-1}{2} \right\rfloor\right) \leq q^{mn}.$$

## simplified Sphere-Packing Bound - New Result [OPB21]

For a linear $[n, k, d]$ sum-rank metric code $\mathcal{C}$, the parameters fulfill

$$q^{mk} \cdot q^{(m+\eta-\frac{1}{\ell}\lfloor\frac{d-1}{2}\rfloor)\lfloor\frac{d-1}{2}\rfloor-\frac{\ell}{4}} \cdot \gamma_q^{-\ell} \leq q^{mn}.$$

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### Asymptotic Sphere-Packing Bound - New Result [OPB21]

Let $\mathcal{C}$ be a linear $[n,k,d]$ sum-rank metric code and $\delta := \frac{d}{n}$ the relative minimum distance. Then the code rate $\mathcal{R} = \frac{k}{n}$ is upper bounded by

$$\mathcal{R} < \delta^2 \frac{\eta}{4m} - \delta\left(\frac{1}{2} + \frac{\eta}{m}\left(\frac{1}{2} + \frac{1}{n}\right)\right) + \frac{1}{n}\left(1 + \frac{\eta}{m} + \frac{\eta}{nm}\right) + \frac{1}{\eta m}\left(\frac{1}{4} + \log_q(\gamma_q)\right) + 1.$$

Let $\xi > 0$ be fixed.

- For $m = \eta\xi \to \infty$ we get $\mathcal{R} \sim \delta^2 \frac{1}{4\xi} - \frac{\delta}{2}\left(1 + \frac{1}{\xi}\right) + 1.$
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## Gilbert–Varshamov Bound [BGLR21, Theorem III.11]]

Let $\mathbb{F}_{q^m}$ be a finite field, $\ell, n, k, d \leq \mu\ell$ be positive integers that satisfy

$$q^{m(k-1)} \cdot \mathrm{Vol}_{\mathcal{B}_\ell}(d-1) < q^{mn}.$$

Then, there is a linear code of length $n$, dimension $k$, and minimum $\ell$-sum-rank distance at least $d$.

## Simplified Gilbert–Varshamov Bound - New Result [OPB21]

Let $\mathbb{F}_{q^m}$ be a finite field, $\ell, n, k, d$ be positive integers with $2 < d \leq \mu\ell$ that satisfy

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## Asymptotic Gilbert–Varshamov Bound - New Result [OPB21]

For a finite field $\mathbb{F}_{q^m}$ and positive integers $\ell, n, \mathcal{R}n, d$ with $\delta := \frac{d}{n}$ and $2 < d \leq \mu\ell$ satisfying

$$\mathcal{R} \leq \delta^2 \frac{\eta}{m} - \delta\left(1 + \frac{\eta}{m} + \frac{2\eta}{nm}\right) + 1 + \frac{1}{n} + \frac{\eta}{nm} + \frac{\eta}{n^2 m} - \frac{\sum_{i=1}^{\delta n - 1} \log_q\left(1 + \frac{\ell-1}{i}\right) + \log_q(\delta n - 1)}{mn} - \frac{\log_q(\gamma_q)}{\eta m}$$

there exists a linear $\ell$-sum-rank metric code of rate $\mathcal{R}$ and relative minimum sum-rank distance at least $\delta$. Let $\xi$ be a constant. For $m = \eta\xi \to \infty$ and $m \in \omega(\log_q(\ell))$, we have

$$\mathcal{R} \sim \delta^2 \frac{1}{\xi} - \delta\left(1 + \frac{1}{\xi}\right) + 1.$$

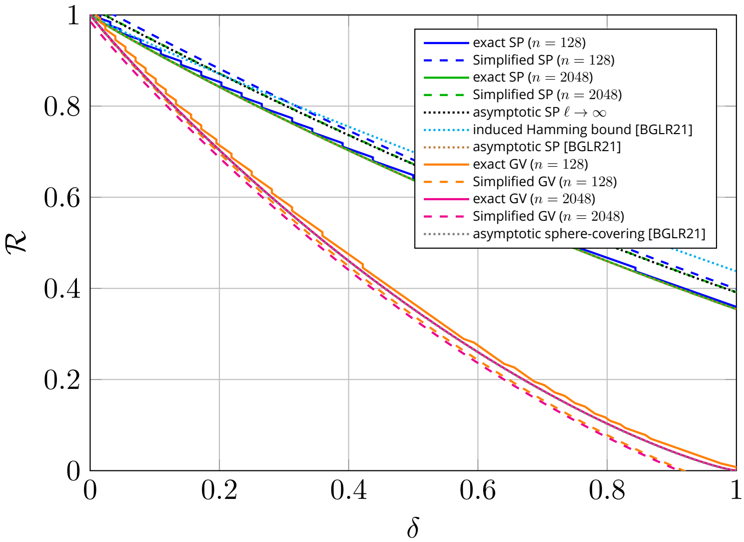



Figure: Comparison of different bounds for fixed value $q = 2$ $\eta = 8$ $m = 16$ for different values of $n$ ($\ell = \frac{n}{\eta}$)

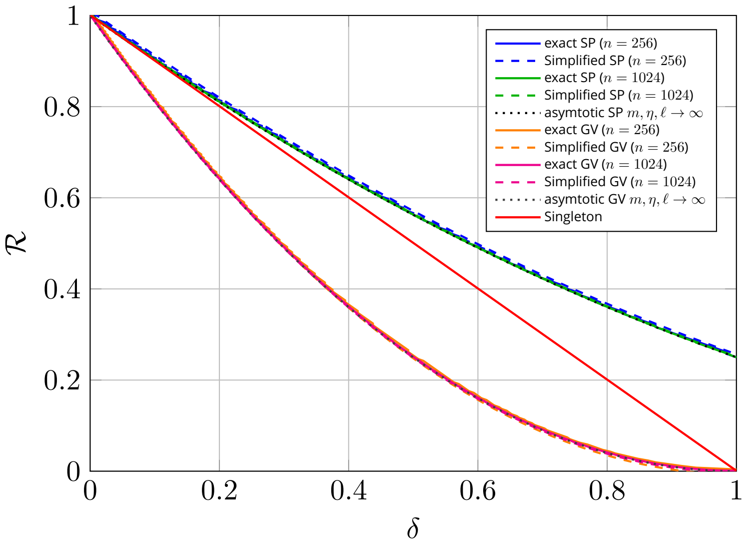
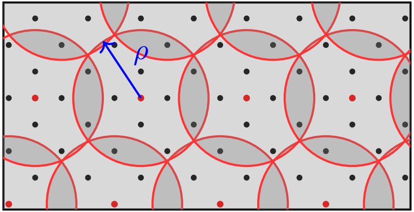



Figure: Comparison of different bounds for fixed value of $q = 16$ and different values of $n$ with $\eta = \ell = m$

## Covering Problem

Find the minimum number of sum-rank balls $\mathcal{B}_\ell(\boldsymbol{x}, \rho)$ of radius $\rho$ (with $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$) that cover the space $\mathbb{F}_{q^m}^n$ entirely. We denote this cardinality by $\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho)$.

$\mathbb{F}_{q^m}^n$




- $\in \mathbb{F}_{q^m}^n$
- $\in \mathcal{C}$

# Covering Problem

## Definition (Covering Radius)

Let $\mathcal{C}$ be a linear $[n,k,d]$ sum-rank metric code over $\mathbb{F}_{q^m}$. The covering radius of $\mathcal{C}$ is the smallest integer $\rho_{SR,\ell}$ such that every vector $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$ has at most sum-rank distance $\rho_{SR,\ell}$ to some codeword $\boldsymbol{c} \in \mathcal{C}$ i.e.,
$\rho_{SR,\ell} = \max_{\boldsymbol{x} \in \mathbb{F}_{q^m}^n} \{\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C})\}$.




## Reformulation of the Covering Problem

Find the minimum cardinality of a code $\mathcal{C} \subset \mathbb{F}_{q^m}^n$ with covering radius $\rho$.

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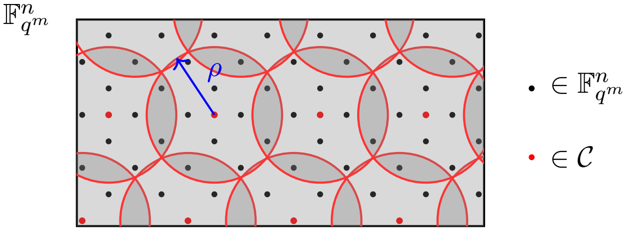


## Reformulation of the Covering Problem

Find the minimum cardinality of a code $\mathcal{C} \subset \mathbb{F}_{q^m}^n$ with covering radius $\rho$.

Extreme cases for the covering radius:

(i) $\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, 0) = q^{mn}$
From $\rho_{SR,\ell} = \max_{x \in \mathbb{F}_{q^m}^n} \{\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C})\} = 0$ it follows that $\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C}) = 0, \forall \boldsymbol{x} \in \mathbb{F}_{q^m}^n$ and therefore $\boldsymbol{x} \in \mathcal{C}$, i.e., $\mathcal{C} = \mathbb{F}_{q^m}^n$.

(ii) $\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \mu\ell) = 1$
Consider $\rho_{SR,\ell} = \max_{x \in \mathbb{F}_{q^m}^n} \{\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C})\} = \mu \cdot \ell$ which means that there exists an $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$ such that $\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C}) = \mu \cdot \ell$. This is already fulfilled by choosing $\mathcal{C} = \{(0, \ldots, 0)\}$.

# Covering Properties in Different Metrics

**New Result [[OLWZ22, Lemma 1]]**

Let $\mathcal{C} \subset \mathbb{F}_{q^m}^n$, then it holds for its corresponding covering radii $\rho_{SR,1}$, $\rho_{SR,\ell}$ and $\rho_{SR,n}$ in the rank, the sum-rank and the Hamming metric that

$$\rho_{SR,1} \leq \rho_{SR,\ell} \leq \rho_{SR,n}.$$

**New Result [OLWZ22, Theorem 2]**

For $0 < \rho < \mu \cdot \ell$, it holds

$$\mathcal{K}_{SR,1}(\mathbb{F}_{q^m}^n, \rho) \leq \mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho) \leq \mathcal{K}_{SR,n}(\mathbb{F}_{q^m}^n, \rho).$$

# Numerical Comparison of Different Covering Bounds

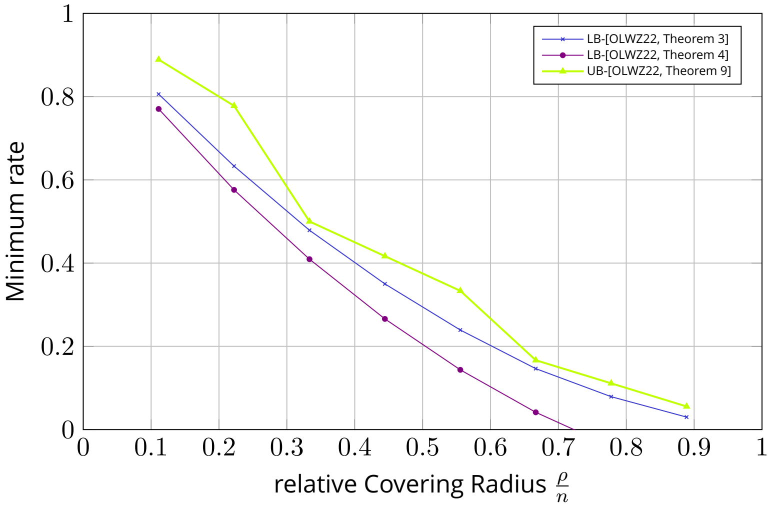


Figure: Comparison of bounds on $\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho)$ for parameters $q = 4, m = 4, \eta = 3, \ell = 3, n = \eta\ell = 9$.

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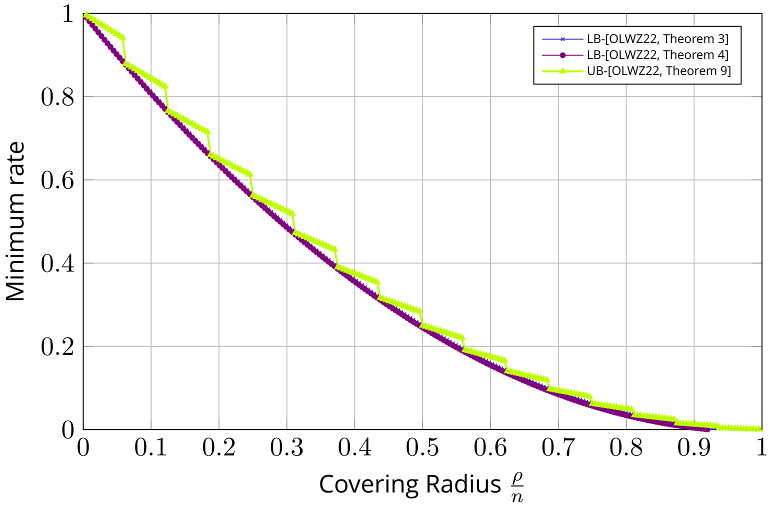


Figure: Comparison of bounds on $\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho)$ for parameters $q = 16, m = 16, \eta = 16, \ell = 14, n = \eta\ell = 224.$

# Conclusion

- the sum-rank metric $\rightarrow \mathrm{wt}_{SR,1}(\boldsymbol{x}) \leq \mathrm{wt}_{SR,\ell}(\boldsymbol{x}) \leq \mathrm{wt}_{SR,n}(\boldsymbol{x})$
- spheres and balls in sum-rank metric
  - exact volume of a ball in sum-rank metric $\rightarrow$ lower and upper bound
  - exact Volume of the intersection of two balls in sum-rank metric
  - two special cases of the intersection of two balls in sum-rank metric

NEW

- bounds on codes in sum-rank metric
  - Singleton like bound
  - Sphere-Packing bound $\rightarrow$ simplified and asymptotic version
  - Gilbert-Varshamov like bound $\rightarrow$ simplified and asymptotic version
  - numerical comparisons of the bounds
- covering problem in the sum-rank metric
  - two extreme cases
  - $\rho_{SR,1} \leq \rho_{SR,\ell} \leq \rho_{SR,n}$
  - $\mathcal{K}_{SR,1}(\mathbb{F}_{q^m}^n, \rho) \leq \mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho) \leq \mathcal{K}_{SR,n}(\mathbb{F}_{q^m}^n, \rho)$
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Let $A \neq \{0\}$ be a unitary ring (i.e., there exists $1 \in A \setminus \{0\}$, such that $1 \cdot a = a \cdot 1 = a, \forall a \in A$) with a unitary endomorphism $\theta$ (i.e., $\theta(1) = 1$).
For any $a, b \in A$, an endomorphism $\theta$ fulfills

- $\theta(a + b) = \theta(a) + \theta(b)$,
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A $\theta$-derivation is a map $\delta : A \to A$ such that, for all $a, b \in A$

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$$R = A[X; \theta, \delta] := \left\{ \sum_{i=0}^{n} a_i X^i | a_i \in A, n \in \mathbb{N} \right\} .$$

The addition in $R$ is the usual polynomial addition and the multiplication is defined using the rule $Xa = \theta(a)X + \delta(a)$ which is extended using associativity and distributivity.

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## Definion: Principal Module Code

Let $\theta$ be an endomorphism of the finite ring $A$, $\delta$ be a $\theta$-derivation of $A$ and $R = A[X;\theta,\delta]$ be the skew polynomial ring. Let $f \in R$ be a **monic** skew polynomial with a right divisor $g \in R$.
A *principal module* $(\theta, \delta)$*-code* $\mathcal{C}$ is defined as

$$(g)_{n,\theta,\delta} := \{\boldsymbol{c} = (c_0, c_1, \dots, c_{n-1}) \mid \forall c_0 + c_1 X + \dots + c_{n-1} X^{n-1} \in Rg/Rf\}.$$

**Setting:** Choose $g$ of degree $n-k$ then we have that $Rg/Rf$ is a free left $A$-submodule of $R/Rf$ of dimension $k = \deg(f) - \deg(g)$ (a linear code of length $n$ and dimension $k$ over $A$).
**Aim:** Find selforthogonal codes

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