Lecture Notes in Economics and Mathematical Systems 679

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Income Modeling and Balancing

A Rigorous Treatment of Distribution Patterns

All in all, progressive taxation leads from a given Lorenz curve to one which is smaller in Lorenz order and identical transfers of all or a fraction of the whole tax revenues leads to an even smaller Lorenz curve. When all tax revenues are transferred, taxation is nothing but redistribution. This agrees with Pigou–Dalton transfers from high to lower incomes. When the taxation rate is equal for all incomes, each income is multiplied by the same factor. This leaves their Lorenz curve unchanged.

5.6 Further Order Relation

An equivalence to the Lorenz order without equality of expectations can be obtained from the harmonic new better than used in expectation order from reliability analysis.

Definition 5.2 Let $0 < EX, EY < \infty$. Then the *harmonic new better than used in expectation* HNBUE order is defined by the condition $\frac{1}{EX}\int_x^\infty P(X > t)\,dt \leq \frac{1}{EY}\int_x^\infty P(Y > t)\,dt$ for all $x > 0$. Notation $X \leq_{HNBUE} Y$.

Theorem 5.4 (Equivalence of Lorenz Order and HNBUE Order Even for Unequal Means)
$X \leq_{HNBUE} Y$ *if and only if* $X \leq_{cx} Y$.

Proof See Borzadaran and Behdani (2008, Theorem 5). ◇

References

Borzadaran GRM, Behdani Z (2008) Notes Due to Lorenz Curve and Lorenz Ordering in View of Weighted Distributions, 374–386, Proceedings of the 9th Iranian Statistical Conference, Department of Statistics, University of Isfahan
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Chapter 6
Societal Utility and the Atkinson Theorem

Abstract The Atkinson theorem is formulated in a rigorous way. Societal utility of an income distribution is to be maximized for an additive utility function such that the mean income is preserved. Then, for finite distributions with rational probabilities, (1) majorization, (2) finite sequences of Pigou-Dalton transfers, (3) the Lorenz order and (4) the convex stochastic order are equivalent. When distributions are no longer finite, majorization and finite sequences of Pigou-Dalton transfers refer to approximate distributions. With these concepts, the Atkinson theorem can be shown to also hold for general distributions.

Inverse formulations of the Atkinson theorem are given additionally. Switching between convex and concave utility functions can hence be thought of as balancing the income distribution of a society, depending on the perceived social state of a society, sometimes too much inequality, sometimes too little.

This chapter repeats some material of the two foregoing chapters in order to elaborate on the so-called Atkinson theorem, see Atkinson (1970). It seems that this important theorem, though stated mathematically, is primarily an issue of interest and passed on among economists. It is a "rich" theorem in context, offers some surprising insights and allows certain interpretations with importance for a better understanding of the social and income structure of a society. Its intention is, among other aspects, to relate Pigou-Dalton transfers, e.g. transfers from rich to poor, to partial orders so that any order relation between two particular income distributions becomes intuitive from the perspective of simple redistribution steps.

Statements in the literature often lack a precise qualification concerning types of distributions covered. Also, not all four dimensions of the theorem, as formulated in full generality below, are always looked at, nor are the generalizations from the discrete to the general case treated in most publications. On top, a "natural" inversion step is most often not dealt with.

The domain of the theorem in its original formulation includes all natural distributions which are the finite discrete distributions whose probabilities are multiples of some common value $1/n$, see Chap. 4. But all finite discrete distributions with rational probabilities or even with arbitrary probabilities might have been targeted as well. Distributions with Lebesgue densities are often included.

As pointed out by Atkinson in a personal communication, the Atkinson theorem was independently discovered by Kolm who phrases it via the "principle of diminishing transfers", see Kolm (1976a) and Kolm (1976b, p. 88). In addition, it seems that the theorem or a very similar result may also have been discovered by Stiglitz and Rothschild who use the terminology of "risk", see Rothschild and Stiglitz (1970, 1973). Several pertinent overlapping publications appeared in the 1970s—some with informal cross-references. The development of inequality measures was the background of all these references.

6.1 Pigou-Dalton Transfers: Revisited

What is the theorem of Atkinson about? The issue is distributional balance for societies: what can be said about acceptable or desirable levels of balance of societies in the sense of corresponding income distributions from a very abstract point of view? We accept the concept of an additive utility function u to capture the societal utility of an income distribution in the form of a vector $x = (x_1, \ldots, x_n)$ with associated random variable X as $u(x) = 1/n \cdot u(x_1) + \ldots + 1/n \cdot u(x_n)$. Then a most natural requirement is that vector x is preferred to $y = (y_1, \ldots, y_n)$ with associated random variable Y, if $u(x) \geq u(y)$, i.e. $E\,u(X) \geq E\,u(Y)$ for all monotonically increasing functions u (stochastic order).

However, if we think about optimal distribution issues in a society and about justifiable income transfers, the increasing order is of no help. The reason is that total incomes of X and Y being equal, i.e. $EX = EY$, and X and Y being stochastically ordered (or being in Lorenz order) imply the identity of X and Y.

Can more be said in general terms i.e. involving whole classes of utility functions? Yes—depending on the state in which societies feel they are. Actually, much about politics is about income adjustments. And sometimes the issue is more balance, sometimes the issue is less taxes and more incentives for more entrepreneurship. Income distribution in societies is about balance, and there may be too much or too little balance.

Looking to Chap. 5, the Pigou-Dalton transfers were discussed. Income is moved from a higher income position to a lower one without changing the sum of all incomes and without changing income ranks. The motivation behind these transfers seems to have been a welfare oriented analysis of inequality measures, see the historic perspective given in Atkinson and Brandolini (2014). Pigou-Dalton transfers are sometimes called Robin-Hood transfers ("from rich to poor"). Formally, one can also consider inverse Pigou-Dalton transfers which move income into the opposite direction ("from the wretched to the peer").

Let x be a vector with decreasingly sorted coordinates. Consider a new vector x' which also has decreasingly sorted coordinates and which differs from x in only two coordinates such that $x'_i = x_i + \varepsilon$ and $x'_j = x_j - \varepsilon$ for $\varepsilon > 0$ and $i < j$. Then x' is understood as resulting from x by an inverse Pigou-Dalton transfer. Obviously,

In order to make relations coherent, inverse majorization, also, is formally introduced. When $x \leq_m y$ then the inverse majorization simply is $y \leq_{im} x$. A sequence of inverse Pigou-Dalton steps leads from a vector to one which is smaller in inverse majorization. This is similar to a sequence of ordinary Pigou-Dalton steps leading from a vector to one which is smaller in ordinary majorization.

A version of the Atkinson theorem can be stated as follows.

Theorem 6.1 ("Atkinson Theorem for Natural Distributions") *Let X and Y be finite distributions with equal expectations, rational probabilities and with respective natural vectors x and y. Then the following conditions are equivalent*

1. $x \leq_m y$.
2. *x results from y by a finite sequence of Pigou-Dalton transfers.*
3. $X \leq_L Y$.
4. $X \leq_{cx} Y$ ($\Longleftrightarrow Y \leq_{cv} X$).

Usually, the last condition of Theorem 6.1 is stated in terms of the concave stochastic order. If x results from y be a sequence of Pigou-Dalton steps, then it is more balanced and the associated Lorenz curve lies pointwise everywhere above the original Lorenz curve. This means $X \leq_L Y$ or $L_x(u) \geq L_y(u)$ for all $u \in [0, 1]$. This corresponds to $X \geq_{cv} Y$ which means that for a concave utility function, X produces greater expectations than Y so that the expected utility of X is better than that of Y.

Proof of Theorem 6.1 The equivalence of conditions 1 and 2 follows from Lemma 5.1, equivalence of conditions 1 and 4 follows from Theorem 4.1 and the equivalence of conditions 3 and 4 follows from Theorem 4.3. ◇

Inverse Pigou-Dalton transfers and inverse majorization allow a corresponding version of the Atkinson theorem.

Theorem 6.2 ("Atkinson Theorem for Natural Distributions and Inverse Pigou-Dalton Transfers") *Let X and Y be finite distributions with equal expectations, rational probabilities and with respective natural vectors x and y. Then the following conditions are equivalent*

1. $y \leq_{im} x$.
2. *y results from x by a finite sequence of inverse Pigou-Dalton transfers.*
3. $Y \geq_L X$.
4. $Y \leq_{cv} X$ ($\Longleftrightarrow X \leq_{cx} Y$).

These equivalences of Pigou-Dalton transfers and inverse Pigou-Dalton transfers will now be extended to more complicated distributions. The approach is "dual" to introducing probabilistic versions of Pigou-Dalton transfer as given in Sect. 5.3. There, transfer operations became slightly more complicated to allow relatively simple treatment of more general distributions. Now, Pigou-Dalton transfers are kept to their original simplicity but applications to more general distributions become more complicated.

6.2 Pigou-Dalton Transfers and Distribution Approximations

Applicability of Pigou-Daltorn transfers and inverse Pigou-Dalton transfers is obtained for general distributions as an extension from natural distributions by invoking them to converging replacements in accordance to Theorem 5.2. In particular, converging replacements have rational probabilities only and apply to distributions with equal expectations.

Definition 6.1 X is defined to be smaller in the *Pigou-Dalton relation* than Y if X and Y have respective sequences $(X_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$ of converging replacements such that P^{X_n} results from P^{Y_n} by a finite sequence of Pigou-Dalton transfers. Notation $X \leq_{PD} Y$.

Definition 6.2 Y is defined to be smaller in the *inverse Pigou-Dalton relation* than X if X and Y have respective sequences $(X_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$ of converging replacements such that P^{Y_n} results from P^{X_n} by a finite sequence of inverse Pigou-Dalton transfers. Notation $Y \leq_{iPD} X$.

Both relations are inverses of each other which means that $X \leq_{PD} Y$ is equivalent to $Y \leq_{iPD} X$. Moreover, in the same way as majorization for vectors corresponds to the convex stochastic order, the Pigou-Dalton relation corresponds to that order.

Lemma 6.1 *Let X and Y be non-negative random variables with $EX = EY$. Then*

1. *$X \leq_{PD} Y$ if and only if $X \leq_{cx} Y$.*
2. *$Y \leq_{iPD} X$ if and only if $Y \leq_{cv} X$.*

Proof

Part 1. "$\Longrightarrow$". Let $X \leq_{PD} Y$. Then, by definition, there exist approximating sequences $X_n \to X$ and $Y_n \to Y$ such that $X_n \leq_{cx} Y_n$ and $EX = EX_n = EY_n = EY$ for all n. Since all random variables are non-negative, the convex stochastic order carries over to the limit according to Müller and Stoyan (2002, Theorem 1.5.9). This means $X \leq_{cx} Y$.

Part 1. "$\Longleftarrow$". $X \leq_{cx} Y$ entails a sequence of approximating replacements with finite support, rational probabilities and $X_n \leq_{cx} Y_n$ according to Theorem 5.2. The natural vector for X_n then results from a finite sequence of Pigou-Dalton transfers from the natural vector of Y_n according to Lemma 5.1. Thus, $X \leq_{PD} Y$ by definition.

Part 2. Analogous. ◇

As a suitable, natural vector of X_n from a converging replacement results from finite many Pigou-Dalton transfers from a suitable, natural vector of Y_n, the former is majorized by the latter. This allows to generalize majorization along converging replacements.

Definition 6.3 X is defined to be *majorized in the general sense* by Y if X and Y have respective sequences $(X_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$ of converging replacements such that common natural vectors are majorized as $x(X_n, Y_n) \leq y(X_n, Y_n)$. Notation $X \leq_{gm} Y$.

All definitions now admit a version of the Atkinson theorem for general distributions that is formally alike the Atkinson theorem for natural distributions.

Theorem 6.3 ("Atkinson Theorem for General Distributions") *Let X and Y be distributions with equal expectations. Then the following conditions are equivalent*

1. *$X \leq_{gm} Y$.*
2. *$X \leq_{PD} Y$.*
3. *$X \leq_L Y$.*
4. *$X \leq_{cx} Y$ ($\Longleftrightarrow Y \leq_{cv} X$).*

Proof Conditions 1 and 2 being equivalent can be seen from convergent replacements: majorization is equivalent to a finite sequence of Pigou-Dalton transfers for common natural vectors of the convergent replacements. Conditions 2 and 3 are equivalent according to Lemma 6.1 and conditions 3 and 4 are equivalent according to Theorem 4.3. ◇

The Atkinson theorem for general distributions allows a formulation in terms of inverse Pigou-Dalton transfers. This is in analogy to natural distributions. Generalized majorization is therefore inverted as follows. Inverse generalized majorization $Y \leq_{igm} X$ is understood to be equivalent to $X \leq_{gm} Y$.

Theorem 6.4 ("Atkinson Theorem for General Distributions and Inverse Pigou-Dalton Transfers") *Let X and Y be distributions with equal expectations. Then the following conditions are equivalent*

1. *$Y \leq_{igm} X$.*
2. *$Y \leq_{iPD} X$.*
3. *$Y \geq_L X$.*
4. *$Y \leq_{cv} X$ ($\Longleftrightarrow X \leq_{cx} Y$).*

6.3 Economic Interpretation

It is here that the Atkinson theorem clarifies the situation completely. As a consequence of Atkinsons theorem in its general versions, distributions with equal expectations being Lorenz ordered $X \leq_L Y$ is in total generality equivalent to $Eu(X) \geq Eu(Y)$ for all increasing and concave functions. The integration functions are sometimes called welfare functions in relation to Atkinsons theorem. The Lorenz order being equivalent to the concave order means that the welfare functions need not even be increasing but concave only. Yet, non-monotone welfare functions may have a limited economical meaning. So the essential case covered from a societal point of view is concave and monotone increasing.

The theorem then tells the truly surprising insight that improving social utility w.r.t. concave utility functions is fully understandable as iterated Pigou-Dalton steps, possibly involving weakly convergent approximations in the non-rational case. Concavity or convexity as mathematical concepts with a "continuous character" are fully covered in these local operations, if weak limits are included.

However, there remains one huge interpretation problem in a societal context: The limit of ever more suitable Pigou-Dalton steps is the Egalitarian distribution. Empirically and analytically this distribution is not beneficial to a society and this obvious shortcoming to some extend limited the full appreciation of this theorem and the insight it makes possible. How can this "deficit" be overcome and social reality be better accommodated with the Atkinson theorem? The answer is as follows.

Eventually, societies will switch their collective utility function from concave to convex. More differentiation will then seem better than more equality—to allow for differentiation in contributions, abilities, fortune, risk taking etc., but also to allow for capital accumulation, to make the financing of innovations and investments easier or possible at all. Now, inverse Pigou-Dalton steps, iterated and maybe involving weak limits, are the rule of the day.

However, the limit of inverse Pigou-Dalton transfers amounts to an always higher concentration of income with always fewer very rich people. This will also not work in society. So, eventually, social preference of a society will again switch—back to concavity, back to Pigou-Dalton transfers—a dynamic fluid balance of societal development.

The Atkinson theorem is a marvel—from a mathematical as well as a social science point of view. The ongoing, sometimes surprising fluctuations in preferences in society concerning more equality or more differentiation become better understandable, at least as far as mere distributional issues are concerned. This leaves out aspects of power, economic growth potential and market dynamics, which, of course, also carry a huge societal importance in this context.

References

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Part II
Lorenz Curves and Models