Task 7.1 (Differential Entropy)

a) Determine the differential entropy $H(x)$ if X is uniformly distributed in the interval $[a, b]$.

b) Determine the differential entropy $H(x)$ with

$$f_X = \begin{cases} 1, & \text{if } x \in [1, 2] \\ 0, & \text{otherwise.} \end{cases}$$

c) Determine the differential entropy $H(x)$ with

$$f_X = \begin{cases} \frac{1}{100}, & \text{if } x \in [100, 200] \\ 0, & \text{otherwise.} \end{cases}$$

Task 7.2 (Gaussian Distribution maximizes Entropy)

Given a real and continuous random variable X with mean square $E(X^2) = \sigma^2$ and $E(X) = 0$. Show that the differential entropy $H(X)$ is maximized if X is Gaussian distributed. What is the maximum entropy of X? Give a proof.

Hint: Use that the Kullback-Leibler distance of two continuous probability density functions $f_X(x)$ and $g_X(x)$ (over the same probability space) is defined as

$$D\Big(f_X(x) \Big|\Big| g_X(x)\Big) = \int_{-\infty}^{\infty} f_X(x) \log_2 \frac{f_X(x)}{g_X(x)}\, dx,$$

where we define $0 \cdot \log_2 \frac{0}{0} = 0$ and assume that $g_X(x) = 0 \Rightarrow f_X(x) = 0$. Moreover, the Kullback-Leibler distance is non-negative (see script section 2.4).
Use also equation (5.7) in the script.

Task 7.3 (Waterfilling)

We consider n independent, parallel, time-discrete and additive Gaussian channels. The noise variances are given by $N = \{N_1, N_2, \dots N_n\}$. The sum power of the input signal is constrained by

$$\sum_{i=1}^{n} S_i \leq E,$$

where S_i indicates the power used in subchannel i. The sum capacity of the channel is given by

$$C = \sum_{i=1}^{n} \frac{1}{2} \log_2 \left(1 + \frac{S_i}{N_i}\right)$$

and is achieved by Gaussian distributed input signals with zero mean and variance S_i. The capacity is maximized, if the variances S_i are chosen according to the "Waterfilling" method:

$$\begin{aligned} S_i + N_i &= B \quad \text{for } N_i < B \\ S_i &= 0 \quad \text{for } N_i \geq B \end{aligned}$$

and B is chosen such that $\sum_{i=1}^{n} S_i = E$.

Here, we consider the following example: $n = 5,\ N = \{1, 2, 4, 9, 16\},\ E = 20$

a) Calculate the sum capacity of this parallel combination, if the transmit power is equally distributed on all subchannels

b) Calculate the sum capacity, if the transmit power is distributed on the subchannels according to the "Waterfilling" method.