Solution to Task 1.1 (Introduction to probability theory)

a) As all three coin tosses are independent, the probability for a specific result is the product of the probabilities of three single coin tosses.

ω_i	$P(\omega_i)$	X	Y	XY
HHH	$\frac{1}{27}$	3	1	3
HHT	$\frac{2}{27}$	2	1	2
HTH	$\frac{2}{27}$	2	1	2
HTT	$\frac{4}{27}$	1	1	1
THH	$\frac{2}{27}$	2	0	0
THT	$\frac{4}{27}$	1	0	0
TTH	$\frac{4}{27}$	1	0	0
TTT	$\frac{8}{27}$	0	0	0

b)
$$f_X(x) = \frac{8}{27}\delta(x) + \frac{12}{27}\delta(x-1) + \frac{6}{27}\delta(x-2) + \frac{1}{27}\delta(x-3)$$

c)

	$X=0$	$X=1$	$X=2$	$X=3$	$\Sigma = f_Y(y)$
$Y=0$	$\frac{8}{27}$	$\frac{8}{27}$	$\frac{2}{27}$	0	$\frac{2}{3}$
$Y=1$	0	$\frac{4}{27}$	$\frac{4}{27}$	$\frac{1}{27}$	$\frac{1}{3}$
$\Sigma = f_X(x)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$	

Solution to Task 1.2 (Conditional Probability, Statistical Independence)

A: Student passes the Channel Coding exam
B: Student passes the Applied Information Theory exam
$P(A) = 0.8$
$P(B) = 0.85$
$P(B|A) = 0.95$

a) $P(A \cap B) \stackrel{Bayes}{=} P(B|A) \cdot P(A) = 0.95 \cdot 0.8 = 0.76$

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.85 - 0.76 = 0.89$

c) $P(A|B) \stackrel{Bayes}{=} \frac{P(A\cap B)}{P(B)} = \frac{0.76}{0.85} \approx 0.8941$

d) $P(A \cap B) = 0.76 \neq 0.8 \cdot 0.85 = P(A) \cdot P(B) \Rightarrow$ not independent.

Solution to Task 1.3 (Total Probability)

A: Second ball is green
B_1: First ball is green, B_2: First ball is red, B_3: First ball is blue
Law of total probability:
$P(A) = P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2) + P(A|B_3) \cdot P(B_3) = \frac{3}{11} \cdot \frac{4}{12} + \frac{4}{11} \cdot \frac{5}{12} + \frac{4}{11} \cdot \frac{3}{12} = \frac{1}{3}$

Solution to Task 1.4 (Expected value I)

$$E[X] = 0 \cdot 0.5 + 1 \cdot 0.25 + 3 \cdot 0.25 = 1$$

x	$f_X(x) = p_i$	$-\log_2 f_X(x)$	$\big(x - E[X]\big)^2$
0	0.5	1	1
1	0.25	2	0
3	0.25	2	4

$$E\Big[f_X(X)\Big] = \sum_x f_X(x) \cdot f_X(x) = 0.5 \cdot 0.5 + 0.25 \cdot 0.25 + 0.25 \cdot 0.25 = 0.375$$

$$E\Big[-\log(f_X(X))\Big] = -\sum_x f_X(x) \log_2 f_X(x) = 0.5 \cdot 1 + 0.25 \cdot 2 + 0.25 \cdot 2 = 1.5$$

Variance: $E\Big[\big(X - E[X]\big)^2\Big] = \sum_x \big(x_i - E[X]\big)^2 f_X(x) = 1 \cdot 0.5 + 0 \cdot 0.25 + 4 \cdot 0.25 = 1.5$

Solution to Task 1.5 (Expected value II)

$$\text{Recall:} \qquad E[a \cdot g_1(X) + b \cdot g_2(X)] = aE[g_1(X)] + bE[g_2(X)]$$

a)

$$\begin{aligned}
\sigma_X^2 &= E[(X-\mu_X)^2] \\
&= E[X^2 - 2\mu_X X + \mu_X^2] \\
&= E[X^2] - 2\mu_X E[X] + \mu_X^2 \\
&= \mu_{X^2} - \mu_X^2 \\
\sigma_{XY}^2 &= E[(X-\mu_X)(Y-\mu_Y)] \\
&= E[XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y] \\
&= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X\mu_Y \\
&= \mu_{XY} - \mu_X\mu_Y
\end{aligned}$$

b) $Z = X + Y$:
If X and Y are uncorrelated, then $\mu_{XY} = \mu_X\mu_Y$ according to a).

$$\begin{aligned}
\mu_Z &= E[Z] = E[X+Y] \\
&= E[X] + E[Y] \\
&= \mu_X + \mu_Y
\end{aligned}$$

$$\begin{aligned}
\sigma_Z^2 &= E[(Z-\mu_Z)^2] \\
&= E[Z^2 - 2\mu_Z Z + \mu_Z^2] \\
&= E[Z^2] - 2\mu_Z E[Z] + \mu_Z^2 \\
&= E[Z^2] - \mu_Z^2 \\
&= E[(X+Y)^2] - \mu_Z^2 \\
&= \mu_{X^2} + 2\mu_{XY} + \mu_{Y^2} - (\mu_X + \mu_Y)^2 \\
&= \mu_{X^2} - \mu_X^2 + \mu_{Y^2} - \mu_Y^2 + 2\mu_{XY} - 2\mu_X\mu_Y \\
\overset{uncorr.}{\Rightarrow} \sigma_Z^2 &= \mu_{X^2} - \mu_X^2 + \mu_{Y^2} - \mu_Y^2 \\
&\overset{a)}{=} \sigma_X^2 + \sigma_Y^2
\end{aligned}$$

Solution to Task 1.6 (Binomial distribution, Geometric distribution)

a) We use the binomial distribution
$P(X=m)=\binom{n}{m}p^m(1-p)^{n-m}$ (Probability for m successes in n runs)
X: Number of incorrectly transmitted bits

$$P(X=2)=\binom{50}{2}\cdot 0.01^2\cdot(1-0.01)^{48}\approx 0.0756$$

b) We again use binomial distribution.

$$\begin{aligned} P(X\geq 1) &= 1-P(X=0)\\ P(X=0) &= \binom{50}{0}\cdot 0.01^0\cdot(1-0.01)^{50}\approx 0.6050\\ P(X\geq 1) &= 1-P(X=0)=1-0.6050\approx 0.395 \end{aligned}$$

c)

$$\begin{aligned} P(X\geq 100) &= (1-0.01)^{100}\\ &= 0.99^{100}\\ &\approx 0.3660 \end{aligned}$$

Alternative solution: We also can use the geometric distribution
$P(X=m)=p(1-p)^m$ (first success in run $(m+1)$ after m failures)
X: number of correctly transmitted bits before the first erroneous bit

$$\begin{aligned} P(X\geq 100) &= 1-P(X\leq 99)\\ &= 1-\sum_{k=0}^{99}P(X=k)\\ &= 1-\sum_{k=0}^{99}(1-p)^k\cdot p\\ &\overset{(*)}{=} 1-p\cdot\frac{1-(1-p)^{100}}{1-(1-p)}\\ &= 1-(1-(1-p)^{100})\\ &= 0.99^{100}\\ &\approx 0.3660 \end{aligned}$$

$(*)$ Here, the geometrical sum is used:

$$\sum_{i=0}^{n}z^i=\frac{1-z^{n+1}}{1-z}\qquad\forall z\neq 1$$