Solution to Task 10.1 (Broadcast vs. TDMA, AWGN Channel)

1. We consider TDMA with a rate $R_{A,TD} = 0.5$ for user A. This gives us a fraction $\lambda$ of time for user A as:

$$\lambda \frac{1}{2} \log_2 \left( 1 + \frac{P}{N_A} \right) = 0.5 \quad \Rightarrow \lambda = 0.63.$$ 

The resulting rate $R_{B,TD}$ for user B, whose time fraction is $(1 - \lambda)$, is

$$R_{B,TD} = (1 - \lambda) \frac{1}{2} \log_2 \left( 1 + \frac{P}{N_B} \right) = 0.64.$$

2. We consider a Broadcast case with rate $R_{A,BC} = 0.5$ for user A. As user A has a higher error probability, we treat user B’s messages as additional noise for user A and assume that user B can decode user A’s messages. This gives us a power fraction of user A as:

$$\frac{1}{2} \log_2 \left( 1 + \frac{(1 - \alpha)P}{\alpha P + N_A} \right) = 0.5 \quad \Leftrightarrow \quad \frac{(1 - \alpha)P}{\alpha P + N_A} = 1 \quad \Rightarrow \alpha = 0.25.$$ 

The resulting rate $R_{B,BC}$ for user B, whose power fraction is $(1 - \alpha)$, is:

$$R_{B,BC} = \frac{1}{2} \log_2 \left( 1 + \frac{\alpha P}{N_B} \right) = 0.90.$$

A direct comparison yields

$$\frac{R_{B,BC}}{R_{B,TD}} = 1.41.$$ 

In the broadcast case, user B achieves a rate which is 41% higher, while the overall power stays the same.
Solution to Task 10.2 (TDMA in the Multiple-Access Channel)

a) If the users access the channels exclusively they would achieve the single-user capacities

\[ C_1 = \frac{1}{2} \log_2 \left( 1 + \frac{P_1}{N} \right) \quad \text{or} \quad C_2 = \frac{1}{2} \log_2 \left( 1 + \frac{P_2}{N} \right), \]

respectively.

b) While the users are not transmitting, their transmit power is zero and thus we have

\[ E(X^2_1) = \alpha \cdot \tilde{P}_1 + (1 - \alpha) \cdot 0 = \alpha \cdot \tilde{P}_1 \]
\[ E(X^2_2) = \alpha \cdot 0 + (1 - \alpha) \cdot \tilde{P}_2 = (1 - \alpha) \cdot \tilde{P}_2 \]

c) \[ \tilde{P}_{1max} = \frac{P_1}{\alpha} \]
\[ \tilde{P}_{2max} = \frac{P_2}{1 - \alpha} \]

d) \[ R_1 = \alpha \cdot \frac{1}{2} \log_2 \left( 1 + \frac{P_1}{\alpha N} \right) \]
\[ R_2 = (1 - \alpha) \cdot \frac{1}{2} \log_2 \left( 1 + \frac{P_2}{(1 - \alpha)N} \right) \]

e) If the achievable sum rate

\[ R_\Sigma = \frac{1}{2} \left[ \alpha \log_2 \left( 1 + \frac{P_1}{\alpha N} \right) + (1 - \alpha) \log_2 \left( 1 + \frac{P_2}{(1 - \alpha)N} \right) \right] \]

is differentiated with respect to alpha, we obtain

\[ \frac{dR_\Sigma}{d\alpha} = \frac{1}{2} \log_2 e \left[ \ln \left( 1 + \frac{P_1}{\alpha N} \right) - \ln \left( 1 + \frac{P_2}{(1 - \alpha)N} \right) - \frac{P_1}{\alpha N + P_1} + \frac{P_2}{(1 - \alpha)N + P_2} \right] \]
\[ \frac{d^2 R_\Sigma}{d\alpha^2} = -\frac{1}{2} \log_2 e \left[ \frac{P_1^2}{\alpha(\alpha N + P_1)^2} + \frac{P_2^2}{(1 - \alpha)((1 - \alpha)N + P_2)^2} \right]. \]
Obviously, for $\alpha = \alpha^* = \frac{P_1}{P_1 + P_2}$, we have $(1 - \alpha) = \frac{P_2}{P_1 + P_2}$ and $\frac{P_1}{\alpha N} = \frac{P_2}{(1 - \alpha) N} = \frac{P_1 + P_2}{N}$. Thus the first derivation is zero. Moreover it holds that $\frac{d^2 R_S}{d\alpha^2} < 0$ for $0 < \alpha < 1$. Thus, $\alpha^*$ has to achieve the global maximum of $R_S$. The maximum is

$$R_S|_{\alpha = \alpha^*} = \frac{1}{2} \log_2 \left(1 + \frac{P_1 + P_2}{N}\right),$$

which is equal to the maximum sum-rate the successive decoding scheme achieves. (see script section 6.4, example 6.3)

f) The capacity regions can be visualized as follows for $P_1 = 15$, $P_2 = 7$, $N = 1$.

TDMA given by the curve:

$$(R_1, R_2) = \left(\frac{1}{2} \log_2 \left(1 + \frac{5}{\alpha}\right), \frac{1}{2} (1 - \alpha) \log_2 \left(1 + \frac{7}{(1 - \alpha)}\right)\right).$$

Successive decoding going through the points: (see script section 6.4, example 6.3)

$$(0, \frac{1}{2} \log_2 (1 + 7)), \left(\frac{1}{2} \log_2 \left(1 + \frac{15}{7 + 1}\right), \frac{1}{2} \log_2 (1 + 7)\right), \left(\frac{1}{2} \log_2 (1 + 15), \frac{1}{2} \log_2 \left(1 + \frac{7}{15 + 1}\right)\right), \left(\frac{1}{2} \log_2 (1 + 15), 0\right).$$