Solution to Task 9.1 (Broadcast vs. TDMA, AWGN Channel)

1. We consider TDMA with a rate $R_{A,TD} = 0.5$ for user A. This gives us a fraction λ of time for user A as:

$$\lambda \frac{1}{2} \log_2 \left(1 + \frac{P}{N_A}\right) = 0.5 \qquad \Rightarrow \lambda = 0.63.$$

The resulting rate $R_{B,TD}$ for user B, whose time fraction is $(1 - \lambda)$, is

$$R_{B,TD} = (1 - \lambda) \frac{1}{2} \log_2 \left(1 + \frac{P}{N_B}\right) = 0.64.$$

2. We consider a Broadcast case with rate $R_{A,BC} = 0.5$ for user A. As user A has a higher error probability, we treat user B's messages as additional noise for user A and assume that user B can decode user A's messages. This gives us a power fraction of user A as:

$$\frac{1}{2} \log_2 \left(1 + \frac{(1-\alpha)P}{\alpha P + N_A}\right) = 0.5 \quad \Leftrightarrow \quad \frac{(1-\alpha)P}{\alpha P + N_A} = 1 \qquad \Rightarrow \alpha = 0.25.$$

The resulting rate $R_{B,BC}$ for user B, whose power fraction is $(1 - \alpha)$, is:

$$R_{B,BC} = \frac{1}{2} \log_2 \left(1 + \frac{\alpha P}{N_B}\right) = 0.90.$$

A direct comparison yields

$$\frac{R_{B,BC}}{R_{B,TD}} = 1.41.$$

In the broadcast case, user B achieves a rate which is 41% higher, while the overall power stays the same.

Solution to Task 9.2 (TDMA in the Multiple-Access Channel)

a) If the users access the channels exclusively they would achieve the single-user capacities

$$C_1 = \frac{1}{2}\log_2\left(1+\frac{P_1}{N}\right) \quad \text{or}$$
$$C_2 = \frac{1}{2}\log_2\left(1+\frac{P_2}{N}\right),$$

respectively.

b) While the users are not transmitting, their transmit power is zero and thus we have

$$E(X_1^2) = \alpha\cdot\widetilde{P_1} + (1-\alpha)\cdot 0 = \alpha\cdot\widetilde{P_1}$$
$$E(X_2^2) = \alpha\cdot 0 + (1-\alpha)\cdot\widetilde{P_2} = (1-\alpha)\cdot\widetilde{P_2}$$

c)

$$\widetilde{P}_{1max} = \frac{P_1}{\alpha}$$
$$\widetilde{P}_{2max} = \frac{P_2}{1-\alpha}$$

d)

$$R_1 = \alpha\cdot\frac{1}{2}\log_2\left(1+\frac{P_1}{\alpha N}\right)$$
$$R_2 = (1-\alpha)\cdot\frac{1}{2}\log_2\left(1+\frac{P_2}{(1-\alpha)N}\right)$$

e) If the achievable sum rate

$$R_\Sigma = \frac{1}{2}\left[\alpha\log_2\left(1+\frac{P_1}{\alpha N}\right) + (1-\alpha)\log_2\left(1+\frac{P_2}{(1-\alpha)N}\right)\right]$$

is differentiated with respect to alpha, we obtain

$$\frac{dR_\Sigma}{d\alpha} = \frac{1}{2}\log_2 e\left[\ln\left(1+\frac{P_1}{\alpha N}\right) - \ln\left(1+\frac{P_2}{(1-\alpha)N}\right) - \frac{P_1}{\alpha N + P_1} + \frac{P_2}{(1-\alpha)N+P_2}\right]$$
$$\frac{d^2R_\Sigma}{d\alpha^2} = -\frac{1}{2}\log_2 e\left[\frac{P_1^2}{\alpha(\alpha N+P_1)^2} + \frac{P_2^2}{(1-\alpha)((1-\alpha)N+P_2)^2}\right].$$

Obviously, for $\alpha = \alpha^* = \frac{P_1}{P_1+P_2}$, we have $(1-\alpha) = \frac{P_2}{P_1+P_2}$ and $\frac{P_1}{\alpha N} = \frac{P_2}{(1-\alpha)N} = \frac{P_1+P_2}{N}$. Thus the first derivation is zero. Moreover it holds that $\frac{d^2 R_\Sigma}{d\alpha^2} < 0$ for $0 < \alpha < 1$. Thus, α^* has to achieve the global maximum of R_Σ. The maximum is

$$R_\Sigma|_{\alpha=\alpha^*} = \frac{1}{2}\log_2\left(1 + \frac{P_1+P_2}{N}\right),$$

which is equal to the maximum sum-rate the successive decoding scheme achieves. (see script section 6.4, example 6.3)

f) The capacity regions can be visualized as follows for $P_1 = 15, P_2 = 7, N = 1$.
TDMA given by the curve:

$$(R_1, R_2) = \left(\frac{1}{2}\alpha\log_2\left(1+\frac{5}{\alpha}\right), \quad \frac{1}{2}(1-\alpha)\log_2\left(1+\frac{7}{(1-\alpha)}\right)\right).$$

Successive decoding going through the points: (see script section 6.4, example 6.3)

$$\left(0, \quad \frac{1}{2}\log_2(1+7)\right), \qquad \left(\frac{1}{2}\log_2\left(1+\frac{15}{7+1}\right), \quad \frac{1}{2}\log_2(1+7)\right),$$
$$\left(\frac{1}{2}\log_2(1+15), \quad \frac{1}{2}\log_2\left(1+\frac{7}{15+1}\right)\right), \qquad \left(\frac{1}{2}\log_2(1+15), \quad 0\right).$$