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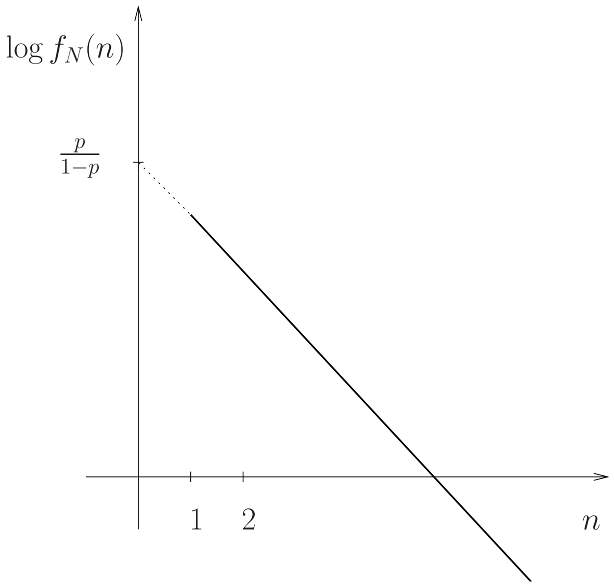
Solutions to exercise sheet 2

Solution to Task 2.1 (Entropy of the geometric distribution)

The probability density function of a geometrically distributed random variable N is:

$$\begin{aligned} f_N(n) &= p(1-p)^{n-1} \\ &= \exp\left(n\log(1-p) + \log\frac{p}{1-p}\right), \end{aligned}$$

for $n = 1, 2, \ldots$ and $0 < p < 1$. The probability $f_N(n)$ decreases exponentially with n.



a) Calculation of the expected value of a geometrically distributed random variable N:

$$\begin{aligned} \mu_N = E[N] &= \sum_{n=1}^{\infty} np(1-p)^{n-1} \\ &= \frac{p}{1-p}\sum_{n=1}^{\infty} n(1-p)^n \\ &= \frac{p}{1-p}\cdot\frac{1-p}{p^2} \\ &= \frac{1}{p} \end{aligned}$$

using

$$\sum_{n=1}^{\infty} n\cdot x^n = \frac{x}{(x-1)^2} \qquad \forall |x| < 1.$$

b) Calculation of the entropy of a geometrically distributed random variable N:

$$
\begin{aligned}
H(N) &= E[-\log_2 f_N(n)] = -\sum_{n=1}^{\infty} f_N(n) \log_2\Big(p(1-p)^{n-1}\Big) \\
&= -\sum_{n=1}^{\infty} f_N(n)\Big(\log_2(p) + (n-1)\log_2(1-p)\Big) \\
&= -\log_2(p)\underbrace{\sum_{n=1}^{\infty} f_N(n)}_{=1} \\
&\quad -\log_2(1-p)\left(\underbrace{\sum_{n=1}^{\infty} n f_N(n)}_{=E[N]=1/p} - \underbrace{\sum_{n=1}^{\infty} f_N(n)}_{=1}\right) \\
&= \frac{1}{p}\left(-p\log_2(p) - (1-p)\log_2(1-p)\right) \\
&= \mu_N \cdot h(p)
\end{aligned}
$$

Solution to Task 2.2 (Conditional entropies)

From the table, we obtain

$X\backslash Y$	0	1	$f_X(x)$	$f_{X\|Y}(x\|Y=0)$	$f_{X\|Y}(x\|Y=1)$
0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{2}$
1	0	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{2}$
$f_Y(y)$	$\frac{1}{3}$	$\frac{2}{3}$			

a) $H(X) = H(Y) = h(\frac{1}{3}) = \frac{1}{3}\log_2 3 + \frac{2}{3}\log_2\frac{3}{2} = 0.9183\ (= \log_2 3 - \frac{2}{3})$

b) $H(X|Y) = f_Y(0)\cdot H(X|Y=0) + f_Y(1)\cdot H(X|Y=1) = \frac{1}{3}h(0) + \frac{2}{3}\cdot h(\frac{1}{2}) = \frac{2}{3}$
$H(Y|X) = f_X(0)\cdot H(Y|X=0) + f_X(1)\cdot H(Y|X=1) = \frac{2}{3}\cdot 1 + 0 = \frac{2}{3}$

c) $H(X,Y) \overset{chain\ rule}{=} H(X) + H(Y|X) = 0.9183 + \frac{2}{3} = 1.585\ (= \log_2 3)$

d) $H(Y) - H(Y|X)\Big(= I(X;Y)\Big) = 0.9183 - \frac{2}{3} = 0.2516$

Solution to Task 2.3 (Conditional mutual information)

a) Let $X = Z$, then we have $I(X;Y|Z) = H(X|Z) - H(X|Y,Z) = 0$, since the two terms are equal. This is due to the fact, that conditioning on Y does not reduce the entropy, if $Z = X$ is given, too. However, $I(X;Y)$ is larger than zero if X and Y are not independent.

b) We consider a channel with an additional noise Z. Let $Y = X + Z$ be the output and let X be independent of Z, then $I(X;Y|Z) = H(Y|Z) - H(Y|X,Z) = H(X)$. The unconditioned mutual information is $I(X;Y) = H(X) - H(X|Y)$, which is less than $H(X)$ if $H(X|Y) > 0$.

Solution to Task 2.4 (Entropy and mutual information)

(i) $H(X_1) = -f_X(0)\cdot\log_2 f_X(0) - f_X(1)\cdot\log_2 f_X(1) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1$

(ii) $H(X_1X_2) = \sum_{x_1x_2\in\{00,01,10,11\}} -f_{X_1X_2}(x_1,x_2)\log_2 f_{X_1X_2}(x_1,x_2) = -\frac{1}{4}\log_2\frac{1}{4}\cdot 4 = 2$

(iii) $H(X_2|X_1) = H(X_1X_2) - H(X_1) = 1$

(iv) $H(X_1X_2X_3) = -\frac{1}{4}\log\frac{1}{4}\cdot 4 = 2$

(v) $H(X_3|X_1X_2) = -\sum_{x_1x_2}\sum_{x_3}\underbrace{f_{X_1X_2X_3}(x_1,x_2,x_3)}_{=0\ \forall(x_1x_2x_3)\notin\Omega}\log_2\underbrace{f_{X_3|X_1X_2}(x_3|x_1x_2)}_{=1\ \forall(x_1x_2x_3)\in\Omega} = 0$

(vi) $H(X_3) = -\frac{1}{2}\log\frac{1}{2}\cdot 2 = 1$

(vii) $I(X_1;X_3) = H(X_1) + H(X_3) - H(X_1X_3) = 0$

(viii) $I(X_1X_2;X_3) = H(X_1X_2) + H(X_3) - H(X_1X_2X_3) = 1$

Solution to Task 2.5 (Kullback-Leibler distance and average mutual information)

$$D_{KL}\Big(f_{XY}(x,y)\Big|\Big|f_X(x)f_Y(x)\Big) \overset{def}{=} \sum_x\sum_y f_{XY}(x,y)\log\frac{f_{XY}(x,y)}{f_X(x)f_Y(y)}$$

$$\overset{Bayes}{=} \sum_x\sum_y f_{XY}(x,y)\log\frac{f_{X|Y}(x|y)}{f_X(x)} \overset{def}{=} I(X;Y)$$