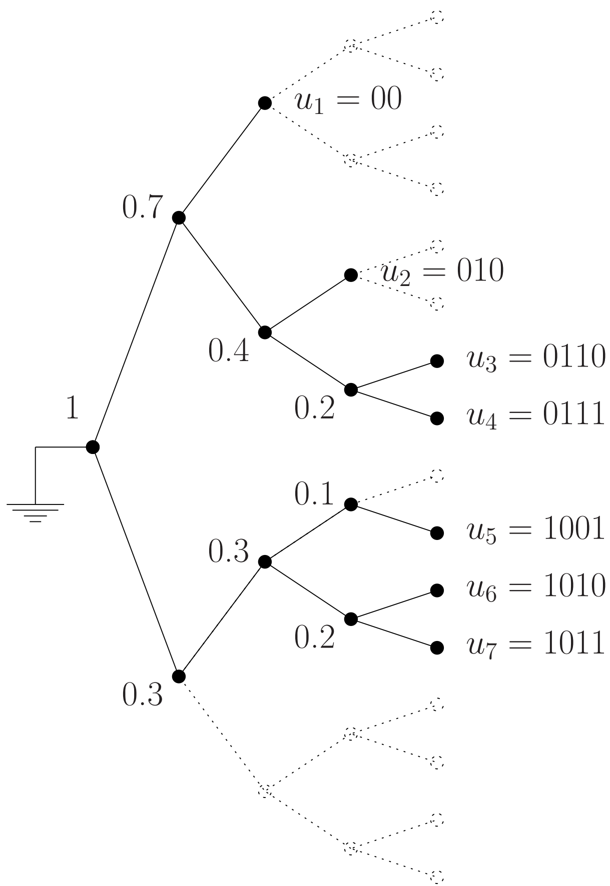
Solution to Task 4.1 (Shannon-Fano Code)

a) Codeword lengths for Shannon-Fano code:

u	$P_U(u)$	$\left\lceil \log_2 \frac{1}{P_U(u)} \right\rceil$
u_1	0.3	2
u_2	0.2	3
u_3	0.1	4
u_4	0.1	4
u_5	0.1	4
u_6	0.1	4
u_7	0.1	4

The existence of a prefix-free code with these lengths is ensured.
We start with a full tree of depth 4:



b) Pathlength-Lemma:

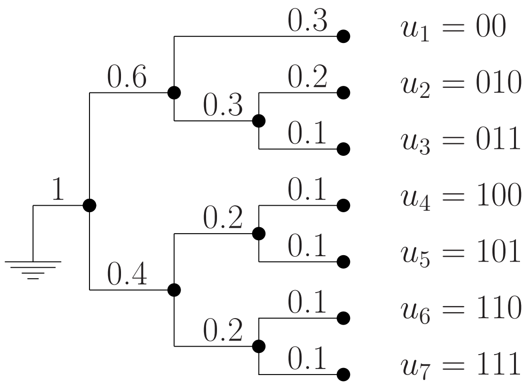
$$E[W] = \sum P_i = 1 + 0.7 + 0.4 + 2 \cdot 0.3 + 2 \cdot 0.2 + 0.1 = 3.2$$

and it holds that:

$$\begin{aligned} & H(U) \le E[W] \le H(U) + 1 \\ \Leftrightarrow \quad & E[W] - 1 \le H(U) \le E[W] \\ \Leftrightarrow \quad & 2.2 \le H(U) \le 3.2 \end{aligned}$$

Solution to Task 4.2 (Huffman Code)

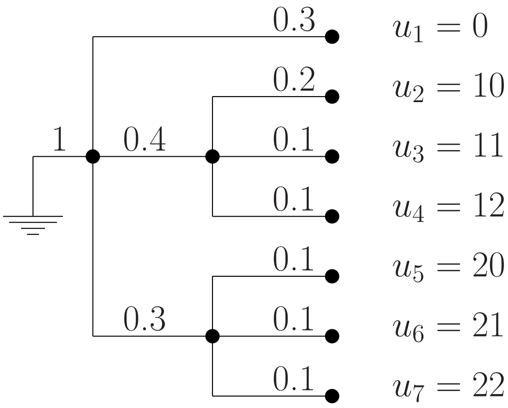
a)



The Pathlength-Lemma holds:

$$E[W] = 1 + 0.4 + 0.6 + 0.3 + 0.2 + 0.2 = 2.7$$

b) $D = 3$:



$$E[W] = 1 + 0.3 + 04 = 1.7$$

and it holds that:
$$\frac{H(U)}{\log_2(3)} = 1.6697 \leq 1.7 \leq \frac{H(U)}{\log_2(3)} + 1 = 2.6697.$$

c) Binary Shannon-Fano code: $\nu_{BSF} = 2.6464/3.2 = 0.8270$
Binary Huffman code: $\nu_{BH} = 2.6464/2.7 = 0.9802$

d) Ternary Huffman Code: $\nu_{TH} = \frac{2.6464}{1.7 \log 3} = 0.9822$
The efficiency of the ternary Huffman Code is higher here.

Solution to Task 4.3 (Tunstall Algorithm)

Given:

- $P(0) = 0.2$
- $P(1) = 0.8$
- $k = 2$ (binary source)
- $n = 3$ (codeword length)
- $q = 2$ (codeword alphabet)
- $q^n \geq k$

Number of extensions:
$$e = \left\lfloor \frac{q^n - k}{k - 1} \right\rfloor = \left\lfloor \frac{2^3 - 2}{2 - 1} \right\rfloor = 6$$

Number of sequences (= number of leaves) :

$$m = k + e(k - 1) = 2 + 6(2 - 1) = 8$$

Sequences	**Codewords**
0	000
10	001
110	011
1110	010
11110	110
111110	111
1111110	101
1111111	100

Solution to Task 4.4 (Lempel-Ziv-Algorithm)

a) Parsing:

0	01	00	1	010	001	10
1	2	3	4	5	6	7

Encoding:

Parsed sequence	0	01	00	1	010	001	10
Coded sequence	$(0,0)$	$(1,1)$	$(1,0)$	$(0,1)$	$(2,0)$	$(3,1)$	$(4,0)$

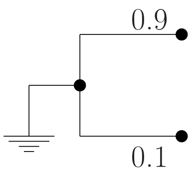
b) Decoding:

Coded sequence	$(0,0)$	$(0,1)$	$(1,0)$	$(2,1)$	$(4,0)$	$(1,1)$	$(5,1)$
Decoded sequence	0	1	00	11	110	01	1101

Resulting code sequence: 010011110011101

Solution to Task 4.5 (Efficiency of different source coding schemes)

u	$P_U(u)$	codeword
0	0.9	0
1	0.1	1



$$H(U) = h(0.9) = 0.469$$

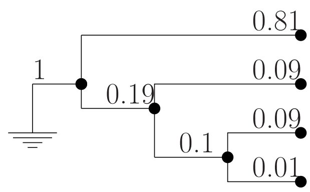
a) <u>Block to variable length</u>: We use the Huffman-Coding to construct optimal codes.

$L = 1$:(no coding)

$$E[W] = 1 \Rightarrow \nu = 0.469$$

$L = 2$:

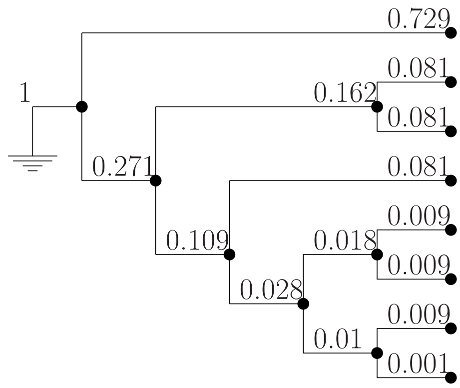
$\hat{u}$	$P_{\hat{U}}(\hat{u})$	codeword
00	0.81	0
01	0.09	10
10	0.09	110
11	0.01	111



$$E[W] = 1.29 \Rightarrow \nu = 0.727$$

$L = 3$:

$\tilde{u}$	$P_{\tilde{U}}(\tilde{u})$	codeword
000	0.729	0
001	0.081	100
010	0.081	101
011	0.009	11100
100	0.081	110
101	0.009	11101
110	0.009	11110
111	0.001	11111



$$E[W] = 1.598 \Rightarrow \nu = 0.881$$

b) Variable length to block: We use the Tunstall-Algorithm to construct the codes.

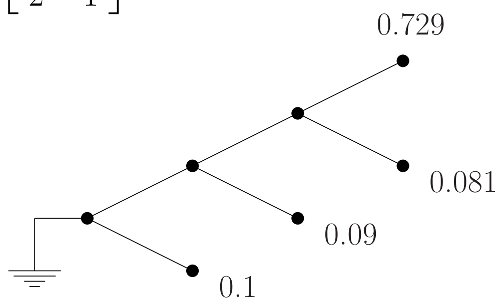
$N = 1$:(no coding)

$$E[Y] = 1 \Rightarrow \nu = 0.469$$

$N = 2$:
Number of expansions

$$e = \left\lfloor \frac{q^N - k}{k-1} \right\rfloor = \left\lfloor \frac{2^2 - 2}{2-1} \right\rfloor = 2$$

$\hat{u}$	$P_{\hat{U}}(\hat{u})$	codeword
000	0.729	00
001	0.081	01
01	0.09	10
1	0.1	11

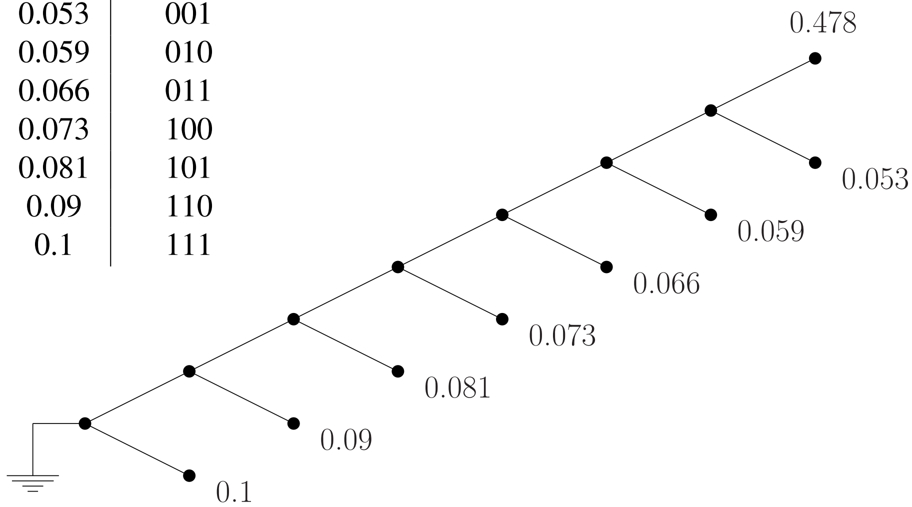


$$E[Y] = 2.71 \Rightarrow \nu = 0.635$$

$N = 3$:
Number of expansions:

$$e = \left\lfloor \frac{q^N - k}{k-1} \right\rfloor = \left\lfloor \frac{2^3 - 2}{2-1} \right\rfloor = 6$$

$\tilde{u}$	$P_{\tilde{U}}(\tilde{u})$	codeword
0000000	0.478	000
0000001	0.053	001
000001	0.059	010
00001	0.066	011
0001	0.073	100
001	0.081	101
01	0.09	110
1	0.1	111



$$E[Y] = 5.217 \Rightarrow \nu = 0.816$$

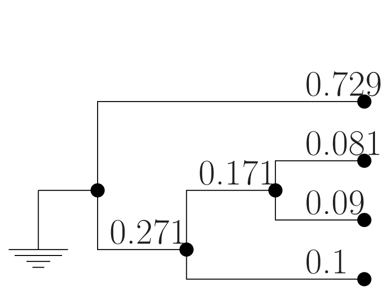
c) Variable length to variable length: We construct Huffman-Codes for Tunstall-parsed sources.

$M = 2$: (no coding)

$$E[Y] = E[W] = 1 \Rightarrow \nu = 0.469$$

$M = 4$:

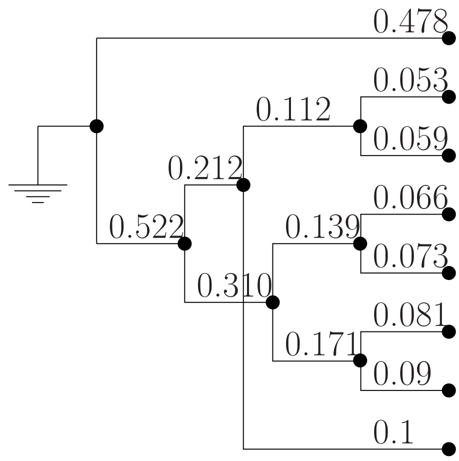
$\hat{u}$	$P_{\hat{U}}(\hat{u})$	codeword
000	0.729	0
001	0.081	100
01	0.09	101
1	0.1	11



$$\begin{aligned} E[Y] &= 2.71 \\ E[W] &= 1.442 \qquad \Rightarrow \nu = 0.881 \end{aligned}$$

$M = 8$:

$\tilde{u}$	$P_{\tilde{U}}(\tilde{u})$	codeword
0000000	0.478	0
0000001	0.053	1000
000001	0.059	1001
00001	0.066	1100
0001	0.073	1101
001	0.081	1110
01	0.09	1111
1	0.1	101



$$E[Y] = 5.217$$
$$E[W] = 2.465 \qquad \Rightarrow \nu = 0.993$$