Solution to Task 5.1 (MAP- and ML-Decoding)

a) The number of detectable errors is $d-1=1$, the number of correctable errors is $\lfloor \frac{d-1}{2} \rfloor = 0$. This means a bounded minimum distance decoder (BMD) can not correct any errors.

b) The Hamming-distances of $y = (111011)$ to the codewords $c \in \mathcal{C}$ are given by the following table.

c	$c-y$	$wt(c-y)$
000000	111011	5
001111	110100	3
010111	101100	3
100111	011100	3
110000	001011	3
101000	010011	3
011000	100011	3
111111	000100	1

Hence the decoding decision is

$$\hat{c} = \arg\min_{c} \Big\{ wt(c-y) \Big\} = (111111).$$

c) The conditional probabilities $f_{Y|C}(y|c)$ for $c \in \mathcal{C}$ are given in the following table.

c	$c-y$	$f_{Y\|C}(y\|c)$
000000	111011	$p^5 \cdot (1-p)$
001111	110100	$p^3 \cdot (1-p)^3$
010111	101100	$p^3 \cdot (1-p)^3$
100111	011100	$p^3 \cdot (1-p)^3$
110000	001011	$p^3 \cdot (1-p)^3$
101000	010011	$p^3 \cdot (1-p)^3$
011000	100011	$p^3 \cdot (1-p)^3$
111111	000100	$p \cdot (1-p)^5$

Case 1: For $0 \leq p < \frac{1}{2}$ we have

$$p^5 \cdot (1-p) < p^3 \cdot (1-p)^3 < p \cdot (1-p)^5$$

and thus the decoder is equal to the decoder given in part b).

$$\hat{c} = \arg\max_{c} \Big\{ f_{Y|C}(y|c) \Big\} = \arg\min_{c} \Big\{ wt(c-y) \Big\} = (111111).$$

Case 2: For $\frac{1}{2} < p \leq 1$ we have

$$p \cdot (1-p)^5 < p^3 \cdot (1-p)^3 < p^5 \cdot (1-p)$$

and thus the decoder is equal to the inverse of decoder given in part b). The decision is

$$\hat{c} = \arg\max_c \left\{ f_{Y|C}(y|c) \right\} = \arg\max_c \left\{ wt(c-y) \right\} = (000000).$$

d) The values $f_{Y|C}(y|c) \cdot f_C(c)$ for $c \in \mathcal{C}_M$ and $p = \frac{1}{3}$ are given in the following table.

i	c	$c-y$	$f_{Y\|C}(y\|c)$	$f_C(c)$	$f_{Y\|C}(y\|c) \cdot f_C(c)$
000	000000	111011	$\frac{1}{3}^5 \cdot \frac{2}{3}$	0.75^3	0.0012
100	100111	011100	$\frac{1}{3}^3 \cdot \frac{2}{3}^3$	$0.75^2 \cdot 0.25$	0.0015
110	110000	001011	$\frac{1}{3}^3 \cdot \frac{2}{3}^3$	$0.75 \cdot 0.25^2$	0.0005
111	111111	000100	$\frac{1}{3} \cdot \frac{2}{3}^5$	0.25^3	0.0007

The decision is

$$\hat{c} = \arg\max_c \left\{ f_{Y|C}(y|c) \cdot f_C(c) \right\} = (100111)$$

which is not equivalent to the decision of a ML- or a Hamming-metric decoder.

Solution to Task 5.2 (Reed-Muller Codes and Incremental Redundancy)

a) The generator matrix G of $\mathcal{C}(8,4,4)$ consist of the generator matrices G_u of $\mathcal{R}(1,2)$, a single parity check code $(4,3,2)$, and G_v of $\mathcal{R}(0,2)$, a repetition code $(4,1,4)$, that are given as

$$G_u = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad G_v = (1\,1\,1\,1).$$

Hence

$$G = \begin{pmatrix} G_u & G_u \\ 0 & G_v \end{pmatrix} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).$$

b) We remove the 2^{nd} and 6^{th} column of G.

$$G_p = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

c) By puncturing the second half of the code $\mathcal{C}$ we are left with $\mathcal{R}(1,2)$ a single parity check code with parameters $(4,3,2)$. (note: the dimension is reduced as i_4 is redundant)

d) The Code $\mathcal{C}(8,4,4)$ can correct $\lfloor \frac{d-1}{2} \rfloor = 1$ error.
The Code $\mathcal{C}_p(6,4,d_p)$ has minumum distance

$$\begin{aligned} d-2 \leq d_p &\leq d \\ 2 \leq d_p &\leq 4 \end{aligned}$$

as two bits were punctured and $\mathcal{C}$ has minimum distance $d=4$. Thus it can detect at least 1 error. In practice the number of detectable errors should always be at least the number of correctable errors.

e) The received word is $y_1 = (101|011)$. Adding the two halves of y_1 gives

$$(101) + (011) = (110)$$

which is not a valid codeword of a punctured repetition code $(3,1,3)$. Hence additional redundancy $y_2 = (01)$ is requested that gives

$$y = (1|0|010|1|11) = (1001|0111).$$

Adding the two halves of y gives

$$(1001) + (0111) = (1110)$$

which can be decoded in the repetition code $(4, 1, 4)$ to $v = (1111)$.
Adding v to the latter half of y yields

$$u_1 = (1001), \qquad u_2 = (0111) + (1111) = 1000.$$

As u_1 is a valid codeword in the single parity check code $(4, 3, 2)$ the decoding result is

$$\begin{aligned} u &= u_1 = (1001) \\ \Rightarrow c &= (u|u+v) = (1001|0110) \\ e &= y - c = (0000|0001). \end{aligned}$$