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Solution to Task 6.1 (DMC properties)

We consider a discrete memoryless channel.

a) Let $A = \{x_1, \dots, x_n\}$ and $B = \{y_1, \dots, y_m\}$ be sets with $n, m \in \mathbb{N}_{<\infty}$. Let X and Y be the Input- and Output- random variables that took values out of the set A and B, respectively. The transitionprobability p_{ij} from x_i to y_j is defined as follows:

$$p_{ij} \coloneqq P(Y = y_j | X = x_i) \quad \forall i \in \{1, \dots, n\}, \quad j \in \{1, \dots, m\}.$$

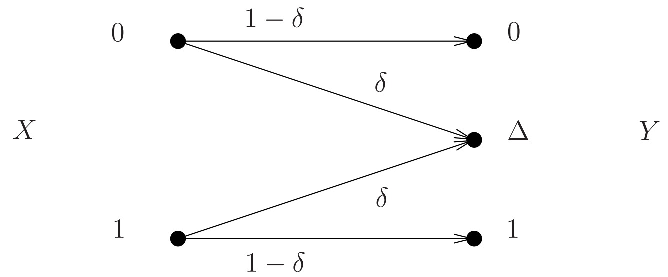
(i) A DMC is called *uniformly dispersive* if the sets $\{p_{i1}, \dots, p_{im}\} = \{p_{k1}, \dots, p_{km}\} \quad \forall x_i, x_k \in A$.

(ii) A DMC is called *uniformly focussing* if the sets $\{p_{1j}, \dots, p_{nj}\} = \{p_{1l}, \dots, p_{nl}\} \quad \forall y_j, y_l \in B$.

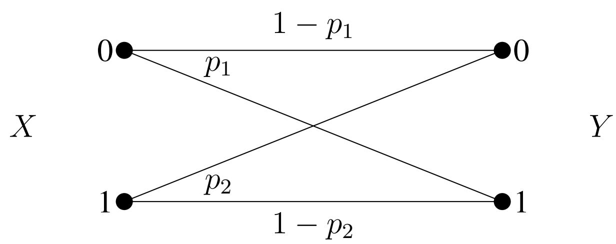
(iii) A DMC is called *strongly symmetric* if the channel is both uniformly dispersive and focussing.

b) Examples:

(i) For the binary erasure channel it holds that the outgoing probabilities are $\{1-\delta, \delta\}$ for $0 \in A$ and $\{\delta, 1-\delta\}$ for $1 \in A$. As these sets are equal the DMC is uniformly dispersive. For $0, 1, \triangle \in B$ it holds that the incoming probabilities are $\{1-\delta\}, \{\delta\}$ and $\{1-\delta\}$, respectively. As the sets are not equal, the DMC ist not unifomly focussing.

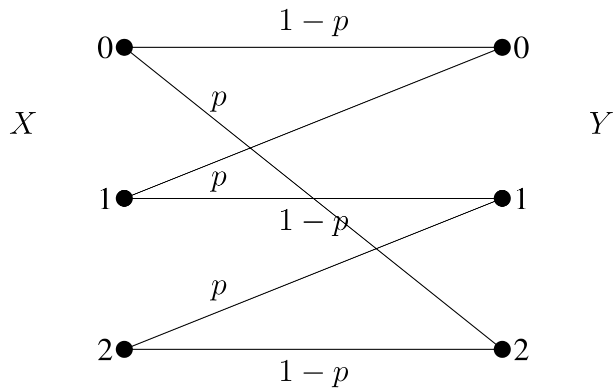


(ii) We consider the following binary channel:



For $0, 1 \in A$ it holds that the outgoing probabilities are $\{1-p_1, p_1\}$ and $\{p_2, 1-p_2\}$, respectively. As the sets are not equal, the DMC ist not unifomly disperive. For $0, 1 \in B$ the incoming probabilities are $\{1-p_1, p_2\}$ and $\{p_1, 1-p_2\}$, respectively. As the sets are not equal, the DMC is not unifomly focussing.

(iii) We consider the following ternary channel:



For $0, 1, 2 \in A$ the outgoing probabilities are $\{1-p, p\}$, $\{1-p, p\}$ and $\{1-p, p\}$, respectively. As the sets are equal, the DMC is unifomly dispersive. For $0, 1, 2 \in B$ the incoming probabilities are $\{1-p, p\}$, $\{1-p, p\}$ and $\{1-p, p\}$, respectively. As the sets are equal, the DMC is unifomly focussing. Finally it follows that the DMC is strongly symmetric.

Solution to Task 6.2 (Channel capacity of symmetric channels)

The capacity C for a strongly symmetric channel is

$$C = \log_2(m) + \sum_{j=1}^{m} p_{ij} \cdot \log_2(p_{ij}) \quad \forall i \in \{1, \ldots, n\}.$$

Proof. In general the channel capacity is

$$C = \max_{f_X(x_i)} \{I(X;Y)\} = \max_{f_X(x_i)} \{H(Y) - H(Y|X)\}. \tag{1}$$

For $X = x_i$ the conditional Entropy is

$$H(Y|X = x_i) = -\sum_{j=1}^{m} f_{Y|X}(y_j|x_i) \cdot \log_2(f_{Y|X}(y_j|x_i)) = -\sum_{j=1}^{m} p_{ij} \cdot \log_2(p_{ij}). \tag{2}$$

As the channel is uniformly dispersive it follows that

$$H(Y|X = x_i) = -\sum_{j=1}^{m} p_{ij} \cdot \log_2(p_{ij}) = -\sum_{j=1}^{m} p_{kj} \cdot \log_2(p_{kj}) = H(Y|X = x_k) \quad \forall i, k \in \{1, \ldots, n\}.$$

Therefore

$$H(Y|X) = \sum_{i=1}^{n} f_X(x_i) \cdot H(Y|X = x_i) = H(Y|X = x_i) \cdot \underbrace{\sum_{i=1}^{n} f_X(x_i)}_{=1}$$
$$= H(Y|X = x_i) \overset{(2)}{=} -\sum_{j=1}^{m} p_{ij} \cdot \log_2(p_{ij}) \quad \forall i \in \{1, \ldots, n\}.$$

So we have that the conditional Entropy $H(Y|X)$ is independent of the distirbution $f_X(x_i)$. With (1) it follows, that

$$C = \max_{f_X(x_i)} \{I(X;Y)\} = \max_{f_X(x_i)} \{H(Y) - H(Y|X)\} = \max_{f_X(x_i)} \{H(Y)\} - H(Y|X)$$
$$= \max_{f_X(x_i)} \{H(Y)\} + \sum_{j=1}^{m} p_{ij} \cdot \log_2(p_{ij}) \quad \forall i \in \{1, \ldots, n\}. \tag{3}$$

To maximize $H(Y)$ we assume that $f_X(x_i) = \frac{1}{n} \quad \forall x_i \in A$. It holds that

$$f_Y(y_j) = \sum_{i=1}^{n} f_{Y|X}(y_j|x_i) \cdot f_X(x_i) = \frac{1}{n} \sum_{i=1}^{n} f_{Y|X}(y_j|x_i) = \frac{1}{n} \underbrace{\sum_{i=1}^{n} p_{ij}}_{=1} = \frac{1}{n}.$$

(Note that $\{p_{1j}, \ldots, p_{nj}\} = \{p_{1l}, \ldots, p_{nl}\} \quad \forall j, l \in \{1, \ldots, m\}$, since the channel is uniformly focussing.) Further it holds that for a strongly symmetric channel the number of inputs n is equal to the number of outputs m. So for $f_X(x_i) = \frac{1}{n}$ it follows that

$$H(Y) = -\sum_{j=1}^{m} f_Y(y_j) \log_2(f_Y(y_j)) = \sum_{j=1}^{m} \frac{1}{n} \log_2(n) = m \cdot \frac{1}{n} \log_2(n) \overset{n=m}{=} \log_2(m).$$

With Theorem 2.3 in the scriptum (Bounds of the uncertainty) we know that $H(Y) \leq \log_2(m)$. So actually

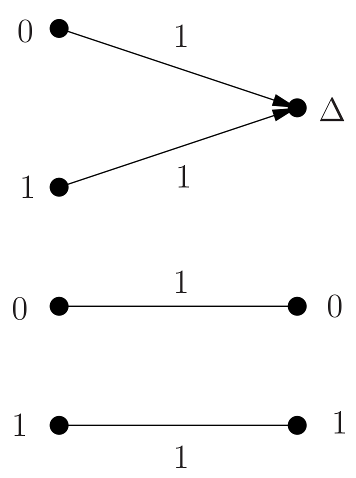
$$\max_{f_X(x_i)} \{H(Y)\} = \log_2(m).$$

Now with (3) we have that

$$C = \max_{f_X(x_i)} \{H(Y)\} + \sum_{j=1}^{m} p_{ij} \cdot \log_2(p_{ij}) = \log_2(m) + \sum_{j=1}^{m} p_{ij} \cdot \log_2(p_{ij}) \quad \forall i \in \{1, \ldots, n\}.$$

□

a) BEC is not strongly symmetric but a partition into strongly symmetric channels is possible:



$q_1 = \delta$

$C_1 = 0$

$q_2 = 1 - \delta$

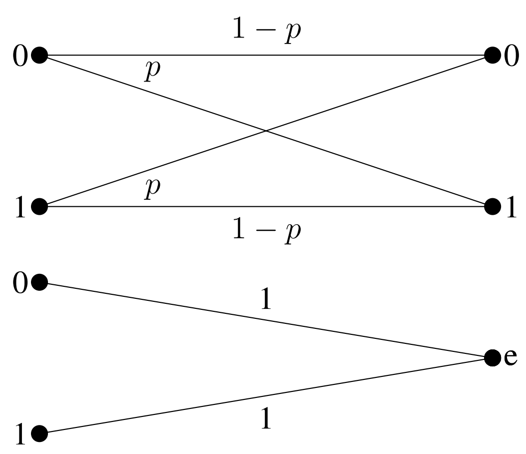
$C_2 = 1$

b) Using the partition from a) we have

$$
\begin{aligned}
C_{BEC} &= \sum_{i=1}^{2} q_i C_i \quad \Big(= E\big[C_i\big]\Big) \\
&= \delta \cdot 0 + (1-\delta)1 \\
&= 1 - \delta
\end{aligned}
$$

Solution to Task 6.3 (Binary Symmetric Erasure Channel)

a) Following the ideas of the previous task we split the channel into two strongly symmetric channels.



$q_1 = 1 - \alpha$

$C_1 = C_{BSC} = 1 - h(p)$

$q_2 = \alpha$

$C_2 = 0$

The probability p of the symmetric channel can be calculated via

$$q_1 \cdot p = \epsilon \qquad \Leftrightarrow \qquad p = \frac{\epsilon}{1-\alpha}.$$

The resulting capacity C is thus

$$\begin{aligned} C &= q_1 C_1 + q_2 C_2 \\ &= (1-\alpha)\left(1 - h\left(\frac{\epsilon}{1-\alpha}\right)\right) + \alpha \cdot 0 \\ &= (1-\alpha)\left(1 - h\left(\frac{\epsilon}{1-\alpha}\right)\right) \end{aligned}$$

b) For $\alpha = 0$, we have

$$C = 1 - h(\epsilon) = C_{BSC}$$

c) For $\epsilon = 0$, we have

$$C = (1-\alpha)(1-h(0)) = 1 - \alpha = C_{BEC}$$