Solution to Task 7.1 (Differential Entropy)

a)

$$f_X = \begin{cases} \frac{1}{b-a}, & \text{if } x \in [a,b] \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} H(X) &= -\int_{-\infty}^{\infty} f_X(x) \log_2 \Big(f_X(x)\Big)\, dx = -\int_a^b \frac{1}{b-a} \log_2 \left(\frac{1}{b-a}\right) dx \\ &= \log_2 \big(b-a\big) \end{aligned}$$

b)

$$H(X) = -\int_1^2 1 \cdot \log_2(1)\, dx = 0$$

c)

$$H(X) = -\int_{100}^{200} \frac{1}{100} \cdot \log_2(\frac{1}{100})\, dx = \frac{1}{100} \cdot \log_2(100) \cdot (200 - 100) = \log_2(100) \approx 6,644$$

Solution to Task 7.2 (Gaussian Distribution maximizes Entropy)

Note: To be mathematically precise, different density functions $f_{X_i}(x)$ must refer to different random variables X_i. For a better readability we neglect this in the following solution)

The Kullback-Leibler distance of two continuous probability density functions $f_X(x)$ and $g_X(x)$ (over the same probability space) is defined as

$$D\Big(f_X(x) \Big|\Big| g_X(x)\Big) = \int_{-\infty}^{\infty} f_X(x) \log_2 \frac{f_X(x)}{g_X(x)}\, dx,$$

where we define $0 \cdot \log_2 \frac{0}{0} = 0$ and assume that $g_X(x) = 0 \Rightarrow f_X(x) = 0$. Moreover, the Kullback-Leibler distance is non-negative.(see script section 2.4)

Now let $\phi_X(x)$, $f_X(x)$, and $g_X(x)$ be probability density functions of a random variable X with $E(X^2) = \sigma^2$ and $E(X) = 0$, where $\phi_X(x)$ is a Gaussian distribution, i.e.,

$$\phi_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$
$$\log_2 \phi_X(x) = \frac{1}{2}\log_2 \frac{1}{2\pi\sigma^2} - \left(\frac{x^2}{2\sigma^2}\right)\log_2 e,$$

and $f_X(x)$ and $g_X(x)$ are arbitrary distributions of the random variable X which lead to the same mean and variance. Then, we can use the following lemma from the script (formula 5.7):

$$\int_{-\infty}^{\infty} f_X(x)\log_2 \phi_X(x)\, dx = \int_{-\infty}^{\infty} g_X(x)\log_2 \phi_X(x)\, dx \tag{1}$$

For completeness we give the proof:

$$\int_{-\infty}^{\infty} f_X(x)\log_2 \phi_X(x)\, dx = \frac{1}{2}\log_2 \frac{1}{2\pi\sigma^2} \underbrace{\int_{-\infty}^{\infty} f_X(x)\, dx}_{1} - \frac{\log_2 e}{2\sigma^2} \underbrace{\int_{-\infty}^{\infty} f_X(x)x^2\, dx}_{E[X^2]=\sigma^2}$$
$$= \frac{1}{2}\log_2 \frac{1}{2\pi\sigma^2} \underbrace{\int_{-\infty}^{\infty} g_X(x)\, dx}_{1} - \frac{\log_2 e}{2\sigma^2} \underbrace{\int_{-\infty}^{\infty} g_X(x)x^2\, dx}_{E[X^2]=\sigma^2} = \int_{-\infty}^{\infty} g_X(x)\log_2 \phi_X(x)\, dx \qquad \square$$

With the lemma, it is easy to show that $\phi_X(x)$ maximizes the entropy:

$$0 \le D\Big(f_X(x)\,\Big|\Big|\,\phi_X(x)\Big) = \int_{-\infty}^{\infty} f_X(x)\log_2 f_X(x)\, dx \; - \int_{-\infty}^{\infty} f_X(x)\log_2 \phi_X(x)\, dx$$
$$\overset{(1)}{=} -H_f(X) - \int_{-\infty}^{\infty} \phi_X(x)\log_2 \phi_X(x)\, dx = -H_f(X) + H_\phi(X).$$

Thus, we have

$$H_f(X) \le H_\phi(X) = \frac{1}{2}\log_2(2\pi e\sigma^2)$$

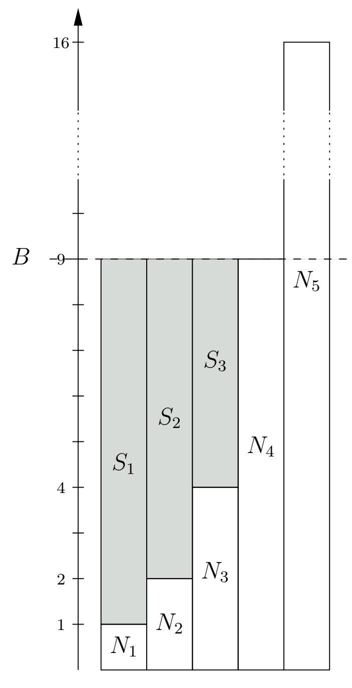
for any distribution $f_X(x)$ with the same mean and variance as $\phi_X(x)$.

Solution to Task 7.3 (Waterfilling)

a) Constant transmit power in all subchannels $S_1 = S_2 = \ldots = S_5 = 4$:

$$
\begin{aligned}
C &= \sum_{i=1}^{5} \frac{1}{2}\log_2(1+\frac{S_i}{N_i}) \\
&= \frac{1}{2}\log_2(1+\frac{4}{1}) + \frac{1}{2}\log_2(1+\frac{4}{2}) + \frac{1}{2}\log_2(1+\frac{4}{4}) + \frac{1}{2}\log_2(1+\frac{4}{9}) + \frac{1}{2}\log_2(1+\frac{4}{16}) \\
&= 2.8797
\end{aligned}
$$

b) Waterfilling:



Algorithm:

1. Sort the channels in ascending order according to their noise powers
2. Set all transmit powers to zero at the beginning:
   $S_1 = 0, S_2 = 0, \ldots S_n = 0$
3. Let E_{free} be the remaining transmit power, which is initialized with the total transmit power E:
   $E_{free} = E$

4. **for** $i = 1$ **to** $n - 1$ **do**
   if $(E_{free} - i \cdot (N_{i+1} - N_i) < 0$ **then break**;
   $E_{free} = E_{free} - i \cdot (N_{i+1} - N_i)$;
   end
   $B = N_i + E_{free}/i$;
5. **for** $i = 1$ **to** n **do**
   if $(N_i < B)$ **then** $S_i = B - N_i$
   else $S_i = 0$;
   end

i	E_{free}
1	$20 - 1 \cdot (2 - 1) = 19$
2	$19 - 2 \cdot (4 - 2) = 15$
3	$15 - 3 \cdot (9 - 4) = 0$
4	$0 - 4 \cdot (16 - 9) < 0$ (break!)

$B = N_4 + \frac{E_{free}}{i} = 9$

i	Condition	S_i
1	$1 < 9$	$9 - 1 = 8$
2	$2 < 9$	$9 - 2 = 7$
3	$4 < 9$	$9 - 4 = 5$
4	$9 < 9$	0
5	$16 < 9$	0

$C = \frac{1}{2}\left(\log_2(1 + \frac{8}{1}) + \log_2(1 + \frac{7}{2}) + \log_2(1 + \frac{5}{4})\right) = 3.2549$