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Geometrical Properties of Balls in Sum-Rank Metric and their Applications

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Outline

Preliminaries of the Sum-Rank Metric

Codes in Sum-Rank-Metric

- $\mathbb{F}_{q^m}$ Extension Field of $\mathbb{F}_q$
- Codelength $n = \eta \cdot \ell$ splitted into ℓ blocks, each of size η
- Linear Code $\mathcal{C} \subset \mathbb{F}_{q^m}^n$ subspace of dimension k

$$\boldsymbol{c} = [\ \underbrace{\boldsymbol{c}_1}_{\in \mathbb{F}_{q^m}^{\eta}} \mid \boldsymbol{c}_2 \mid \dots \mid \boldsymbol{c}_\ell\ ] \in \mathbb{F}_{q^m}^n$$

$$\boldsymbol{C} = [\ \underbrace{\boldsymbol{C}_1}_{\in \mathbb{F}_q^{m \times \eta}} \mid \boldsymbol{C}_2 \mid \dots \mid \boldsymbol{C}_\ell\ ] \in \mathbb{F}_q^{m \times n}$$

ℓ-sum-rank weight/distance:

$$\mathrm{wt}_{SR,\ell}(\boldsymbol{c}) := \sum_{i=1}^{\ell} \mathrm{rk}_{\mathbb{F}_q}(\boldsymbol{C}_i) \leq \ell \cdot \underbrace{\mu}_{:=\min\{m,\eta\}}$$

$$\mathrm{d}_{SR,\ell}(\boldsymbol{c}, \boldsymbol{c}') := \mathrm{wt}_{SR,\ell}(\boldsymbol{c} - \boldsymbol{c}')$$

Preliminaries of the Sum-Rank Metric

Relation between the different metrics

Relation between sum-rank weight and Hamming weight

For $x \in \mathbb{F}_{q^m}$ it holds that $\mathrm{wt}_{SR,\ell}(x) \leq \mathrm{wt}_{SR,n}(x)$.

Proof: $\boldsymbol{x} = [\boldsymbol{x}_1 | \dots | \boldsymbol{x}_\ell] \in \mathbb{F}_{q^m}^n$ with $\mathrm{wt}_{SR,n}(\boldsymbol{x}) = n - t = \eta - t_1 + \ldots + \eta - t_\ell$ where $\sum_{i=1}^{\ell} t_i = t$ and each $\boldsymbol{x}_i$ has t_i zero entries. For the sum-rank weight one gets $\mathrm{wt}_{SR,\ell}(\boldsymbol{x}) = \sum_{i=1}^{\ell} \mathrm{rk}_q(\boldsymbol{x}_i) \leq \sum_{i=1}^{\ell} \min\{m, \eta - t_i\} \leq \sum_{i=1}^{\ell} (\eta - t_i) = n - t = \mathrm{wt}_{SR,n}(\boldsymbol{x})$.

Preliminaries of the Sum-Rank Metric

Relation between the different metrics

Relation between sum-rank weight and rank weight

For $x \in \mathbb{F}_{q^m}$ it holds that $\mathrm{wt}_{SR,1}(x) \leq \mathrm{wt}_{SR,\ell}(x)$.

Proof: Assume w.l.o.g. $n \leq m$ and let $\mathrm{wt}_{SR,\ell}(\boldsymbol{x}) = t = t_1 + \ldots + t_\ell$ (i.e., each $\boldsymbol{x}_i$ has t_i $\mathbb{F}_q$-linearly independent columns for $i \in \{1, \ldots \ell\}$) $\Rightarrow$ $\boldsymbol{x}$ has at most t $\mathbb{F}_q$-linearly independent columns in the union of all blocks, which corresponds to the rank weight of $\boldsymbol{x}$.

Spheres and Balls in Sum-Rank-Metric

Let $\tau \in \mathbb{Z}_{\geq 0}$ with $0 \leq \tau \leq \ell \cdot \mu$ and $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$. The sum-rank-metric sphere with radius τ and center $\boldsymbol{x}$ is defined as

$$\mathcal{S}_\ell(\boldsymbol{x}, \tau) := \{\boldsymbol{y} \in \mathbb{F}_{q^m}^n \mid \mathrm{d}_{SR,\ell}(\boldsymbol{x}, \boldsymbol{y}) = \tau\}.$$

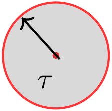
Analogously, we define the ball of sum-rank radius τ with center $\boldsymbol{x}$ by

$$\mathcal{B}_\ell(\boldsymbol{x}, \tau) := \bigcup_{i=0}^{\tau} \mathcal{S}_\ell(\boldsymbol{x}, i).$$

We also define the following cardinalities:

$$\mathrm{Vol}_{\mathcal{S}_\ell}(\tau) := |\{\boldsymbol{y} \in \mathbb{F}_{q^m}^n \mid \mathrm{wt}_{SR,\ell}(\boldsymbol{y}) = \tau\}|,$$

$$\mathrm{Vol}_{\mathcal{B}_\ell}(\tau) := \sum_{i=0}^{\tau} \mathrm{Vol}_{\mathcal{S}_\ell}(i).$$



Spheres and Balls in Sum-Rank-Metric

Volume of a Single Ball in the Sum-Rank Metric

The number of $m \times n$ matrices over $\mathbb{F}_q$ for a given rank $t \leq \min\{m, n\}$ is

$$\mathrm{NM}_q(n, m, t) := \begin{bmatrix} n \\ t \end{bmatrix}_q \cdot \prod_{i=0}^{t-1} (q^m - q^i)$$

(see e.g., [MMO04]), where $\begin{bmatrix} n \\ t \end{bmatrix}_q := \prod_{i=1}^{t} \frac{q^{n-t+i}-1}{q^i-1}$ denotes the q-Gaussian binomial coefficient, which is defined by the number of t-dimensional subspaces of $\mathbb{F}_q^n$.

Spheres and Balls in Sum-Rank-Metric

Volume of a Single Ball in the Sum-Rank Metric

The set of ordered partitions with bounded number of bounded summands is

$$\tau_{t,\ell,\mu} := \left\{ \boldsymbol{t} = [t_1, \ldots, t_\ell] \mid \sum_{i=1}^{\ell} t_i = t \wedge t_i \leq \mu, \forall i \right\} \ .$$

Its cardinality corresponds to the number of possibilities how to partition the sum-rank weight t of a vector into ℓ blocks of at most rank μ. By common combinatorial methods, we obtain (see also [Rat08, Lemma 1.1])

$$|\tau_{t,\ell,\mu}| = \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \binom{t+\ell-(\mu+1)i}{\ell-1} \leq \binom{t+\ell-1}{\ell-1}.$$

Spheres and Balls in Sum-Rank-Metric

Volume of a Single Ball in the Sum-Rank Metric

Volume of Spheres and Balls in Sum-Rank-Metric

$$\mathrm{Vol}_{\mathcal{S}_\ell}(t) = \sum_{\substack{\boldsymbol{t}=[t_1,\dots,t_\ell] \\ \in \mathcal{T}_{t,\ell,\mu}}} \prod_{i=1}^{\ell} \mathrm{NM}_q(\eta, m, t_i)$$

$$\mathrm{Vol}_{\mathcal{B}_\ell}(t) = \sum_{j=0}^{t} \sum_{\substack{\boldsymbol{t}=[t_1,\dots,t_\ell] \\ \in \mathcal{T}_{j,\ell,\mu}}} \prod_{i=1}^{\ell} \mathrm{NM}_q(\eta, m, t_i)$$

$\mathrm{Vol}_{\mathcal{S}_\ell}(t)$ can be computed with complexity $\tilde{\mathcal{O}}\big(\ell^2 t^3 + \ell d^t (m+\eta)\log(q)\big)$ using the efficient algorithm for computing $\mathrm{Vol}_{\mathcal{S}_\ell}$ in [PRR22, Theorem 6 and Algorithm 1].

Spheres and Balls in Sum-Rank-Metric

Bounds on the Volume of a Single Ball in the Sum-Rank Metric

Upper Bound

$$\mathrm{Vol}_{\mathcal{S}_\ell}(\rho) \leq \binom{\ell+\rho-1}{\ell-1}\gamma_q^\ell q^{\rho(m+\eta-\frac{\rho}{\ell})} \quad \text{[PRR20, Theorem 5].}$$

Since $\mathrm{Vol}_{\mathcal{B}_\ell}(\rho) = \sum_{\rho'=0}^{\rho} \mathrm{Vol}_{\mathcal{S}_\ell}(\rho') \leq \rho \mathrm{Vol}_{\mathcal{S}_\ell}(\rho)$ for $\rho > 1$, this gives an upper bound on $\mathrm{Vol}_{\mathcal{B}_\ell}(\rho)$.

Lower Bound [OPB21, Lemma 2]

$$\mathrm{Vol}_{\mathcal{B}_\ell}(t) \geq \mathrm{Vol}_{\mathcal{S}_\ell}(t) \geq q^{(m+\eta-\frac{t}{\ell})t-\frac{\ell}{4}} \cdot \gamma_q^{-\ell}.$$

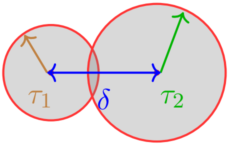
Spheres and Balls in Sum-Rank Metric

Intersection of two balls

We can define the volume of the intersection of two balls of radii $|\mathcal{B}_\ell(\boldsymbol{x}_1, \tau_1) \cap \mathcal{B}_\ell(\boldsymbol{x}_2, \tau_2)|$ independently of their centers but only dependent on their radii τ_1, τ_2 and the distance $\delta := \mathrm{d}_{SR,\ell}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ between their respective centers as follows:

$$\mathrm{Vol}_{\mathcal{I}_\ell}(\tau_1, \tau_2, \delta) := |\{\boldsymbol{y} \in \mathbb{F}_{q^m}^n | \mathrm{wt}_{SR,\ell}(\boldsymbol{y}) \leq \tau_1 \wedge \mathrm{d}_{SR,\ell}(\boldsymbol{y}, \boldsymbol{d}) \leq \tau_2\}|,$$

where $\boldsymbol{d} \in \mathbb{F}_{q^m}^n$ arbitrary but fix with $\mathrm{wt}_{SR,\ell}(\mathrm{d}) = \delta$. Obviously if $\delta > \tau_1 + \tau_2$, then $\mathrm{Vol}_{\mathcal{I}_\ell}(\tau_1, \tau_2, \delta) = 0$.



Spheres and Balls in Sum-Rank Metric

Volume of the Intersection of two Balls in rank metric

In [GY09] the number of vectors lying in the intersection of two spheres in the rank metric of radii u and s and distance t between their centers was derived, that is

$$\mathcal{J}(u,s,t,n,m) := \frac{\sum_{i=0}^{n} \mathrm{NM}_q(n,m,i)\mathcal{K}_u(i,n,m)\mathcal{K}_s(i,n,m)\mathcal{K}_t(i,n,m)}{q^{mn}\mathrm{NM}_q(n,m,t)},$$

where $\mathcal{K}_j(i,n,m)$ is a q-Krawtchouk polynomial (see [Del76]) and defined as

$$\mathcal{K}_j(i,n,m) := \sum_{l=0}^{j} (-1)^{j-l} q^{lm+\binom{j-l}{2}} \begin{bmatrix} n-l \\ n-j \end{bmatrix}_q \begin{bmatrix} n-i \\ l \end{bmatrix}_q .$$

Spheres and Balls in Sum-Rank Metric

Volume of the Intersection of two Balls in rank metric

Moreover the cardinality of all vectors lying in the intersection of two balls with rank metric radii u and s and distance t between their centers was given in [GY09],

$$\mathcal{I}(u,s,t,n,m) \coloneqq \sum_{i=0}^{u}\sum_{j=0}^{s}\mathcal{J}(i,j,t,n,m).$$

Spheres and Balls in Sum-Rank Metric

Volume of the Intersection of two Balls in sum-rank metric

[OLWZ23, Theorem 1]

Let u, s, t be positive integers such that $u + s \geq t$. The number of vectors $\boldsymbol{v} \in \mathbb{F}_{q^m}^n$ lying in the intersection of two balls with sum-rank radii u and s and sum-rank distance t between their centers is

$$\mathrm{Vol}_{\mathcal{I}_\ell}(u, s, t) = \sum_{\substack{\boldsymbol{u}=[u_1,\ldots,u_\ell] \\ \in \mathcal{T}_{u,\ell,\mu}}} \sum_{\substack{\boldsymbol{s}=[s_1,\ldots,s_\ell] \\ \in \mathcal{T}_{s,\ell,\mu}}} \sum_{\substack{\boldsymbol{t}=[t_1,\ldots,t_\ell] \\ \in \mathcal{T}_{t,\ell,\mu}}} \prod_{i=1}^{\ell} \mathcal{I}(u_i, s_i, t_i, \eta, m).$$

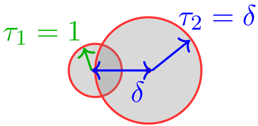
Spheres and Balls in Sum-Rank Metric

Volume of the Intersection of two Balls: Special case 1

[OLWZ23, Theorem 2]

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_{q^m}^n$ such that $\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \boldsymbol{y}) = \delta$. Then

$$\begin{aligned}\mathrm{Vol}_{\mathcal{I}_\ell}(, \delta, 1, \delta) =& |\mathcal{B}_\ell(\boldsymbol{x}, , \delta) \cap \mathcal{B}_\ell(\boldsymbol{y}, 1|\\ =& 1 + \frac{(q^m - 1)(q^n - 1)}{q - 1}\\ & - \sum_{\substack{\boldsymbol{\delta}=[\delta_1,\ldots,\delta_\ell]\\ \in \mathcal{T}_{\delta,\ell,\mu}}} \sum_{i=1}^{\ell} \frac{(q^\eta - q^{\delta_i}) \cdot (q^m - q^{\delta_i})}{q - 1} .\end{aligned}$$



Spheres and Balls in Sum-Rank Metric

Volume of the Intersection of two Balls: Special case 2

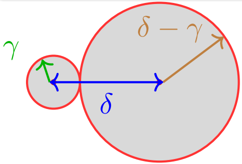
[OLWZ23, Theorem 1]

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_{q^m}^n$ such that $\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \boldsymbol{y}) = \delta$ then

$$\mathrm{Vol}_{\mathcal{I}_\ell}(\gamma, \delta - \gamma, \delta) = |\mathcal{B}_\ell(\boldsymbol{x}, \gamma) \cap \mathcal{B}_\ell(\boldsymbol{y}, \delta - \gamma)|$$

$$= \sum_{\substack{\boldsymbol{\delta}=[\delta_1,\dots,\delta_\ell] \\ \in \mathcal{T}_{\delta,\ell,\mu}}} \sum_{\substack{\boldsymbol{\gamma}=[\gamma_1,\dots,\gamma_\ell] \\ \in \mathcal{T}_{\gamma,\ell,\boldsymbol{\delta}}}} \sum_{i=1}^{\ell} q^{\gamma_i \cdot (\delta_i - \gamma_i)} \cdot \begin{bmatrix} \delta_i \\ \gamma_i \end{bmatrix}_q$$

for $0 \leq \gamma \leq \delta$.



Bounds on Codes in Sum-Rank-Metric

Sphere-Packing Bound

Sphere-Packing Bound[BGLR20, Theorem III.6]

For a linear $[n, k, d]$ sum-rank metric code $\mathcal{C}$, it holds that

$$q^{mk} \cdot \mathrm{Vol}_{\mathcal{B}_\ell}\left(\left\lfloor \frac{d-1}{2} \right\rfloor\right) \leq q^{mn}.$$

Both sides of the bounds can be computed in complexity $\tilde{\mathcal{O}}\left(\ell^2 d^3 + \ell d^4 (m+\eta)\log(q)\right)$ using the efficient algorithm for computing $\mathrm{Vol}_{\mathcal{S}_\ell}$ in [PRR20, Theorem 5 and Algorithm 1].

simplified Sphere-Packing Bound[OPB21, Theorem 4]

For a linear $[n, k, d]$ sum-rank metric code $\mathcal{C}$, the parameters fulfill

$$q^{mk} \cdot q^{(m+\eta-\frac{1}{\ell}\lfloor\frac{d-1}{2}\rfloor)\lfloor\frac{d-1}{2}\rfloor-\frac{\ell}{4}} \cdot \gamma_q^{-\ell} \leq q^{mn}.$$

Bounds on Codes in Sum-Rank-Metric

Sphere-Packing Bound

asymptotic Sphere-Packing Bound[OPB21, Theorem 5]

Let $\mathcal{C}$ be a linear $[n, k, d]$ sum-rank metric code and $\delta := \frac{d}{n}$ the relative minimum distance. Then the code rate $\mathcal{R} = \frac{k}{n}$ is upper bounded by

$$\mathcal{R} < \delta^2 \frac{\eta}{4m} - \delta\left(\frac{1}{2} + \frac{\eta}{m}\left(\frac{1}{2} + \frac{1}{n}\right)\right) + \frac{1}{n}\left(1 + \frac{\eta}{m} + \frac{\eta}{nm}\right) + \frac{1}{\eta m}\left(\frac{1}{4} + \log_q(\gamma_q)\right) + 1.$$

Let $\xi > 0$ be fixed.

- For $m = \eta\xi \to \infty$ we get $\mathcal{R} \sim \delta^2 \frac{1}{4\xi} - \frac{\delta}{2}\left(1 + \frac{1}{\xi}\right) + 1.$
- For $\ell \to \infty$ one get $\mathcal{R} \sim \delta^2 \frac{\eta}{4m} - \frac{\delta}{2}\left(1 + \frac{\eta}{m}\right) + \frac{1}{\eta m}\left(\frac{1}{4} + \log_q(\gamma_q)\right) + 1.$

Bounds on Codes in Sum-Rank-Metric

Gilbert–Varshamov Bound

Gilbert–Varshamov Bound[BGLR20, Theorem III.11]]

Let $\mathbb{F}_{q^m}$ be a finite field, $\ell, n, k, d \leq \mu\ell$ be positive integers that satisfy

$$q^{m(k-1)} \cdot \mathrm{Vol}_{\mathcal{B}_\ell}(d-1) < q^{mn}.$$

Then, there is a linear code of length n, dimension k, and minimum ℓ-sum-rank distance at least d.

simplified Gilbert–Varshamov Bound [OPB21, Theorem 6]

Let $\mathbb{F}_{q^m}$ be a finite field, ℓ, n, k, d be positive integers with $2 < d \leq \mu\ell$ that satisfy

$$q^{m(k-1)} \cdot (d-1)\binom{\ell+d-2}{\ell-1}\gamma_q^\ell q^{(d-1)(m+\eta-\frac{d-1}{\ell})} < q^{mn}.$$

Then, there is a linear code of length n, dimension k, and minimum ℓ-sum-rank distance at least d.

Bounds on Codes in Sum-Rank-Metric

Gilbert–Varshamov Bound

asymptotic Gilbert–Varshamov Bound [OPB21, Theorem 7]

For a finite field $\mathbb{F}_{q^m}$ and positive integers $\ell, n, \mathcal{R}n, d$ with $\delta := \frac{d}{n}$ and $2 < d \leq \mu\ell$ satisfying

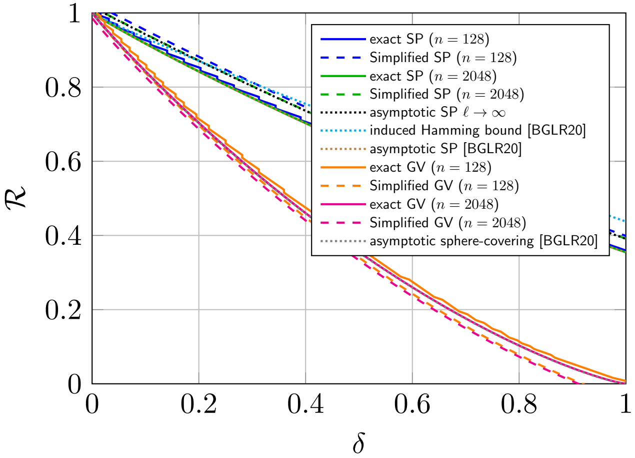
$$\mathcal{R} \leq \delta^2 \frac{\eta}{m} - \delta\left(1 + \frac{\eta}{m} + \frac{2\eta}{nm}\right) + 1 + \frac{1}{n} + \frac{\eta}{nm} + \frac{\eta}{n^2 m} - \frac{\sum_{i=1}^{\delta n - 1} \log_q\left(1 + \frac{\ell - 1}{i}\right) + \log_q(\delta n - 1)}{mn} - \frac{\log_q(\gamma_q)}{\eta m}$$

there exists a linear ℓ-sum-rank metric code of rate $\mathcal{R}$ and relative minimum sum-rank distance at least δ. Let ξ be a constant. For $m = \eta\xi \to \infty$ and $m \in \omega(\log_q(\ell))$ we have

$$\mathcal{R} \sim \delta^2 \frac{1}{\xi} - \delta\left(1 + \frac{1}{\xi}\right) + 1.$$

Bounds on Codes in Sum-Rank-Metric

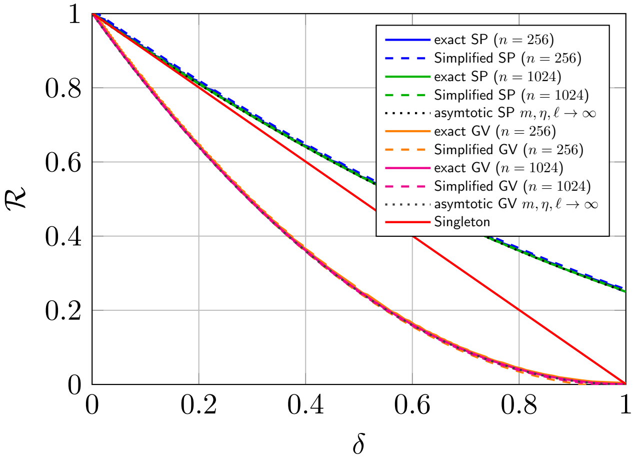
Numerical Comparisons: Bounded Blocksize



Comparison of different bounds for fixed value $q = 2$ $\eta = 8$ $m = 16$ for different values of n ($\ell = \frac{n}{\eta}$)

Simplified and Asymptotic Bounds

Numerical Comparisons: Growing Blocksize



Comparison of different bounds for fixed value of $q = 16$ and different values of n with $\eta = \ell = m$

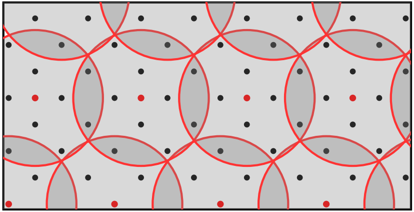
Covering Properties

Covering radius

Definition

Let $\mathcal{C}$ be a linear $[n, k, d]$ sum-rank metric code over $\mathbb{F}_{q^m}$. The covering radius of $\mathcal{C}$ is the smallest integer $\rho_{SR,\ell}$ such that every vector $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$ has at most sum-rank distance $\rho_{SR,\ell}$ to some codeword $\boldsymbol{c} \in \mathcal{C}$ i.e., $\rho_{SR,\ell} = \max_{\boldsymbol{x} \in \mathbb{F}_{q^m}^n} \{\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C})\}$.

$\mathbb{F}_{q^m}^n$



- $\in \mathbb{F}_{q^m}^n$
- $\in \mathcal{C}$

Covering Properties

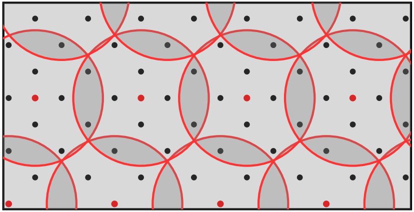
Covering Problem for the sum-rank metric

Ee denote the minimum cardinality of a code $\mathcal{C} \subset \mathbb{F}_{q^m}^n$ with sum-rank covering radius ρ by $\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho)$.

Problem

Find the minimum number of sum-rank balls $\mathcal{B}_\ell(\boldsymbol{x}, \rho)$ of radius ρ (with $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$) that cover the space $\mathbb{F}_{q^m}^n$ entirely. This problem is equivalent to determining the minimum cardinality $\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho)$ of a code $\mathcal{C} \subset \mathbb{F}_{q^m}^n$ with sum-rank covering radius ρ.

$\mathbb{F}_{q^m}^n$



- $\in \mathbb{F}_{q^m}^n$
- $\in \mathcal{C}$

Covering Properties

Covering Problem: Extreme Cases

There are two extreme cases for the covering radius:

(i) $\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, 0) = q^{mn}$, since from $\rho_{SR,\ell} = \max_{x \in \mathbb{F}_{q^m}^n} \{\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C})\} = 0$ it follows that $\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C}) = 0, \forall \boldsymbol{x} \in \mathbb{F}_{q^m}^n$ and therefore $\boldsymbol{x} \in \mathcal{C}$, i.e., $\mathcal{C} = \mathbb{F}_{q^m}^n$.

(ii) $\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \mu\ell) = 1$. Consider $\rho_{SR,\ell} = \max_{x \in \mathbb{F}_{q^m}^n} \{\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C})\} = \mu \cdot \ell$ which means that there exists an $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$ such that $\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C}) = \mu \cdot \ell$. This is already fulfilled by choosing $\mathcal{C} = \{(0, \ldots, 0)\}$.

Covering Properties

Covering radii in different metrics

Relation of covering radii in diffent metrics [OLWZ22, Lemma 1]

Let $\mathcal{C} \subset \mathbb{F}_{q^m}^n$ then it holds for its corresponding covering radii $\rho_{SR,1}$, $\rho_{SR,\ell}$ and $\rho_{SR,n}$ in the rank, the sum-rank and the Hamming metric that

$$\rho_{SR,1} \leq \rho_{SR,\ell} \leq \rho_{SR,n}.$$

Proof.

Since $\mathrm{wt}_{SR,1}(\boldsymbol{x}) \leq \mathrm{wt}_{SR,\ell}(\boldsymbol{x}) \leq \mathrm{wt}_{SR,n}(\boldsymbol{x})$ for a fix $\boldsymbol{x} \in \mathbb{F}_{q^m}$ it follows that $\mathrm{d}_{SR,1}(\boldsymbol{x}, \mathcal{C}) \leq \mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C}) \leq \mathrm{d}_{SR,n}(\boldsymbol{x}, \mathcal{C})$ and hence $\max_{\boldsymbol{x} \in \mathbb{F}_{q^m}^n} \{\mathrm{d}_{SR,1}(\boldsymbol{x}, \mathcal{C})\} \leq \max_{\boldsymbol{x} \in \mathbb{F}_{q^m}^n} \{\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \mathcal{C})\} \leq \max_{\boldsymbol{x} \in \mathbb{F}_{q^m}^n} \{\mathrm{d}_{SR,n}(\boldsymbol{x}, \mathcal{C})\}$. □

Covering Properties

Comparison of the different metrics

Relation of $\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho)$ in different metrics [OLWZ22, Theorem 2]

For $0 < \rho < \mu \cdot \ell$, it holds
$\mathcal{K}_{SR,1}(\mathbb{F}_{q^m}^n, \rho) \leq \mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho) \leq \mathcal{K}_{SR,n}(\mathbb{F}_{q^m}^n, \rho)$.

Lower Bounds for the Sphere Covering Problem

Sphere Covering Bound

Sphere Covering Bound

For the minimum cardinality of a code $\mathcal{C} \subset \mathbb{F}_{q^m}^n$ with sum-rank covering radius $0 < \rho < \mu \cdot \ell$ the following inequality holds:

$$\frac{q^{mn}}{\mathrm{Vol}_{\mathcal{B}_\ell}(\rho)} \leq \mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho).$$

Simplified Sphere Covering Bound [OLWZ22, Theorem 4]

For $0 < \rho < \mu \cdot \ell$ the following inequality holds:

$$\frac{q^{mn-\rho(m+\eta-\frac{\rho}{\ell})}}{\rho \cdot \binom{\ell+\rho-1}{\ell-1}\gamma_q^\ell} \leq \mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho)$$

Lower Bounds for the Sphere Covering Problem

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Theorem [OLWZ22, Theorem 6]

Let $0 < \rho < \mu \cdot \ell$ and $0 < k \leq \lfloor \log_{q^m}(\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho)) \rfloor$ then

$$\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho) \geq \frac{1}{\mathrm{Vol}_{\mathcal{B}_\ell}(\rho) - \mathrm{Vol}_{\mathcal{I}_\ell}(\rho, \rho, \mu\ell - \frac{\mu}{\eta}k)}$$

$$\cdot \Big(q^{mn} - q^{km} \mathrm{Vol}_{\mathcal{I}_\ell}(\rho, \rho, \mu\ell - \frac{\mu}{\eta}k) + \mathrm{Vol}_{\mathcal{I}_\ell}(\rho, \rho, \mu\ell - \frac{\mu}{\eta}k + 1)$$

$$\cdot \textstyle\sum_{k'=\max\{1, n-2\frac{\eta}{\mu}\rho+1\}}^{k} (q^{k'm} - q^{(k'-1)m}) \Big).$$

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