Geometrical Properties of Balls in Sum-Rank Metric

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Outline

NEW

Preliminaries of the Sum-Rank Metric

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ℓ-sum-rank weight/distance:

$$\mathrm{wt}_{SR,\ell}(\boldsymbol{c}) := \sum_{i=1}^{\ell} \mathrm{rk}_{\mathbb{F}_q}(\boldsymbol{C}_i) \leq \ell \cdot \underbrace{\mu}_{:=\min\{m,\eta\}}$$

$$\mathrm{d}_{SR,\ell}(\boldsymbol{c}, \boldsymbol{c}') := \mathrm{wt}_{SR,\ell}(\boldsymbol{c} - \boldsymbol{c}')$$

Relation between sum-rank weight and Hamming weight

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For the sum-rank weight one gets $\mathrm{wt}_{SR,\ell}(\boldsymbol{x}) = \sum_{i=1}^{\ell} \mathrm{rk}_q(\boldsymbol{x}_i) \leq \sum_{i=1}^{\ell} \min\{m, \eta - t_i\} \leq \sum_{i=1}^{\ell} (\eta - t_i) = n - t = \mathrm{wt}_{SR,n}(\boldsymbol{x})$.

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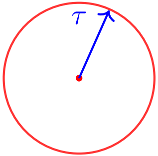
$\Rightarrow$ $\boldsymbol{x}$ has at most t $\mathbb{F}_q$-linearly independent columns in the union of all blocks, which corresponds to the rank weight of $\boldsymbol{x}$.

Spheres and Balls in Sum-Rank-Metric

Let $\tau \in \mathbb{Z}_{\geq 0}$ with $0 \leq \tau \leq \ell \cdot \mu$ and $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$. A sphere in sum-rank metric with radius τ and center $\boldsymbol{x}$ is defined as

$$\mathcal{S}_\ell(\boldsymbol{x}, \tau) := \{\boldsymbol{y} \in \mathbb{F}_{q^m}^n \mid \mathrm{d}_{SR,\ell}(\boldsymbol{x}, \boldsymbol{y}) = \tau\}.$$



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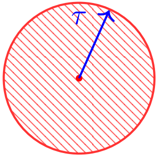
Analogously, we define a ball of sum-rank radius τ with center $\boldsymbol{x}$ by

$$\mathcal{B}_\ell(\boldsymbol{x}, \tau) := \bigcup_{i=0}^{\tau} \mathcal{S}_\ell(\boldsymbol{x}, i).$$



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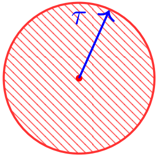

We also define the following cardinalities:

$$\mathrm{Vol}_{\mathcal{S}_\ell}(\tau) := |\{\boldsymbol{y} \in \mathbb{F}_{q^m}^n \mid \mathrm{wt}_{SR,\ell}(\boldsymbol{y}) = \tau\}|,$$

$$\mathrm{Vol}_{\mathcal{B}_\ell}(\tau) := \sum_{i=0}^{\tau} \mathrm{Vol}_{\mathcal{S}_\ell}(i).$$

The number of $m \times n$ matrices over $\mathbb{F}_q$ for a given rank $t \leq \min\{m, n\}$ is

$$\mathrm{NM}_q(n, m, t) := \begin{bmatrix} n \\ t \end{bmatrix}_q \cdot \prod_{i=0}^{t-1} (q^m - q^i)$$

(see e.g., [MMO04]), where $\begin{bmatrix} n \\ t \end{bmatrix}_q := \prod_{i=1}^{t} \frac{q^{n-t+i}-1}{q^i-1}$ denotes the q-Gaussian binomial coefficient, which is defined by the number of t-dimensional subspaces of $\mathbb{F}_q^n$.

The set of ordered partitions with bounded number of bounded terms is

$$\tau_{t,\ell,\mu} := \left\{ \boldsymbol{t} = [t_1, \ldots, t_\ell] \mid \sum_{i=1}^{\ell} t_i = t \wedge t_i \leq \mu, \forall i \right\} .$$

Its cardinality corresponds to the number of possibilities how to partition the sum-rank weight t of a vector into ℓ blocks of at most rank μ. By common combinatorial methods, we obtain (see also [Rat08, Lemma 1.1])

$$\left|\tau_{t,\ell,\mu}\right| = \sum_{i=0}^{\lfloor \frac{t}{\mu+1} \rfloor} (-1)^i \binom{\ell}{i} \binom{t+\ell-1-(\mu+1)i}{\ell-1} \leq \binom{t+\ell-1}{\ell-1}.$$

Example: $t = 5, \mu = 4, \ell = 3$

$$\tau_{5,3,4} \coloneqq \left\{ \boldsymbol{t} = [t_1, t_2, t_3] \mid \sum_{i=1}^{3} t_i = 5 \wedge t_i \leq 4, \forall i \right\} .$$

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$2+2+1=5$	$3+1+1=5$	$4+1+0=5$	$0+2+3=5$
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$$|\tau_{t,\ell,\mu}| = \sum_{i=0}^{\lfloor \frac{t}{\mu+1} \rfloor} (-1)^i \binom{\ell}{i} \binom{t+\ell-1-(\mu+1)i}{\ell-1} = \sum_{i=0}^{1} (-1)^i \binom{3}{i} \binom{7-5i}{2} = \binom{7}{2} - 3\binom{2}{2} = \frac{7 \cdot 6}{2} - 3 = 18.$$

Volume of Spheres and Balls in Sum-Rank-Metric

$$\mathrm{Vol}_{\mathcal{S}_\ell}(t) = \sum_{\substack{\boldsymbol{t}=[t_1,\ldots,t_\ell] \\ \in \mathcal{T}_{t,\ell,\mu}}} \prod_{i=1}^{\ell} \mathrm{NM}_q(\eta, m, t_i)$$

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$\mathrm{Vol}_{\mathcal{S}_\ell}(t)$ can be computed with complexity $\tilde{\mathcal{O}}\left(\ell^2 t^3 + \ell d^t (m+\eta)\log(q)\right)$ using the efficient algorithm for computing $\mathrm{Vol}_{\mathcal{S}_\ell}$ in [PRR22, Theorem 6 and Algorithm 1].

Upper Bound

$$\mathrm{Vol}_{\mathcal{S}_\ell}(\rho) \leq \binom{\ell+\rho-1}{\ell-1} \gamma_q^\ell q^{\rho(m+\eta-\frac{\rho}{\ell})} \quad \text{[PRR20, Theorem 5].}$$

Since $\mathrm{Vol}_{\mathcal{B}_\ell}(\rho) = \sum_{\rho'=0}^{\rho} \mathrm{Vol}_{\mathcal{S}_\ell}(\rho') \leq \rho \mathrm{Vol}_{\mathcal{S}_\ell}(\rho)$ for $\rho > 1$, this gives an upper bound on $\mathrm{Vol}_{\mathcal{B}_\ell}(\rho)$.

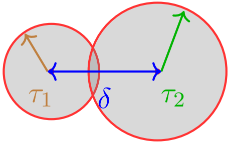
Lower Bound [OPB21, Lemma 2]

$$\mathrm{Vol}_{\mathcal{B}_\ell}(t) \geq \mathrm{Vol}_{\mathcal{S}_\ell}(t) \geq q^{(m+\eta-\frac{t}{\ell})t-\frac{\ell}{4}} \cdot \gamma_q^{-\ell}.$$

We can define the volume of the intersection of two balls $|\mathcal{B}_\ell(\boldsymbol{x}_1, \tau_1) \cap \mathcal{B}_\ell(\boldsymbol{x}_2, \tau_2)|$ independently of their centers but only dependent on their radii τ_1, τ_2 and the distance $\delta := \mathrm{d}_{SR,\ell}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ between their respective centers as follows:

$$\mathrm{Vol}_{\mathcal{I}_\ell}(\tau_1, \tau_2, \delta) := |\{\boldsymbol{y} \in \mathbb{F}_{q^m}^n | \mathrm{wt}_{SR,\ell}(\boldsymbol{y}) \leq \tau_1 \wedge \mathrm{d}_{SR,\ell}(\boldsymbol{y}, \boldsymbol{d}) \leq \tau_2\}|,$$

where $\boldsymbol{d} \in \mathbb{F}_{q^m}^n$ arbitrary but fix with $\mathrm{wt}_{SR,\ell}(\mathrm{d}) = \delta$. Obviously if $\delta > \tau_1 + \tau_2$, then $\mathrm{Vol}_{\mathcal{I}_\ell}(\tau_1, \tau_2, \delta) = 0$.



In [GY09] the number of vectors lying in the intersection of two spheres in the rank metric of radii u and s and distance t between their centers was derived, that is

$$\mathcal{J}(u,s,t,n,m) := \frac{\sum_{i=0}^{n} \mathrm{NM}_q(n,m,i)\mathcal{K}_u(i,n,m)\mathcal{K}_s(i,n,m)\mathcal{K}_t(i,n,m)}{q^{mn}\mathrm{NM}_q(n,m,t)},$$

where $\mathcal{K}_j(i,n,m)$ is a q-Krawtchouk polynomial (see [Del76]) and defined as

$$\mathcal{K}_j(i,n,m) := \sum_{l=0}^{j}(-1)^{j-l}q^{lm+\binom{j-l}{2}} \begin{bmatrix} n-l \\ n-j \end{bmatrix}_q \begin{bmatrix} n-i \\ l \end{bmatrix}_q .$$

Moreover the cardinality of all vectors lying in the intersection of two balls with rank metric radii u and s and distance t between their centers was given in [GY09],

$$\mathcal{I}(u,s,t,n,m) := \sum_{i=0}^{u}\sum_{j=0}^{s}\mathcal{J}(i,j,t,n,m).$$

New Result

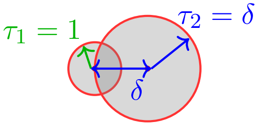
Let u, s, t be positive integers such that $u + s \geq t$. The number of vectors $\boldsymbol{v} \in \mathbb{F}_{q^m}^n$ lying in the intersection of two balls with sum-rank radii u and s and sum-rank distance t between their centers is

$$\mathrm{Vol}_{\mathcal{I}_\ell}(u,s,t) = \sum_{\substack{\boldsymbol{u}=[u_1,\dots,u_\ell] \\ \in \mathcal{T}_{u,\ell,\mu}}} \sum_{\substack{\boldsymbol{s}=[s_1,\dots,s_\ell] \\ \in \mathcal{T}_{s,\ell,\mu}}} \sum_{\substack{\boldsymbol{t}=[t_1,\dots,t_\ell] \\ \in \mathcal{T}_{t,\ell,\mu}}} \prod_{i=1}^{\ell} \mathcal{I}(u_i, s_i, t_i, \eta, m).$$

New Result

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_{q^m}^n$ such that $\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \boldsymbol{y}) = \delta$. Then

$$\begin{aligned}\mathrm{Vol}_{\mathcal{I}_\ell}(, \delta, 1, \delta) =& |\mathcal{B}_\ell(\boldsymbol{x}, , \delta) \cap \mathcal{B}_\ell(\boldsymbol{y}, 1|\\ =& 1 + \frac{(q^m-1)(q^n-1)}{q-1}\\ &- \sum_{\substack{\boldsymbol{\delta}=[\delta_1,\dots,\delta_\ell]\\ \in \tau_{\delta,\ell,\mu}}} \sum_{i=1}^{\ell} \frac{(q^\eta - q^{\delta_i}) \cdot (q^m - q^{\delta_i})}{q-1}.\end{aligned}$$



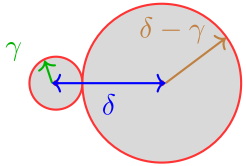
New Result

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_{q^m}^n$ such that $\mathrm{d}_{SR,\ell}(\boldsymbol{x}, \boldsymbol{y}) = \delta$ then

$$\mathrm{Vol}_{\mathcal{I}_\ell}(\gamma, \delta - \gamma, \delta) = |\mathcal{B}_\ell(\boldsymbol{x}, \gamma) \cap \mathcal{B}_\ell(\boldsymbol{y}, \delta - \gamma)|$$

$$= \sum_{\substack{\boldsymbol{\delta}=[\delta_1,\dots,\delta_\ell] \\ \in \mathcal{T}_{\delta,\ell,\mu}}} \sum_{\substack{\boldsymbol{\gamma}=[\gamma_1,\dots,\gamma_\ell] \\ \in \mathcal{T}_{\gamma,\ell,\boldsymbol{\delta}}}} \sum_{i=1}^{\ell} q^{\gamma_i \cdot (\delta_i - \gamma_i)} \cdot \begin{bmatrix} \delta_i \\ \gamma_i \end{bmatrix}_q$$

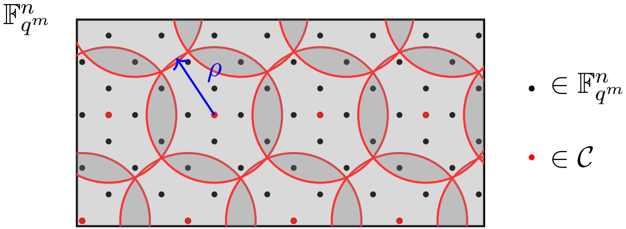
for $0 \leq \gamma \leq \delta$.



Application: Covering Problem

Problem

Find the minimum number of sum-rank balls $\mathcal{B}_\ell(\boldsymbol{x}, \rho)$ of radius ρ (with $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$) that cover the space $\mathbb{F}_{q^m}^n$ entirely. We denote this cardinality by $\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho)$.



Lower Bounds for the Sphere Covering Problem

Sphere Covering Bound

For the minimum cardinality of a code $\mathcal{C} \subset \mathbb{F}_{q^m}^n$ with sum-rank covering radius $0 < \rho < \mu \cdot \ell$ the following inequality holds:

$$\frac{q^{mn}}{\mathrm{Vol}_{\mathcal{B}_\ell}(\rho)} \leq \mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho).$$

Simplified Sphere Covering Bound [OLWZ22, Theorem 4]

For $0 < \rho < \mu \cdot \ell$ the following inequality holds:

$$\frac{q^{mn-\rho(m+\eta-\frac{\rho}{\ell})}}{\rho \cdot \binom{\ell+\rho-1}{\ell-1}\gamma_q^\ell} \leq \mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho)$$

Theorem [OLWZ22, Theorem 6]

Let $0 < \rho < \mu \cdot \ell$ and $0 < k \leq \lfloor \log_{q^m}(\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho)) \rfloor$ then

$$\mathcal{K}_{SR,\ell}(\mathbb{F}_{q^m}^n, \rho) \geq \frac{1}{\mathrm{Vol}_{\mathcal{B}_\ell}(\rho) - \mathrm{Vol}_{\mathcal{I}_\ell}(\rho, \rho, \mu\ell - \frac{\mu}{\eta}k)}$$
$$\cdot \Big(q^{mn} - q^{km}\mathrm{Vol}_{\mathcal{I}_\ell}(\rho, \rho, \mu\ell - \frac{\mu}{\eta}k) + \mathrm{Vol}_{\mathcal{I}_\ell}(\rho, \rho, \mu\ell - \frac{\mu}{\eta}k + 1)$$
$$\cdot \textstyle\sum_{k'=\max\{1, n-2\frac{\eta}{\mu}\rho+1\}}^{k} (q^{k'm} - q^{(k'-1)m})\Big).$$

Conclusion

- $\mathrm{wt}_{SR,1}(\boldsymbol{x}) \leq \mathrm{wt}_{SR,\ell}(\boldsymbol{x}) \leq \mathrm{wt}_{SR,n}(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$
- exact Volume of a Ball in sum-rank metric
  - $\rightarrow$ Bounds on the Volume of a sum-rank Ball

NEW

- exact Volume of the Intersection of two balls in sum-rank metric
- two special cases
- Application: Bounds on the Covering Problem

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