Advanced Factorization Strategies for Lattice-Reduction-Aided Preequalization

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***Abstract*—Lattice-reduction-aided preequalization (LRA PE) is a powerful technique for interference handling on the multi-user multiple-input/multiple-output (MIMO) broadcast channel. However, recent advantages in the strongly related field of compute-and-forward and integer-forcing equalization have raised the question, if the factorization task present in LRA PE is really solved in an optimum way. In this paper, advanced factorization strategies are presented, significantly increasing the transmission performance. Specifically, the signal constellation and its related lattice as well as the factorization task/strategy are discussed. The impact of dropping the common unimodularity constraint in LRA PE is studied. Numerical simulations are given to show the effectiveness of all presented strategies.**

I. Introduction

Multiple-input/multiple-output (MIMO) transmission has become one of the most important approaches in state-of-the-art digital communication systems. In research, a special interest is still on multi-user MIMO techniques. Thereby, in downlink transmission several users are simultaneously supplied by a joint transmitter. This is known as *MIMO broadcast channel*.

In order to cope with the interference present on the MIMO broadcast channel, well-known schemes like linear preequalization or Tomlinson-Harashima precoding (THP) have initially been applied. However, alternative strategies had to be found in order to overcome the limitation to diversity order one. Since it achieves the MIMO channel's diversity [15], *lattice-reduction-aided* equalization [19] has soon become popular—first for receiver-side equalization (uplink). Via uplink-downlink duality [17], the related transmitter-side techniques lattice-reduction-aided preequalization (LRA PE) and precoding could be derived [18], [13]. In LRA (pre)equalization, the inversion of the channel matrix is performed in a *suited basis*. This basis is usually achieved by a decomposition of the channel matrix into a reduced one and an unimodular (complex) integer matrix via the Lenstra-Lenstra-Lovász (LLL) algorithm (*shortest basis problem*).

Advantages in the similar field of *compute-and-forward* (CF) [8] and *integer-forcing* (IF) (pre)equalization [20], [7] have recently found their way into LRA equalization: when instead solving the *shortest independent vector problem* as in IF, the usually requested unimodularity of the integer matrix is dropped. For LRA equalization on the multiple-access channel, this may result in a gain [5]. In [12], this strategy has been adapted to LRA PE, however, with a restriction to special *algebraic signal constellations* not suited for direct mapping from bits to signal points (as, e.g., for square-QAM ones).

In this paper, advanced factorization strategies suitable for *conventional* LRA PE are presented, i.e., a direct bit-mapping to 2^m-ary signal constellations is possible ($m \in \mathbb{N}$). In particular, constellations over the (complex) hexagonal lattice are contrasted to equivalent square-QAM ones w.r.t. the channel factorization. Different factorization strategies are reviewed and/or dualized from receiver-side LRA equalization. The effect on (non-algebraic) constellations when dropping the unimodularity constraint in LRA PE is studied and the relation of the related factorization task to the shortest independent vector problem is enlightened. The presented strategies are covered by means of numerical simulations.

The paper is structured as follows: In Sec. II, the system model for LRA PE is specified. Advanced factorization strategies are both reviewed and proposed in Sec. III. Sec. IV provides numerical performance evaluations. A summary of the paper's content and conclusions are given in Sec. V.

II. System Model

In LRA (pre)equalization—contrary to IF—channel coding and equalization can be completely decoupled, i.e., channel coding can be applied on top of channel equalization. Hence, we concentrate on the handling of the multi-user interference, which is equivalent to treating an uncoded system. As usual, a discrete-time complex-baseband transmission over the MIMO broadcast channel is considered: a joint transmitter (N antennas) is assumed to supply K single-antenna non-cooperating receivers ($N \geq K$). The related system model, depicted in Fig. 1, is detailed in the following.[1]

A. Transmitter-Side Processing

Independent streams of binary source symbols (bits) have to be transmitted to their assigned user $k = 1, \ldots, K$. In each modulation step, blocks of bits $\boldsymbol{q}_k = [q_{k,1}, \ldots, q_{k,\log_2(M)}]$ are mapped to data symbols $a_1, \ldots, a_K$ of a zero-mean signal constellation $\mathcal{A}$ with cardinality $M \overset{\text{def}}{=} |\mathcal{A}|$ (mapping $\mathcal{M}$). In order to enable a joint LRA PE, the users' data symbols are combined into the vector $\boldsymbol{a} = [a_1 \ldots, a_K]^\mathsf{T}$.

[1]Notation: $\mathrm{E}\{\cdot\}$ denotes expectation. $\boldsymbol{A}^\mathsf{T}$ is the transpose and $\boldsymbol{A}^\mathsf{H}$ the Hermitian of matrix $\boldsymbol{A}$; $\boldsymbol{I}$ the identity matrix. $\boldsymbol{A}^+ = \boldsymbol{A}^\mathsf{H}(\boldsymbol{A}\boldsymbol{A}^\mathsf{H})^{-1}$ denotes the right pseudoinverse of $\boldsymbol{A}$.

This work has been supported by Deutsche Forschungsgemeinschaft (DFG) within the framework COIN under grant Fi 982/4-3.

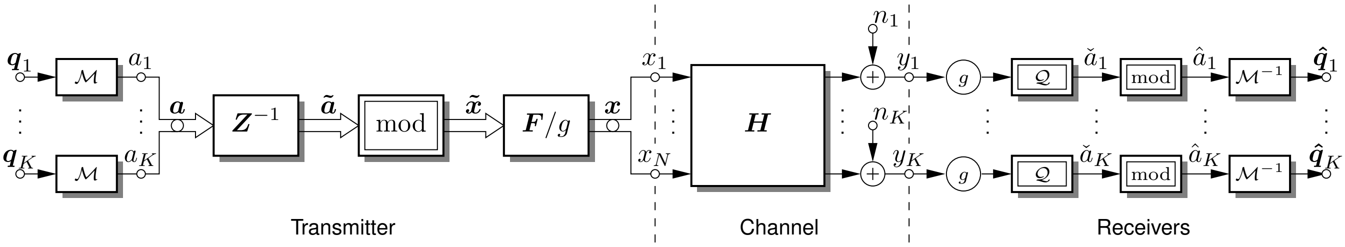

Fig. 1. System model for lattice-reduction-aided preequalization: joint transmitter-side processing, MIMO channel and separated receiver-side processing.

The basic idea of LRA PE is to equalize not the channel itself, but a more suited representation of it. This is achieved by a factorization $\boldsymbol{H} = \boldsymbol{Z}\boldsymbol{H}_{\mathrm{r}}$, where $\boldsymbol{H}$ is the $K \times N$ MIMO channel matrix and $\boldsymbol{H}_{\mathrm{r}}$ the *reduced* channel matrix of same dimensions which is conventionally (pre)equalized instead of $\boldsymbol{H}$. The integer matrix $\boldsymbol{Z} \in \boldsymbol{\Lambda}_{\mathrm{a}}^{K\times K}$ consists of elements from the signal-point lattice $\boldsymbol{\Lambda}_{\mathrm{a}}$ [3], i.e., the lattice the data symbols are drawn from ($\mathcal{A} \subset \boldsymbol{\Lambda}_{\mathrm{a}}$).

In the first step of LRA PE, the integer part of the interference is canceled via the inverse integer matrix resulting in a vector of preequalized symbols $\tilde{\boldsymbol{a}} = \boldsymbol{Z}^{-1}\boldsymbol{a}$. Following this, each preequalized symbol is modulo-reduced to precoded symbols $\tilde{x}_k = \mathrm{mod}_{\boldsymbol{\Lambda}_{\mathrm{p}}}(\tilde{a}_k) \overset{\text{def}}{=} \tilde{a}_k - \mathcal{Q}_{\boldsymbol{\Lambda}_{\mathrm{p}}}(\tilde{a}_k)$, where $\mathcal{Q}_{\boldsymbol{\Lambda}_{\mathrm{p}}}(\cdot)$ denotes the quantization to the precoding lattice $\boldsymbol{\Lambda}_{\mathrm{p}}$ [3]. Sharing the philosophy of THP by applying a modulo operation to limit the transmit power, the signal constellation $\mathcal{A}$ has to be *periodically extendable*. The choice of both signal and precoding lattice will be discussed in the following section.

As a final step, the residual non-integer interference (w.r.t. $\boldsymbol{H}_{\mathrm{r}}$) is equalized via the $N \times K$ matrix $\boldsymbol{F}$. The transmit symbols $[x_1, \dots, x_N]^{\mathsf{T}} = \boldsymbol{x} = g^{-1}\boldsymbol{F}\tilde{\boldsymbol{x}}$ to be radiated from the antennas are scaled by the scalar $1/g$ to keep the transmit power fixed to $N\sigma_x^2 = K\sigma_a^2$ for each channel realization. The variances of data and transmit symbols are defined by $\sigma_x^2 \overset{\text{def}}{=} \mathrm{E}\{|x_l|^2\}$, $l = 1, \dots, N$, and $\sigma_a^2 \overset{\text{def}}{=} \mathrm{E}\{|a_k|^2\}$, $k = 1, \dots, K$.

B. Channel Model

A flat-fading MIMO broadcast channel is considered, where the $K \times N$ channel matrix $\boldsymbol{H}$ is constant over a block of symbols (block fading). The system equation is given by

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n} \ . \qquad (1)$$

The coefficients of $\boldsymbol{H}$ are assumed to be i.i.d. zero-mean unit-variance complex Gaussian. The vector $\boldsymbol{n} = [n_1, \dots, n_K]^{\mathsf{T}}$ denotes zero-mean additive white complex Gaussian noise present at receiver $k = 1, \dots, K$, also assumed to be i.i.d. with variance $\sigma_n^2 \overset{\text{def}}{=} \mathrm{E}\{|n_k|^2\}$ per component. The receive symbols are combined into the vector $\boldsymbol{y} = [y_1, \dots, y_K]^{\mathsf{T}}$.

C. Receiver-Side Processing

At each of the K receivers, the following operations are performed: First, the transmitter-side scaling is reversed by a multiplication with g. Then, in coded transmission, channel decoding is performed. In the uncoded case, this corresponds to a quantization of the noisy symbols w.r.t. the signal-point lattice $\boldsymbol{\Lambda}_{\mathrm{a}}$, leading to estimated receive symbols $\check{a}_k = \mathcal{Q}_{\boldsymbol{\Lambda}_{\mathrm{a}}}(g\, y_k)$, $k = 1, \dots, K$. From these modulo-congruent estimated receive symbols, estimated data symbols $\hat{a}_1, \dots, \hat{a}_K \in \mathcal{A}$ can be obtained by the modulo operation $\hat{a}_k = \mathrm{mod}_{\boldsymbol{\Lambda}_{\mathrm{p}}}(\check{a}_k)$. Applying the inverse mapping $\mathcal{M}^{-1}$ results in the estimated blocks of $\log_2(M)$ bits, denoted as $\hat{\boldsymbol{q}}_k = [\hat{q}_{k,1}, \dots, \hat{q}_{k,\log_2(M)}]$.

III. Advanced Factorization Strategies

In this section, advanced factorization strategies are presented and their advantages in comparison to state-of-the-art approaches are enlightened. This includes a discussion on the signal-point lattice and the factorization strategy/algorithm.

A. Choice of the Signal-Point Lattice and Constellation

Usually, the signal-point lattice is given by the Gaussian integers $\mathbb{G} = \mathbb{Z} + \mathrm{j}\,\mathbb{Z}$, i.e., the integer lattice in the complex plane. The conventional approach is to choose the precoding lattice as a scaled version of $\boldsymbol{\Lambda}_{\mathrm{a}} = \mathbb{G}$. Defining the constellation as $\mathcal{A} = \mathcal{R}_{\mathrm{V}}(\boldsymbol{\Lambda}_{\mathrm{p}}) \cap \boldsymbol{\Lambda}_{\mathrm{a}}$ [3], where $\mathcal{R}_{\mathrm{V}}(\boldsymbol{\Lambda}_{\mathrm{p}})$ is the Voronoi region of the precoding lattice, the choice of $\boldsymbol{\Lambda}_{\mathrm{p}} = \sqrt{M}\mathbb{G}$, $\sqrt{M} \in \mathbb{N}$, results in an M-ary square-QAM constellation.[2] A direct mapping from bits to signal points (i.e., $M = 2^m$, $m \in \mathbb{N}$) is possible. In Fig. 2 (left), a 16QAM one ($\mathcal{A} = \mathcal{R}_{\mathrm{V}}(4\mathbb{G}) \cap (\mathbb{G} + \boldsymbol{o})$) including Gray labeling for bit mapping is exemplarily shown.

An alternative strategy is to choose the Eisenstein lattice $\mathbb{E} = \mathbb{Z} + \omega\mathbb{Z}$ (*Eisenstein unit* $\omega = \mathrm{e}^{\mathrm{j}2/3\pi}$) which represents the hexagonal lattice in the complex plane.[3] As above, the principle of *nested lattices* can be applied: taking the signal point lattice $\boldsymbol{\Lambda}_{\mathrm{a}} = \mathbb{E}$ and a related precoding lattice $\boldsymbol{\Lambda}_{\mathrm{p}} = \sqrt{M}\mathbb{E}$ (hexagonal shaping region), an M-ary periodically extendable signal constellation is defined by $\mathcal{A} = \mathcal{R}_{\mathrm{V}}(\sqrt{M}\mathbb{E}) \cap \mathbb{E}$. For $\sqrt{M} \in \mathbb{N}$, these constellations are always zero-mean. A hexagonal quantizer $\mathcal{Q}_{\mathbb{E}}(\cdot)$ is efficiently implementable on the basis of a Gaussian one $\mathcal{Q}_{\mathbb{G}}(\cdot)$ [14]. In Fig. 2 (right), the respective 16-ary Eisenstein constellation is illustrated.

A disadvantage of the Eisenstein lattice is the increase of a signal point's nearest neighbors from 4 to 6 in comparison to $\mathbb{G}$. As a consequence, a Gray labeling is not possible any more. Nevertheless, optimal labelings can be given, where the

[2] In order to obtain a zero-mean constellation, for $\sqrt{M}$ even, the signal-point lattice has to be shifted by the offset $\boldsymbol{o} = (1 + \mathrm{j})/2$. The signal constellation is hence given as $\mathcal{A} = \mathcal{R}_{\mathrm{V}}(\sqrt{M}\mathbb{G}) \cap (\mathbb{G} + \boldsymbol{o})$ and the quantization in the receivers is performed w.r.t. $\mathbb{G} + \boldsymbol{o}$; all other operations stay the same.

[3] Noteworthy, $\mathbb{G}$ is isomorphic to the lattice $\mathbb{Z}^2$ and $\mathbb{E}$ to the lattice A_2 [1].

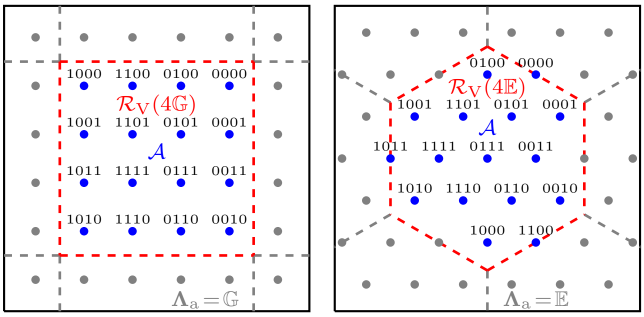

Fig. 2. 16-ary Gaussian integer (left) and Eisenstein integer (right) constellations defined by $\mathcal{A} = \mathcal{R}_\mathrm{V}(\boldsymbol{\Lambda}_\mathrm{p}) \cap \boldsymbol{\Lambda}_\mathrm{a}$. Some points of the constellations' periodical extensions and a suited mapping from bits to symbols are shown.

average number of distinct bits is 1.5 (as, e.g., in Fig. 2). In contrast, the lowered variance of the constellation[4] is advantageous. It is induced by a *packing gain* due to the densest (and hence optimum) packing in two dimensions [1], as well as the *shaping gain* of the hexagonal boundary region. Most important, regarding LRA PE, a *factorization gain* is present: due to the higher packing density of the Eisenstein integers, the choice of the integer matrix' elements is more flexible ($\boldsymbol{Z} \in \mathbb{E}^{K\times K}$ vs. $\boldsymbol{Z} \in \mathbb{G}^{K\times K}$; cf. [12]).

In CF/IF, periodically extendable constellations are not sufficient. There, p-ary constellations, p a prime, have to be employed and a match between the arithmetic in the complex and finite-field domain has to be guaranteed [8], [16]. A direct mapping from bits to signal points is no longer possible [12].

B. Choice of the Factorization Strategy

To solve the factorization task for LRA PE, different approaches have either directly been given in literature or can be dualized from receiver-side LRA equalization. This—in the first step—concerns the choice of the matrix that has to be factorized into an integer and residual non-integer part.

1) Classical Approach: In the initial papers on LRA pre-equalization/precoding [18], [13], the factorization task

$$\boldsymbol{H} = \boldsymbol{Z}\boldsymbol{H}_\mathrm{r} \tag{2}$$

is solved. The factorization is consequently directly applied on the channel matrix $\boldsymbol{H}$. The reduced channel matrix reads $\boldsymbol{H}_\mathrm{r} = \boldsymbol{Z}^{-1}\boldsymbol{H}$ and, employing the zero-forcing (ZF) criterion, the residual equalization is performed by $\boldsymbol{F} = \boldsymbol{H}_\mathrm{r}^{+}$.

2) Advanced Approaches: Applying the minimum mean-square error (MMSE) instead of the ZF criterion/solution [4], the transmission performance can be increased. In this case, the $K \times (N+K)$ *augmented* channel matrix $\boldsymbol{\mathcal{H}} \overset{\text{def}}{=} [\boldsymbol{H},\ \sqrt{\zeta}\boldsymbol{I}]$ with parameter $\zeta = \sigma_n^2/\sigma_a^2$ is factorized.The matrix for MMSE linear (residual) preequalization is the $N \times K$ upper part $\boldsymbol{F} = \boldsymbol{H}_\mathrm{r}^\mathsf{H}(\boldsymbol{H}_\mathrm{r}\boldsymbol{H}_\mathrm{r}^\mathsf{H} + \zeta\boldsymbol{Z}^{-1}\boldsymbol{Z}^{-\mathsf{H}})^{-1}$ of its $(N+K)\times K$ augmented variant $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{H}}_\mathrm{r}^{+} = ([\boldsymbol{Z}^{-1}\boldsymbol{H},\ \sqrt{\zeta}\boldsymbol{Z}^{-1}])^{+}$.

Another strategy is the factorization of $(\boldsymbol{H}^{+})^\mathsf{H}$, i.e., the Hermitian of the pseudoinverse of $\boldsymbol{H}$ instead of the channel matrix itself [15]. This incorporates the fact that the transmission performance does not directly depend on $\boldsymbol{H}_\mathrm{r}$ but on its inverse $\boldsymbol{F} = \boldsymbol{H}_\mathrm{r}^{+}$ used to equalize the reduced channel [5]. Dualizing this approach to LRA PE, the factorization task reads $(\boldsymbol{H}^{+})^\mathsf{H} = \boldsymbol{Z}^{-\mathsf{H}}\boldsymbol{F}^\mathsf{H}$. The ZF linear residual equalization matrix is straightly given by $\boldsymbol{F} = (\boldsymbol{F}^\mathsf{H})^\mathsf{H}$, and $\boldsymbol{Z}^{-1} = (\boldsymbol{Z}^{-\mathsf{H}})^\mathsf{H}$.

Combining the advantages of both MMSE solution and factorization of $(\boldsymbol{H}^{+})^\mathsf{H}$, the factorization of the *Hermitian of the inverse augmented channel matrix* (cf. dual variant in [5])

$$(\boldsymbol{\mathcal{H}}^{+})^\mathsf{H} = \boldsymbol{Z}^{-\mathsf{H}}\boldsymbol{\mathcal{F}}^\mathsf{H} \tag{3}$$

is suited to minimize the mean-square error directly.[5] The equalization matrix $\boldsymbol{F}$ is immediately the $N \times K$ upper part of $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{H}}_\mathrm{r}^{+} = ([(\boldsymbol{Z}^{-\mathsf{H}})^\mathsf{H}\boldsymbol{H},\ \sqrt{\zeta}(\boldsymbol{Z}^{-\mathsf{H}})^\mathsf{H}])^{+}$.

In CF/IF (transmitter-side) schemes, e.g., [8], the optimization of $\boldsymbol{Z}$ is performed on the $K \times K$ square-root $\boldsymbol{L}$ of

$$\boldsymbol{L}\boldsymbol{L}^\mathsf{H} = (\boldsymbol{H}\boldsymbol{H}^\mathsf{H} + \zeta\boldsymbol{I})^{-1}\ . \tag{4}$$

Thereby, $\boldsymbol{L}$ can be any square-root, e.g., a Cholesky factor. This approach is, however, equivalent to (3): Since $\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^\mathsf{H} = \boldsymbol{H}\boldsymbol{H}^\mathsf{H} + \zeta\boldsymbol{I}$, another possible square-root of $(\boldsymbol{H}\boldsymbol{H}^\mathsf{H} + \zeta\boldsymbol{I})^{-1}$ is $\boldsymbol{L} = (\boldsymbol{\mathcal{H}}^{+})^\mathsf{H}$, as[6] $\boldsymbol{L}\boldsymbol{L}^\mathsf{H} = (\boldsymbol{\mathcal{H}}^{+})^\mathsf{H}\boldsymbol{\mathcal{H}}^{+} = (\boldsymbol{H}\boldsymbol{H}^\mathsf{H} + \zeta\boldsymbol{I})^{-1}$.

The different approaches are listed in terms of their properties in Table I. It is the dual variant of the overview in [5].

TABLE I
FACTORIZATION STRATEGIES DUALIZED TO LRA PE (CF. [5]).

based on	$\boldsymbol{H}$ (*ZF solution*)	$\boldsymbol{\mathcal{H}}$ (*MMSE solution*)
$\boldsymbol{H}$	$\boldsymbol{H} = \boldsymbol{Z}\boldsymbol{H}_\mathrm{r}$	$\boldsymbol{\mathcal{H}} = \boldsymbol{Z}\boldsymbol{\mathcal{H}}_\mathrm{r}$
$(\boldsymbol{H}^{+})^\mathsf{H}$	$(\boldsymbol{H}^{+})^\mathsf{H} = \boldsymbol{Z}^{-\mathsf{H}}\boldsymbol{F}^\mathsf{H}$	$(\boldsymbol{\mathcal{H}}^{+})^\mathsf{H} = \boldsymbol{Z}^{-\mathsf{H}}\boldsymbol{\mathcal{F}}^\mathsf{H}$

C. Choice of the Factorization Task and Algorithm

To solve the abovementioned factorization approaches—in principle—any lattice-reduction algorithm can be employed. The factorization then results in an unimodular integer matrix ($|\det(\boldsymbol{Z})| = 1$) with integer inverse $\boldsymbol{Z}^{-1} \in \boldsymbol{\Lambda}_\mathrm{a}^{K\times K}$. The reduced channel $\boldsymbol{H}_\mathrm{r}$ constitutes a *suited basis* of the lattice spanned by $\boldsymbol{H}$, where $\boldsymbol{Z}$ describes the change of basis. The related factorization criterion may, e.g., be the minimization of the maximum squared Euclidean norm of the basis vectors (*shortest basis problem*). Most often, the LLL algorithm (here: its complex-valued variant [6]) is applied for lattice basis reduction, efficiently computing a near-optimum short basis.[7]

Recently, inspired by integer-forcing schemes, e.g., [20], the unimodularity constraint in LRA (receiver-side) equalization has been queried [5]. In IF, the restriction to *algebraic*

[4] 16QAM: $\sigma_a^2 = 2.5$; 16-ary Eisenstein constellation: $\sigma_a^2 = 2.25$.

[5] The minimization of the (receiver-side) mean-square error can also be found in MMSE vector precoding [11].

[6] Proof: $\boldsymbol{L}\boldsymbol{L}^\mathsf{H} = (\boldsymbol{\mathcal{H}}^{+})^\mathsf{H}\boldsymbol{\mathcal{H}}^{+} = ((\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^\mathsf{H})^{-\mathsf{H}}\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^\mathsf{H}(\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^\mathsf{H})^{-1})^\mathsf{H} = (\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^\mathsf{H})^{-\mathsf{H}} = (\boldsymbol{H}\boldsymbol{H}^\mathsf{H} + \zeta\boldsymbol{I})^{-\mathsf{H}} = (\boldsymbol{H}\boldsymbol{H}^\mathsf{H} + \zeta\boldsymbol{I})^{-1}$, since $\boldsymbol{H}\boldsymbol{H}^\mathsf{H} + \zeta\boldsymbol{I}$ is a Hermitian matrix.

[7] For the Eisenstein lattice, an adapted definition of LLL reduction can be given [9]. Then, in the LLL algorithm, the condition for size reduction [6, Eq. (5)] and the quantization [6, Line 1 in Alg. 2] are adapted to $\mathcal{R}_\mathrm{V}(\mathbb{E})$.

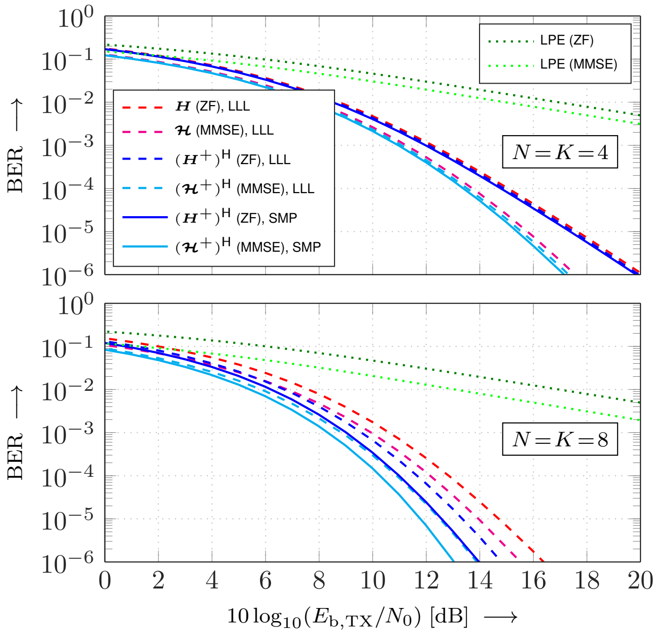

Fig. 3. Bit-error rate of LRA PE in dependency of the SNR in dB. 16-ary QAM constellation ($\mathbf{\Lambda}_\mathrm{a} = \mathbb{G}$). Top: $N = K = 4$. Bottom: $N = K = 8$. Variation of the factorized matrix and the factorization task/algorithm. For comparison, the results for (non-LRA) LPE (ZF/MMSE) are given.

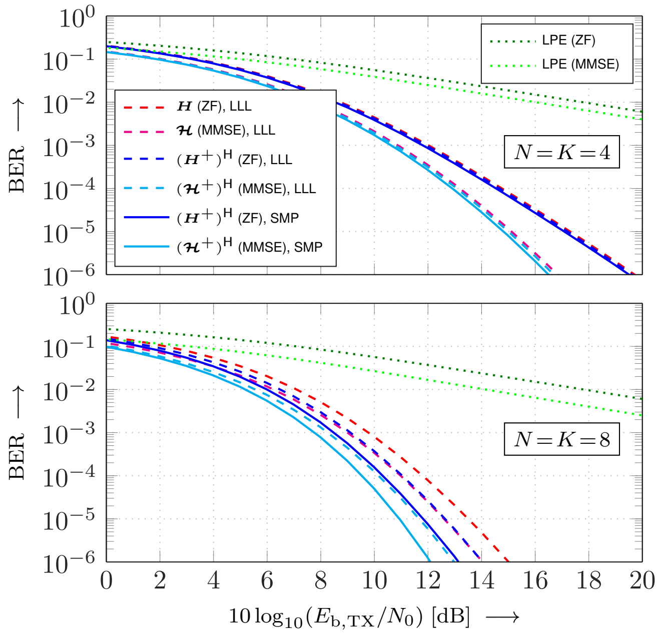

Fig. 4. Bit-error rate of LRA PE in dependency of the SNR in dB. 16-ary Eisenstein constellation ($\mathbf{\Lambda}_\mathrm{a} = \mathbb{E}$). Top: $N = K = 4$. Bottom: $N = K = 8$. Variation of the factorized matrix and the factorization task/algorithm. For comparison, the results for (non-LRA) LPE (ZF/MMSE) are given.

signal constellations allows interference cancellation over p-ary finite fields (cf. [5]). This approach relaxes the shortest basis problem to the so-called *shortest independent vector problem* (SIVP), where the unimodularity constraint on $\boldsymbol{Z}$ is dropped[8] ($|\det(\boldsymbol{Z})| \geq 1$, i.e., $\mathrm{rank}(\boldsymbol{Z}) = K$). Transferring this philosophy to (conventional) LRA PE, a non-unimodular $\boldsymbol{Z}$ results in a $\boldsymbol{Z}^{-1} \in (\delta\mathbf{\Lambda}_\mathrm{a})^{K\times K}$, with $\delta = \det(\boldsymbol{Z})^{-1} < 1$. The preequalized symbols $\tilde{\boldsymbol{a}} = \boldsymbol{Z}^{-1}\boldsymbol{a}$ (cf. Fig. 1) are not drawn from $\mathbf{\Lambda}_\mathrm{a}$ any more, but are elements of a scaled lattice $\delta\mathbf{\Lambda}_\mathrm{a}$. Periodical extensibility of these signal points is, however, kept as $\det(\boldsymbol{Z}) \in \mathbf{\Lambda}_\mathrm{a}$, and hence $\mathbf{\Lambda}_\mathrm{a} \subset \delta\mathbf{\Lambda}_\mathrm{a}$. After the modulo operation the symbols are merely—similar to THP—more uniformly distributed over the shaping region. This scaling is automatically reversed by the channel $\boldsymbol{H} = \boldsymbol{Z}\boldsymbol{H}_\mathrm{r}$, i.e., $\breve{a}_k \in \det(\boldsymbol{Z})\delta\mathbf{\Lambda}_\mathrm{a} = \delta^{-1}\delta\mathbf{\Lambda}_\mathrm{a} = \mathbf{\Lambda}_\mathrm{a}$. In contrast to IF, a restriction to algebraic constellations is not needed when applying a (more flexible) factorization according to the SIVP.

In LRA PE, the channel-dependent scaling factor g^{-1} compensates the enhancement of transmit power induced by the non-integer equalization via $\boldsymbol{F}$. This, however, results in a performance-decreasing enhancement of the receiver-side noise variance by the factor g^2 (inverse scaling by g). Thereby, $g^2 = \sum_{l=1}^{N} ||\boldsymbol{f}_l^\mathsf{H}||^2/N = ||\boldsymbol{F}||_\mathrm{F}^2/N$, where $\boldsymbol{f}_l^\mathsf{H}$ is the lth row of $\boldsymbol{F} = [\boldsymbol{f}_1, \ldots, \boldsymbol{f}_N]^\mathsf{H}$, and $||\cdot||_\mathrm{F}$ denotes the Frobenius norm.[9]

[8]Since in IF $\boldsymbol{Z}$ is substituted by a finite-field variant $\boldsymbol{Z}_\mathbb{F}$ of it, a (finite-field) integer inverse always exists if $\mathrm{rank}(\boldsymbol{Z}_\mathbb{F}) = K$, cf. [5]. A similar finite-field approach for LRA preequalization has been proposed in [12].

[9]The Frobenius norm of a $U \times V$ matrix $\boldsymbol{A} = [a_{u,v}]$ is defined as $||\boldsymbol{A}||_\mathrm{F} = ||\boldsymbol{A}^\mathsf{H}||_\mathrm{F} \stackrel{\mathrm{def}}{=} \sqrt{\sum_{u=1}^{U}\sum_{v=1}^{V} |a_{u,v}|^2}$.

The combination of the minimization of g^2, the factorization approach (3), and the relaxation to $|\det(\boldsymbol{Z})| \geq 1$, results in the sum-mean-square-error minimization

$$\boldsymbol{Z}^\mathsf{H} = \underset{\substack{\boldsymbol{Z}\in\mathbf{\Lambda}_\mathrm{a}^{K\times K},\\ \mathrm{rank}(\boldsymbol{Z})=K}}{\mathrm{argmin}} \sum\nolimits_{k=1}^{K} ||\boldsymbol{z}_k^\mathsf{H}\boldsymbol{L}||^2 = \underset{\substack{\boldsymbol{Z}\in\mathbf{\Lambda}_\mathrm{a}^{K\times K},\\ \mathrm{rank}(\boldsymbol{Z})=K}}{\mathrm{argmin}} ||\boldsymbol{Z}^\mathsf{H}\boldsymbol{L}||_\mathrm{F}^2 . \quad (5)$$

The kth row of $\boldsymbol{Z}^\mathsf{H} = [\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K]^\mathsf{H}$ is denoted as $\boldsymbol{z}_k^\mathsf{H}$ and $\boldsymbol{L} = (\boldsymbol{\mathcal{H}}^+)^\mathsf{H}$ is a square root of $\boldsymbol{L}\boldsymbol{L}^\mathsf{H} = (\boldsymbol{H}\boldsymbol{H}^\mathsf{H} + \zeta\boldsymbol{I})^{-1}$.

The factorization task (5) is similar to the optimum one for LRA/IF receiver-side equalization according to the SIVP, where $\max_k ||\boldsymbol{L}^\mathsf{H}\boldsymbol{z}_k||^2 = \max_k ||\boldsymbol{z}_k^\mathsf{H}\boldsymbol{L}||^2$ has to be minimized [5]. However, the algorithms in [2], [5] additionally solve the *successive minima problem* (SMP) [2], where not only the maximum norm but all of them have to be as small as possible. Thus, the solution to the successive minima problem also minimizes the sum of the norms and the (dualized) algorithms find an optimum solution for (5). Noteworthy, the algorithm in [5] can easily be adapted to $\mathbf{\Lambda}_\mathrm{a} = \mathbb{E}$.

IV. Numerical Results

Numerical simulations have been performed to assess the factorization approaches. The results are averaged over all users and a large number of channel realizations. The signal-to-noise ratio (SNR) is expressed as $E_\mathrm{b,TX}/N_0 = \sigma_a^2/(\log_2(M)\sigma_n^2)$, i.e., as TX energy per bit over the noise power spectral density.

In Fig. 3, the bit-error rate (BER) of LRA PE is depicted over the SNR for a 16QAM transmission ($\mathbf{\Lambda}_\mathrm{a} = \mathbb{G}$). A lattice basis reduction via (complex) LLL algorithm has been applied for all approaches; linear residual equalization has been chosen in accordance (ZF/MMSE). In addition, a factorization w.r.t.

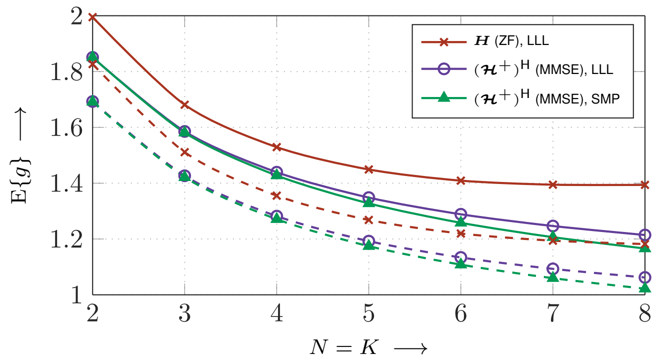

Fig. 5. Expectation of the scaling factor g in dependency of $N = K$ for $\sigma_a^2/\sigma_n^2 \,\hat{=}\, 20$ dB. Variation of the factorized matrix and the task/algorithm. Solid: Gaussian integer lattice $\mathbb{G}$. Dashed: Eisenstein integer lattice $\mathbb{E}$.

the SMP is employed to solve (5) in an optimal way (ZF solution: $\zeta = 0$). Regarding the case $N = K = 4$ (top), the choice of either the ZF or the MMSE solution mainly influences the performance. Factorizing $(\boldsymbol{H}^{+})^{\mathsf{H}}$ or $(\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}}$, respectively, most often results in the same $\boldsymbol{Z}$ compared to $\boldsymbol{H}$ or $\boldsymbol{\mathcal{H}}$; instead a major gain in performance is induced by the MMSE residual linear equalization in lieu of the ZF one. Solving the SMP results in minor additional gains; the vast majority of integer matrices is still unimodular (cf. [5]). An increase in dimension to $N = K = 8$ (Fig. 3 bottom) changes the situation: When factorizing $(\boldsymbol{H}^{+})^{\mathsf{H}}$ or $(\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}}$ instead of $\boldsymbol{H}$ or $\boldsymbol{\mathcal{H}}$ via the LLL, better solutions for $\boldsymbol{Z}$ are found; a gain of about 1 dB is present (high-SNR regime). Solving the SMP instead results in 1 dB additional gain; a significant number of integer matrices is now non-unimodular. Besides, for both ZF and MMSE (non-LRA) linear preequalization (LPE) only diversity order one can be achieved in any case.

Fig. 4 illustrates the results when the 16QAM constellation is substituted by a 16-ary Eisenstein one ($\boldsymbol{\Lambda}_{\mathrm{a}} = \mathbb{E}$). Basically, the same behavior as in the QAM case is visible. Performance is, however, generally increased due to the constellation's packing and shaping gain, and especially due the factorization gain of the Eisenstein lattice. Considering the MMSE solutions in the high-SNR range, the same BER as for QAM is achieved for an approximately 1 dB lower SNR. Combining all advanced strategies (factorization of $(\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}}$, SMP, $\boldsymbol{\Lambda}_{\mathrm{a}} = \mathbb{E}$), for $N = K = 8$, a gain of more than 4 dB is possible compared to the naive approach (factorization of $\boldsymbol{H}$, LLL, $\boldsymbol{\Lambda}_{\mathrm{a}} = \mathbb{G}$).

The illustration of the expectation of g in Fig. 5 for $N = K = 2, \ldots, 8$ complements the theoretical derivations. Restricting to LLL factorization, the best-performing factorization/equalization approach ($(\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}}$, MMSE) lowers the value of g mainly independently from the dimensions. As the computational effort is not increased in comparison to the naive approach ($\boldsymbol{H}$, ZF) this strategy is always advisable. In contrast, a factorization according to the SMP only has relevance for $K \geq 4$. Eisenstein signal constellations (dashed) show a significant gain compared to QAM ones (solid). As a consequence, a positive impact on the BER performance is present even though a Gray labeling is not possible.

V. Summary and Conclusions

Advanced factorization approaches for LRA PE have been presented and assessed. Eisenstein-based signal constellations, the sum-mean-square-error approach, and the factorization according to the SMP (a stricter version of the SIVP) instead of the shortest basis problem have been discussed. Numerical simulations have revealed that even for conventional LLL reduction, the right choice of both the signal point lattice and the factorization approach allows considerable gains. For the high-diversity case, algorithms solving the SMP can further increase performance. Noteworthy, an extension to (THP-type) precoding can directly be performed, cf. [10].

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