

Lattice-Reduction-Aided Precoding for Coded Modulation over Algebraic Signal Constellations

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Abstract—Lattice-reduction-aided preequalization or precoding are powerful techniques for handling the interference on the multi-user MIMO broadcast channel as the channel’s diversity order can be achieved. However, recent advantages in the closely related field of integer-forcing equalization raise the question, if the unimodularity constraint on the integer equalization matrix in LRA schemes is really necessary or if it can be dropped, yielding an additional factorization gain. In this paper, so-called algebraic signal constellations are presented, where the unimodularity is not required anymore. Assuming complex-baseband transmission, particularly q -ary fields of Gaussian primes (complex integer lattice) and Eisenstein primes (complex hexagonal lattice) are considered. Given the signal constellation and the channel code in the same arithmetic over a finite field of order q , a coded modulation approach with straightforward soft-decision decoding metric is applied. Moreover, LRA precoding over algebraic constellations and its advantages as opposed to LRA preequalization are discussed. The theoretical considerations in the paper are covered by means of numerical simulations.

I. INTRODUCTION

In multi-user multiple-input/multiple-output (MIMO) communications, the principle of *lattice-reduction-aided* (LRA) equalization has gained significant interest as the respective schemes are able to achieve the diversity order of the multi-user MIMO channel [30]—in contrast to well-known traditional techniques like linear (pre-)equalization, decision-feedback equalization (DFE) [10] or Tomlinson-Harashima precoding (THP) [31], [16], which have been adapted from the single-user to the multi-user scenario [10], [7].

In LRA schemes, the equalization is performed in a *suited basis* w.r.t. the lattice described by the MIMO channel matrix. This is achieved by factorizing the channel matrix into an unimodular integer part and a “more suited” description of the lattice with basis vectors close to orthogonal and of small norm. This approach has first been presented for receiver-side equalization (multiple-access channel) [38] but could rapidly—via the uplink/downlink duality [33], [34]—be extended to downlink transmission (MIMO broadcast channel) [35], [36], [28].

Recently, inspired by so-called *compute-and-forward* [25], [18] or *integer-forcing* (IF) [39] schemes where the final

resolution of the interference is performed over finite fields, the philosophy of LRA equalization has been considered from a modified perspective [26]: applying signal constellations with algebraic property [9], or more specifically, constellations which represent finite fields over the complex plane, the *shortest basis problem* present in LRA equalization is generalized to the *shortest independent vector problem*. This task, which is always given in IF equalization, drops the unimodularity constraint on the integer matrix. Fields of Gaussian primes [20], [1], [3] (complex integer lattice) or fields of Eisenstein primes [3], [29], [32] (complex hexagonal lattice) are suited algebraic structures [26], not only providing the desired finite-field property, but also directly yielding the precoding lattice or modulo operation inherently accompanied by LRA preequalization or precoding.

In this paper, the approach of applying algebraic signal constellations for LRA preequalization over the MIMO broadcast channel as proposed in [26] is reviewed and extended. This includes a factorization according to the shortest independent vector problem and the assessment of the achievable factorization gain. A coded modulation approach where both the channel code and the signal constellation operate over the same arithmetic [27], specifically a finite field of order q , is applied and implemented via non-binary q -ary LDPC codes. Moreover, the LRA linear preequalization in [26], [27] is replaced by (Tomlinson-Harashima-type) LRA precoding and its advantages w.r.t. signal properties and resulting transmission performance are discussed.

The paper is organized as follows: In Sec. II, the system model of coded modulation in combination with LRA preequalization or precoding for the MIMO broadcast channel is given. Sec. III details the aspects of LRA encoding over algebraic constellations, coded modulation over these constellations and the advantages of LRA precoding instead of LRA preequalization. Numerical results are provided in Sec. IV. The paper closes with a summary and conclusions in Sec. V.

II. SYSTEM MODEL

A discrete-time complex-baseband multi-user MIMO broadcast channel is considered. At the transmitter-side, a joint processing is present to supply N_R non-cooperating single-antenna users via $N_T \geq N_R$ transmit antennas. The system model of LRA precoding in combination with (soft-decision) channel coding is depicted in Fig. 1.

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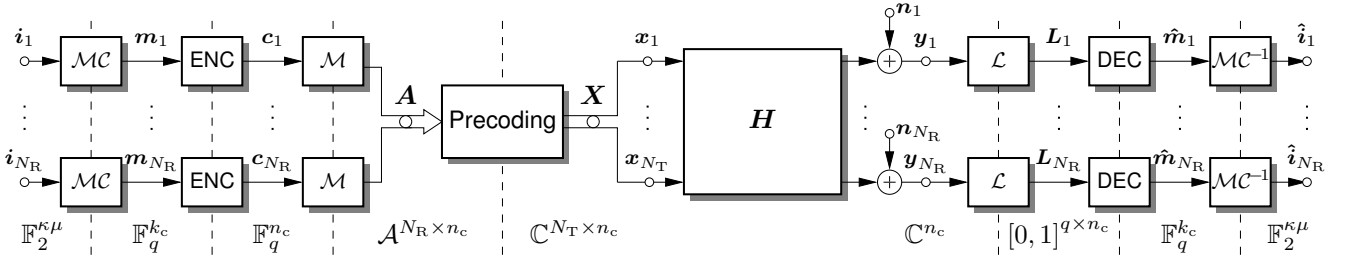


Fig. 1. System model of LRA precoding (transmitter-side and receiver-side processing) for the MIMO broadcast channel with N_T transmit antennas and N_R single-antenna users in combination with soft-decision channel coding. For individual processing, the dimension per user is specified.

A. Transmitter-Side Processing

Independent serial streams of source information symbols are transmitted to user $u = 1, \dots, N_R$. As a block-based channel coding is present, the streams are split into blocks of μ source symbols. Since usually bits are communicated we restrict to the binary case. One block of bits of user u is denoted by $\hat{i}_u = [\hat{i}_{u,1}, \dots, \hat{i}_{u,\mu}] \in \mathbb{F}_2^\mu$.

If the source and channel coding do not share the same arithmetic, i.e., the channel code is represented over a field $\mathbb{F}_q = \{\varphi_1, \dots, \varphi_q\}$ with $q \neq 2$, a *modulus conversion* (denoted as \mathcal{MC}) has to be applied [13], [10], [26]: the block of μ source symbols drawn from \mathbb{F}_2 is converted to a block of ν message symbols drawn from \mathbb{F}_q ($q^\nu \geq 2^\mu$). These symbols are combined into the vector $\mathbf{m}_u = [m_{u,1}, \dots, m_{u,\nu}] \in \mathbb{F}_q^\nu$.

Subsequently, performing the channel encoding (ENC), $k_c = \kappa\nu$, $\kappa \in \mathbb{N}$, (serial) message symbols are encoded to a codeword of length n_c via a $k_c \times n_c$ generator matrix \mathbf{G}_c of a linear block code (rate $R_c \stackrel{\text{def}}{=} k_c/n_c$). This yields the vector of encoded messages $\mathbf{c}_u = [c_{u,1}, \dots, c_{u,n_c}] \in \mathbb{F}_q^{n_c}$. All users are assumed to have the same code properties (length, rate, and code class). However, each encoding is performed independently of the encoding for all other users.

Given the encoded symbols, a (symbol-wise) predefined mapping $\mathcal{M} : c \in \mathbb{F}_q \rightarrow a \in \mathcal{A}$ to data symbols is performed, where $\mathcal{A} \subset \mathbb{C}$ denotes a zero-mean signal constellation with cardinality $M \stackrel{\text{def}}{=} |\mathcal{A}|$ and variance σ_a^2 .

The vectors of data symbols $\mathbf{a}_u = [a_{u,1}, \dots, a_{u,n_c}] \in \mathcal{A}^{n_c}$, $u = 1, \dots, N_R$, are finally combined into the data-symbol matrix $\mathbf{A} \in \mathcal{A}^{N_R \times n_c}$ in order to enable a joint precoding. The precoding results in a matrix of transmit symbols $\mathbf{X} \in \mathbb{C}^{N_T \times n_c}$ that are radiated from the antennas. A sum-power constraint $N_T \sigma_x^2 = N_R \sigma_a^2$ is present, where σ_x^2 is the transmit symbols' variance. The process of precoding will further be explained in Sec. III.

B. Channel Model

The MIMO broadcast channel is expressed by

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{N}. \quad (1)$$

The $N_R \times N_T$ channel matrix \mathbf{H} is assumed to have i.i.d. complex-Gaussian zero-mean unit-variance coefficients. A block-fading channel is assumed, i.e., the channel matrix is constant over the whole block of n_c data symbols. The matrix

$\mathbf{N} \in \mathbb{C}^{N_R \times n_c}$ represents the additive white noise which is present at each receiver. It is assumed to be zero-mean complex Gaussian with variance σ_n^2 ; i.i.d. over the users and time. Finally, $\mathbf{Y} \in \mathbb{C}^{N_R \times n_c}$ denotes the respective matrix of receive symbols.

The signal-to-noise ratio (SNR) is expressed as transmitted energy *per bit* in relation to the noise power spectral density N_0 and given by

$$\frac{E_{b,\text{TX}}}{N_0} = \frac{\sigma_a^2}{\sigma_n^2 R_c \mu / \nu}. \quad (2)$$

C. Receiver-Side Processing

At the receiver-side, for each user $u = 1, \dots, N_R$ a metric $\mathcal{L} : \mathbb{C}^{n_c} \rightarrow [0, 1]^{q \times n_c}$ for soft-decision decoding is calculated from its incoming signal vector $\mathbf{y}_u = [y_{u,1}, \dots, y_{u,n_c}]$. Since the decoding is independently performed in an equivalent way, we drop the user index for simplicity. For each sample y_γ in \mathbf{y} , $\gamma = 1, \dots, n_c$, the metric is represented as q -dimensional vector $\mathbf{l}_\gamma = [l_{1,\gamma}, \dots, l_{q,\gamma}]^\top$. Thereby, $l_{\rho,\gamma} \stackrel{\text{def}}{=} \Pr\{c_\gamma = \varphi_\rho | y_\gamma\}$, $\rho = 1, \dots, q$, i.e., \mathbf{l}_γ is a probability mass function (pmf) of one encoded message symbol w.r.t. all possible elements $\varphi_\rho \in \mathbb{F}_q$.

Combing all n_c metric vectors into $\mathbf{L} \in [0, 1]^{q \times n_c}$, a soft-decision decoding (DEC) is performed. The resulting decoded messages are denoted as $\hat{\mathbf{m}}_u = [\hat{m}_{u,1}, \dots, \hat{m}_{u,k_c}] \in \mathbb{F}_q^{k_c}$.

In a final step, the inverse modulus conversion is applied to obtain the estimated initial information symbols (blocks of ν message symbols are converted to blocks of μ source symbols). It yields κ blocks of estimated bits $\hat{\mathbf{i}} \in \mathbb{F}_2^\mu$.

III. LRA PRECODING FOR CODED MODULATION OVER ALGEBRAIC SIGNAL CONSTELLATIONS

In order to handle the multi-user interference present on the MIMO broadcast channel, both LRA preequalization and (Tomlinson-Harashima-type) precoding share the principle of performing the interference cancellation in a *suited basis*, i.e., a change in basis is realized to reduce the related increase in transmit power. The optimal solution for this strategy is found by solving a *shortest basis problem* (SBP) [35], [36]. In the following, the basic idea behind LRA schemes is reviewed.

Solving the shortest basis problem is equivalent to a factorization of the channel matrix according to

$$\mathbf{H} = \mathbf{Z}\mathbf{H}_{\text{red}}, \quad (3)$$

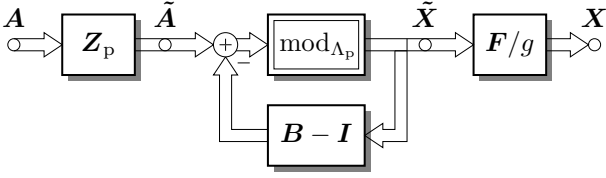


Fig. 2. Block diagram of LRA preequalization (feedback part inactive, i.e., $\mathbf{B} = \mathbf{I}$) and LRA precoding for the MIMO broadcast channel.

where \mathbf{H}_{red} denotes the reduced channel matrix which represents the more suited basis for equalization. The integer matrix $\mathbf{Z} \in \Lambda_a^{N_R \times N_R}$ consists of elements drawn from a given signal-point lattice Λ_a [10], [26] (signal grid of \mathcal{A} , i.e., $\mathcal{A} \subset \Lambda_a$) and describes the change of the basis. It is usually demanded to be unimodular ($|\det(\mathbf{Z})| = 1$) to ensure the existence of an inverse integer matrix $\mathbf{Z}^{-1} \in \Lambda_a^{N_R \times N_R}$.

Concerning transmission performance, it is advantageous to operate on the *augmented channel matrix* [37], [11]

$$\bar{\mathbf{H}} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{H} & \sqrt{\zeta} \mathbf{I} \end{bmatrix}_{N_R \times (N_R + N_T)}, \quad (4)$$

where $\zeta \stackrel{\text{def}}{=} \sigma_n^2 / \sigma_a^2$ and \mathbf{I} denotes the identity matrix (*MMSE solution*). To be precise, the optimum strategy is to factorize the *Hermitian of the pseudo-inverse augmented channel matrix* $(\bar{\mathbf{H}}^+)^H = (\bar{\mathbf{H}}^H (\bar{\mathbf{H}} \bar{\mathbf{H}}^H)^{-1})^H$ according to (cf. [12], [27])

$$(\bar{\mathbf{H}}^+)^H = \mathbf{Z}^{-H} (\bar{\mathbf{H}}_{\text{red}}^+)^H. \quad (5)$$

In this case, the inverse integer matrix is $\mathbf{Z}^{-1} = (\mathbf{Z}^{-H})^H$ and $(\bar{\mathbf{H}}_{\text{red}}^+)^H$ denotes the Hermitian of the right pseudo-inverse of the reduced channel matrix with the same dimensions as $\bar{\mathbf{H}}$.

Given the channel factorization, the LRA preequalization can be performed as depicted in Fig. 2: The LRA preequalization is realized by an integer equalization matrix

$$\mathbf{Z}_p \stackrel{\text{def}}{=} \mathbf{P} \mathbf{Z}^{-1} \quad (6)$$

to obtain a matrix of equalized symbols $\tilde{\mathbf{A}} \in \Lambda_a^{N_R \times n_c}$ from the data symbols \mathbf{A} . Thereby, \mathbf{P} is a permutation matrix enabling an optimized precoding order among the data symbols (detailed below). Employing the element-wise modulo reduction

$$\text{mod}_{\Lambda_p}(z) \stackrel{\text{def}}{=} z - \mathcal{Q}_{\Lambda_p}(z), \quad z \in \mathbb{C}, \quad (7)$$

where $\mathcal{Q}_{\Lambda_p}(\cdot)$ is the quantization to the predefined precoding lattice Λ_p [10], [26], encoded symbols $\tilde{\mathbf{X}} \in \mathbb{C}^{N_R \times n_c}$ are created. In case of LRA precoding, this is done in a successive way as the interference from previously encoded symbols is canceled via the feedback matrix \mathbf{B} (for LRA preequalization: $\mathbf{B} = \mathbf{I}$). Noteworthy, due to the modulo reduction a *periodically extendable* constellation [10], [26]

$$\mathcal{A} \stackrel{\text{def}}{=} \mathcal{R}_V(\Lambda_p) \cap \Lambda_a \quad (8)$$

is required, where $\mathcal{R}_V(\Lambda_p)$ denotes the Voronoi region [10] of the precoding lattice. In the last step, the feedforward matrix \mathbf{F} handles the remaining (non-integer) interference; the factor g ensures that the sum-power constraint is fulfilled when obtaining the matrix of transmit symbols \mathbf{X} . A suited choice of \mathbf{F} will be discussed below.

A. Unimodularity Constraint and Algebraic Constellations

Initially, for LRA preequalization/precoding, conventional square-QAM constellations have been employed [36]: utilizing the Gaussian integers \mathbb{G} [1], [3] (i.e., the integer lattice in the complex plane) as the signal-point lattice¹ Λ_a , the demanded property of a periodical extensibility can be provided by setting $\Lambda_p = \sqrt{M} \mathbb{G}$ [26], i.e., a scaled version of the signal-point lattice (*nested lattices*, cf. (8)).

Quite recently, an alternative strategy [26], [27] inspired by *integer-forcing* schemes [25], [18], [17], [39] was proposed: In IF, the integer interference is canceled over the arithmetic of a finite field, setting the requirement to have a signal constellation which can be represented as *algebraic structure* [9] over the complex numbers. In the following, we briefly review the algebraic structures proposed in [26]; subsequently we discuss the possibility of relaxing the unimodularity constraint in LRA precoding on the basis of these special constellations.

1) *Fields of Gaussian Primes*: For the signal-point lattice $\Lambda_a = \mathbb{G}$, fields of *Gaussian primes* [1], [3], [20] are convenient algebraic structures. A Gaussian prime is a Gaussian integer $\Theta = a + jb$, $a, b \in \mathbb{Z}$, which fulfills the equation $\Theta \Theta^* = |\Theta|^2 = p$, where p is a real-valued prime and $\Theta^* = a - jb$ denotes the complex conjugate of Θ . In particular, primes of the form² $\text{rem}_4(p) = 1$, i.e., $p = 5, 13, 17, \dots$ are suited. In addition, for real-valued primes of the form $\text{rem}_4(p) = 3$, i.e., $p = 3, 7, 11, \dots$, a related real-valued Gaussian prime is directly given by $\Theta = p$.

Choosing the precoding lattice as $\Lambda_p = \Theta \mathbb{G}$ [26], the respective zero-mean signal constellation $\mathcal{A}_{\Theta}^{(\mathbb{G})} \stackrel{\text{def}}{=} \mathcal{R}_V(\Theta \mathbb{G}) \cap \mathbb{G}$ (cf. (8)) represents a finite field over \mathbb{C} . The constellation's cardinality always reads $M = |\Theta|^2$.

2) *Fields of Eisenstein Primes*: As an alternative choice of the signal-point lattice, the Eisenstein integers \mathbb{E} represent the hexagonal lattice over \mathbb{C} ($\Lambda_a = \mathbb{E}$). By analogy to the Gaussian primes, *Eisenstein primes* [3], [29], [32] of the form $\Theta = a + \omega b$ can be found that fulfill $\Theta \Theta^* = p$. Thereby, $\omega = (-1 + j\sqrt{3})/2 = e^{j2\pi/3}$ is the Eisenstein unit. In this case, $\text{rem}_6(p) = 1$ has to hold for the real-valued prime p , i.e., $p = 7, 13, 19, \dots$. When $\text{rem}_3(p) = 2$ is fulfilled instead, i.e., for the case when $p = 2, 5, 11, \dots$, real-valued Eisenstein primes of the form $\Theta = p$ are given.

Since the precoding lattice is now chosen as $\Lambda_p = \Theta \mathbb{E}$, where Θ is an Eisenstein prime, a zero-mean finite-field constellation is formed by $\mathcal{A}_{\Theta}^{(\mathbb{E})} \stackrel{\text{def}}{=} \mathcal{R}_V(\Theta \mathbb{E}) \cap \mathbb{E}$. The cardinality again reads $M = |\Theta|^2$.

The use of Eisenstein constellations additionally enables a packing and shaping gain compared to the Gaussian prime ones [26]: due to the higher packing density of the signal points and the hexagonal shaping region $\mathcal{R}_V(\Theta \mathbb{E})$ instead of the square one $\mathcal{R}_V(\Theta \mathbb{G})$, the power efficiency is increased.

3) *Finite-Field Processing*: Since fields of Gaussian or Eisenstein primes are isomorphic to the finite field of order $q = |\Theta|^2 = M$ [3], [26], an equivalent finite-field equalization

¹If \sqrt{M} is even, a shifted version of \mathbb{G} has to be applied, cf. [26].

² $\text{rem}_d(c) \stackrel{\text{def}}{=} c - d \lfloor c/d \rfloor$, with $c, d \in \mathbb{Z}$.

approach is possible for both cases [26], [27]: Employing these algebraic constellations, the cascade of integer preequalization and modulo operation can be interpreted as a finite-field multiplication with the matrix $\mathbf{P}\mathbf{Z}_{\mathbb{F}}^{-1} \in \mathbb{F}_q^{N_R \times N_R}$, where

$$\mathbf{P}\mathbf{Z}_{\mathbb{F}}^{-1} \simeq \mathbf{P}(\text{mod}_{\Lambda_p}(\mathbf{Z}))^{-1} \in \mathcal{A}_{\Theta}^{N_R \times N_R}. \quad (9)$$

The integer equalization is hence performed with the (integer) inverse of a modulo-reduced finite-field variant of \mathbf{Z} with elements isomorphic to \mathbb{F}_q . The inverse of $\mathbf{Z}_{\mathbb{F}} \in \mathbb{F}_q^{N_R \times N_R}$ always exists as long as $\mathbf{Z}_{\mathbb{F}} \simeq \text{mod}_{\Lambda_p}(\mathbf{Z}) \in \mathcal{A}_{\Theta}^{N_R \times N_R}$ has full rank. As a consequence, the channel factorization can be performed w.r.t. the *shortest independent vector problem* (SIVP): by a relaxation of the unimodularity constraint (i.e., $|\det(\mathbf{Z})| \geq 1$), a factorization gain may be achieved.

B. Coded Modulation over Algebraic Constellations

Due to the isomorphism of q -ary fields of Gaussian and Eisenstein primes to \mathbb{F}_q , a coded modulation approach is straightforward, cf. [27]. Performing the channel coding over the finite field \mathbb{F}_q , the elements $\varphi_1, \dots, \varphi_q$ are mapped to the q -ary constellation $\mathcal{A}_{\Theta}^{(\mathbb{G})} \simeq \mathbb{F}_q$ or $\mathcal{A}_{\Theta}^{(\mathbb{E})} \simeq \mathbb{F}_q$, respectively, where a natural mapping $\mathbb{F}_q \rightarrow \mathcal{A}_{\Theta}^{(\cdot)}$ via modulo reduction (7) can be used [26]. This strategy gives the possibility to operate in the same arithmetic for both channel coding and (integer) channel equalization (a precondition for the application of IF).

Non-binary LDPC codes are a suitable code class for the above coded modulation strategy due to the possibility of soft-decision decoding via non-binary belief-propagation (BP) decoding over \mathbb{F}_q [4] and the flexible code length (e.g., in contrast to Reed-Solomon codes).³ In particular, the subclass of *irregular repeat-accumulate codes* [21] adapted to the non-binary case [23] is of interest, as the parity-check matrix \mathbf{H}_c is guaranteed to have full rank. Consequently, a systematic linear encoding with the related generator matrix \mathbf{G}_c is possible.

An approximate metric for soft-decision decoding can be derived in the following way: According to Bayes' theorem, each element $l_{\rho,\gamma}$, $\rho = 1, \dots, q$, of the q -dimensional probability vector (pmf) for the sample y_γ is given by (cf. Sec. II)

$$l_{\rho,\gamma} = \Pr\{c_\gamma = \varphi_\rho | y_\gamma\} = \frac{\Pr\{y_\gamma | c_\gamma = \varphi_\rho\} \Pr\{c_\gamma = \varphi_\rho\}}{\Pr\{y_\gamma\}}. \quad (10)$$

The factors $\Pr\{c_\gamma = \varphi_\rho\}$ and $1/\Pr\{y_\gamma\}$ are constant $\forall \varphi_\rho$ and the first factor reads

$$\Pr\{y_\gamma | c_\gamma = \varphi_\rho\} = \sum_{\boldsymbol{\lambda} \in \Lambda_p} f_N(y_\gamma - (\mathcal{M}(\varphi_\rho) + \boldsymbol{\lambda})) \cdot \Pr_{\rho,\gamma}^{(\boldsymbol{\lambda})} \quad (11)$$

since an infinite number of modulo-congruent signal points is present at the receiver side [10]. Thereby,

$$f_N(n) = \frac{1}{\pi g^2 \sigma_n^2} \exp\left(\frac{-|n|^2}{g^2 \sigma_n^2}\right), \quad n \in \mathbb{C}, \quad (12)$$

³Literature on non-binary BP decoding is usually focused on the case $q = 2^b$, $b \in \mathbb{N}$, e.g., [5]. For arbitrary fields \mathbb{F}_q , a standard probability-domain BP decoding as explained in [4] can be performed over the related arithmetic. It should be noted that for each element of \mathbb{F}_q , additionally the probability of its additive inverse has to be calculated in the sum-product step (check-node message update). When choosing $q = 2^b$, this is typically neglected as the additive inverse is the element itself.

is the probability density function (pdf) of the scaled noise (factor g) and $\Pr_{\rho,\gamma}^{(\boldsymbol{\lambda})} \stackrel{\text{def}}{=} \Pr\{y_\gamma - n_\gamma = \mathcal{M}(\varphi_\rho) + \boldsymbol{\lambda}\} |_{\mathbf{Z}_{\mathbb{F}}}$ the probability of occurrence for each modulo-congruent signal point which depends on the actual integer equalization matrix. Due to the infinite number of congruent points, (11) has to be approximated. For mid-to-high SNRs, it is sufficient to assume

$$\Pr\{y_\gamma | c_\gamma = \varphi_\rho\} \approx f_N\left(y_\gamma - (\mathcal{M}(\varphi_\rho) + \tilde{\boldsymbol{\lambda}})\right) \cdot \Pr_{\rho,\gamma}^{(\tilde{\boldsymbol{\lambda}})}, \quad (13)$$

i.e., for each element of \mathbb{F}_q , only the neighboring modulo-congruent representative

$$\tilde{\boldsymbol{\lambda}} \stackrel{\text{def}}{=} \underset{\boldsymbol{\lambda} \in \Lambda_p}{\text{argmin}} |y_\gamma - (\mathcal{M}(\varphi_\rho) + \boldsymbol{\lambda})|^2 \quad (14)$$

is taken into account [10], where $\Pr_{\rho,\gamma}^{(\tilde{\boldsymbol{\lambda}})} \approx 1/q$. In summary, the vector $\tilde{\mathbf{l}}_\gamma = [\tilde{l}_{1,\gamma}, \dots, \tilde{l}_{q,\gamma}]^T$ with

$$\tilde{l}_{\rho,\gamma} \stackrel{\text{def}}{=} f_N\left(\min_{\boldsymbol{\lambda} \in \Lambda_p} |y_\gamma - (\mathcal{M}(\varphi_\rho) + \boldsymbol{\lambda})|\right) \quad (15)$$

is calculated and normalized to $\mathbf{l}_\gamma = \tilde{\mathbf{l}}_\gamma / \sum_{\rho=1}^q \tilde{l}_{\rho,\gamma}$.

C. LRA Preequalization over Algebraic Constellations

Since a modulo reduction—in case of LRA preequalization inherently defined by the precoding lattice Λ_p —is one of the basic ideas behind Tomlinson-Harashima-type precoding [31], [16], [10], the combination of LRA (integer) equalization and non-integer precoding / successive interference cancellation is a promising strategy.

1) *LRA Preequalization*: Given the channel factorization $\bar{\mathbf{H}} = \mathbf{Z}\bar{\mathbf{H}}_{\text{red}}$ (cf. (3)), the integer equalization matrix directly reads $\mathbf{Z}_p = \mathbf{Z}^{-1}$, i.e., $\mathbf{P} = \mathbf{I}$. The feedback part (cf. Fig. 2) is deactivated; the feedforward matrix for the residual non-integer equalization according to the minimum mean-square error (MMSE) criterion is the $N_T \times N_R$ upper part \mathbf{F} of⁴

$$\bar{\mathbf{F}} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{F}_{(N_T \times N_R)} \\ \bar{\mathbf{F}}_{(N_R \times N_R)} \end{bmatrix} = \bar{\mathbf{H}}_{\text{red}}^+ = \bar{\mathbf{H}}_{\text{red}}^H \left(\bar{\mathbf{H}}_{\text{red}} \bar{\mathbf{H}}_{\text{red}}^H \right)^{-1}. \quad (16)$$

The factorization tasks (3) or (5) can be solved by any lattice reduction / shortest independent vector algorithm. Employing fields of Gaussian primes ($\Lambda_a = \mathbb{G}$), a complex-valued factorization has to be supported. As an example, the complex variant [14] of the LLL algorithm [22] is suited, however, always resulting in an unimodular Gaussian-integer matrix \mathbf{Z} . For fields of Eisenstein primes ($\Lambda_a = \mathbb{E}$), an adapted version is possible [24], [26], resulting in an unimodular Eisenstein-integer matrix \mathbf{Z} . Recently—for receiver-side IF/LRA equalization—two algorithms which solve the SIVP instead of the SBP have been proposed [6], [12]. The dual variants can directly be applied for the situation at hand. Although both algorithms are designed for Gaussian integers, the strategy in [12] can be adapted to perform the search on Eisenstein integers.⁵

⁴If the factorization task (5) instead of (3) is solved, the feedforward matrix is directly given by the $N_T \times N_R$ upper part of $((\bar{\mathbf{H}}_{\text{red}}^+)^H)^H$.

⁵For the one proposed in [6] this is not possible as real- and imaginary part are separated to a real-valued representation of doubled dimension.

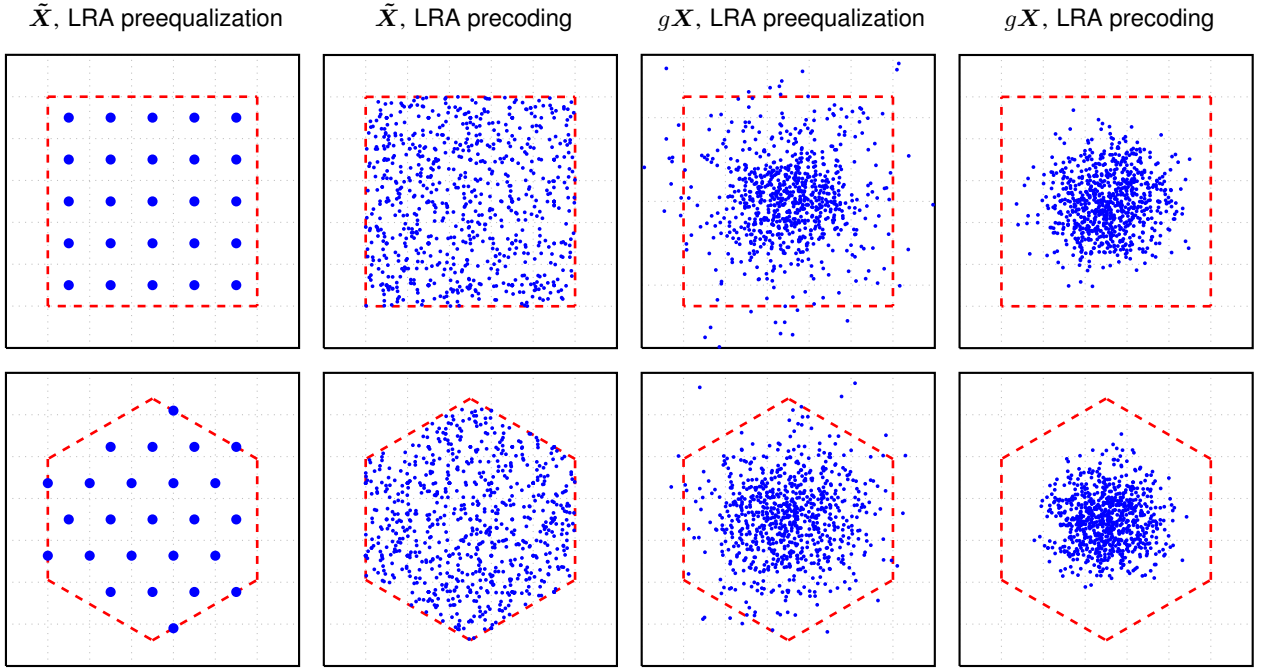


Fig. 3. Comparison of the signal processing for LRA preequalization and LRA precoding (cf. Fig. 2) for one exemplary channel realization. Channel factorization for the lattice \mathbb{G} or \mathbb{E} (MMSE solution); additional sorted QR decomposition for precoding. 25-ary square-QAM (top) and Eisenstein-prime (bottom) constellation. Uncoded transmission ($R_c = 1$); $E_{b, \text{TX}}/N_0 = 10$ dB. $N_T = N_R = 8$ and $k_c = 100$, i.e., 800 samples per illustration.

2) *LRA Precoding*: For the application of LRA precoding, additionally a sorted LQ decomposition according to [11]

$$P\bar{H}_{\text{red}} = L [Q \quad \bar{Q}] \stackrel{\text{def}}{=} L\bar{Q} \quad (17)$$

is necessary. Thereby, P is a $N_R \times N_R$ permutation matrix describing the optimum encoding order among the users. Usually—due to the uplink-downlink duality [33], [34]—the reversed VBLAST sorting [15] is used. The lower triangular $N_R \times N_R$ matrix L with unit main diagonal directly yields the feedback matrix (i.e., $B = L$), and \bar{Q} is a $N_R \times (N_R + N_T)$ matrix with orthogonal rows. The MMSE feedforward matrix for the residual equalization is the upper part \bar{F} of

$$\bar{F} = \begin{bmatrix} \bar{F}_{(N_T \times N_R)} \\ \bar{F}_{(N_R \times N_R)} \end{bmatrix} = \bar{Q}^+ = \bar{Q}^H (\bar{Q}\bar{Q}^H)^{-1}. \quad (18)$$

The integer matrix is now given by $Z_p = PZ^{-1}$ (or by its finite-field variant (9)), including both the integer equalization and the permutation for an optimized encoding order.

In contrast to LRA preequalization, two different factorization tasks have to be solved: first the SBP/SIVP and afterwards the sorted LQ decomposition. Both steps may be combined into a single factorization algorithm (cf. [11]), however, the state-of-the-art approaches are limited to the SBP.

3) *Comparison of Preequalization and Precoding*: Though both LRA preequalization and LRA precoding are performing the same integer-based equalization they differ in how to treat the residual non-integer interference. In the following, this will be discussed with the help of Fig. 3, where LRA preequalization and LRA precoding are exemplarily illustrated for a 25-ary square-QAM and Eisenstein-prime constellation.

Via Z_p , linear combinations of data symbols or lattice points, respectively, are calculated. Performing LRA preequalization, the encoded symbols \bar{A} are simply modulo reduced. This results in symbols \bar{X} that are again elements of the constellation \mathcal{A} (Fig. 3 left). For algebraic constellations $\mathcal{A}_{\mathbb{E}}^{(\cdot)}$ this has the consequence that still elements of the finite field are present. The residual non-integer interference is equalized by the pseudo-inverse of the reduced augmented channel matrix.

In LRA precoding, the feedforward matrix shapes the reduced channel to have a lower triangular structure; consequently the remaining causal interference can perfectly be eliminated by successive interference cancellation. As in conventional THP—due to the modulo operation—this results in approximately uniformly distributed encoded symbols \bar{X} over $\mathcal{R}_V(\Lambda_p)$ (cf. Fig 3). A finite-field property of \bar{X} is no longer present as a part of the non-integer interference is already incorporated.

In Fig. 3 (right side), the transmit symbols X (after feedforward processing) are illustrated for both cases neglecting the scaling factor g which enables a fair comparison. Apparently, on average precoding results in lower signal amplitudes. This is accompanied by a lower scaling factor g for a constant transmit power (dual to the noise enhancement for receiver-side equalization) which finally results in an increase in the receiver-side SNR and hence an improved performance. The gain is induced by a lower row norm [11] of the feedforward matrix when performing precoding instead of a simple inversion of the reduced channel matrix. Moreover, as can be seen from Fig. 3 (both preequalization and precoding), the mean amplitude is even more decreased when the Eisenstein

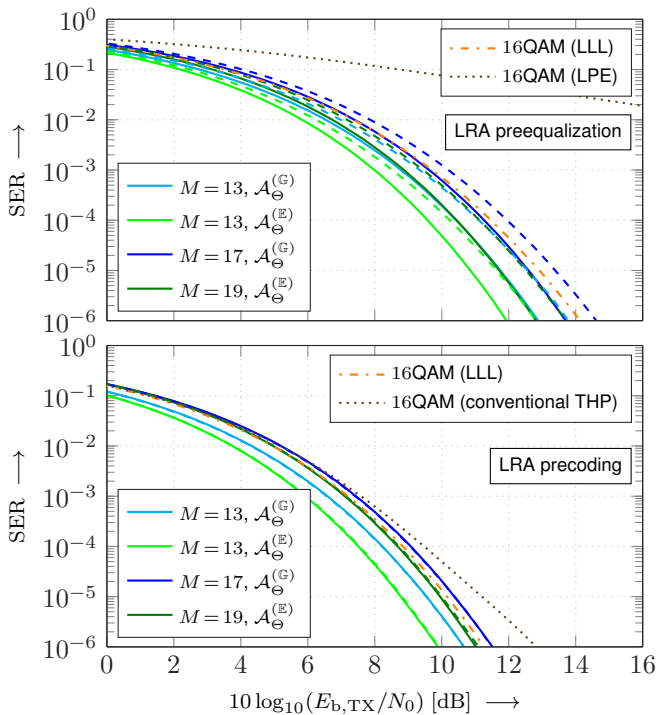


Fig. 4. SER over $E_{b,TX}/N_0$ in dB for LRA preequalization (top) and LRA precoding (bottom) assuming uncoded transmission. Parameter: M ; variation of the signal constellation \mathcal{A} and the related signal-point lattice and precoding lattice. Solid: Factorization according to the SIVP. Dashed: Factorization according to the SBP (LLL algorithm). $N_T = N_R = 8$. The results for a 16QAM constellation are shown when performing both LRA (LLL factorization) and conventional preequalization/precoding.

constellation is applied. This not only results from the packing and shaping gain, but also from a factorization gain due to the higher packing density [26].

IV. NUMERICAL RESULTS

In this section, numerical results for LRA preequalization and LRA precoding over algebraic constellations are provided. The gain in transmission performance is quantified when a factorization according to the SIVP instead of the SBP is applied. Besides, the coded-modulation approach at hand is compared to state-of-the-art techniques. The results are always averaged over all users and a large number of noise and channel realizations. For LRA transmission, the factorization task according to (5) is solved (MMSE solution) in any case.

A. Uncoded Transmission

For a reasonable assessment of the coded-modulation approach and its impact on the transmission performance we first restrict to the uncoded case ($R_c = 1$): the encoding is skipped and the decoding is replaced by a quantization to the signal-point lattice and a modulo operation w.r.t. the precoding lattice (cf. [26]). Hence, the channel equalization is in the focus. In order to achieve a fair comparison among constellations of different cardinalities, the SNR is always considered as energy *per bit* as given in (2). Nevertheless, in the uncoded case the different modulation rates cannot be compensated completely.

TABLE I
SIMULATION PARAMETERS.

\mathcal{A}	Λ_a	Field	μ	ν	n_c	k_c	Info-Bits
$\mathcal{A}_{\Theta}^{(G)}$	G	\mathbb{F}_{13}	37	10	16200	8760	32412
$\mathcal{A}_{\Theta}^{(E)}$	E	\mathbb{F}_{13}	37	10	16200	8760	32412
$\mathcal{A}_{\Theta}^{(G)}$	G	\mathbb{F}_{17}	94	23	16200	7935	32430
$\mathcal{A}_{\Theta}^{(E)}$	E	\mathbb{F}_{19}	497	117	16200	7722	32802
16QAM	G	\mathbb{F}_{16}	4	1	16200	8100	32400
16QAM	G	\mathbb{F}_2	-	-	64800	32400	32400

1) *SIVP and SBP for Preequalization and Precoding*: First, we assess the achievable gains when the finite-field preequalization or precoding approach is performed on the basis of the SIVP. Since—even for the SIVP—a significant number of channel realization is non-unimodular only if $N_R \geq 4$ (cf. [12]), we restrict to the high-diversity case $N_R = N_T = 8$.

In Fig. 4, the symbol-error rate (SER; i.e., $\hat{m} \neq m$) is depicted over the SNR for both LRA preequalization (top) and precoding (bottom) and cardinalities around $M = 16$. Regarding LRA preequalization, the superiority of a factorization according to the SIVP (solid) is clearly visible. For all algebraic constellations, a gain of about 1 dB is present compared to LLL factorization (high-SNR regime). The situation changes when precoding is applied instead: in that case, hardly any gain is possible when applying the SIVP (curves nearly lie on top of each other). Since the transmission performance is generally increased compared to the linear case (about 2 dB), the factorization gain resulting from the SIVP is very low. Noteworthy, all curves converge to diversity eight.

In addition, the results when applying a conventional QAM constellation are shown. In this case, an integer preequalization matrix \mathbf{Z}_p is only present if an unimodular factorization is applied (here: complex LLL algorithm). The performance of both LRA preequalization and LRA precoding over 16QAM is approximately equivalent the one of the Gaussian prime constellations (LLL factorization). Conventional (non-LRA) linear preequalization (LPE) only achieves diversity order one. When applying conventional THP instead, the performance is distinctly improved. Nevertheless, a flattening to diversity order one still occurs.

2) *Binary Transmission*: For the assessment of the factorization and precoding gain, the SER is a suited measure. However, in real systems most often a binary end-to-end transmission is considered, i.e., a modulus conversion has to be performed for q -ary constellations. We hence consider a binary transmission and its related bit-error rate (BER). The simulation parameters are listed in Table I; for the moment the last three columns and last two rows can be omitted.

The simulation results for $N_T = N_R = 8$ are shown in Fig. 5. First—to show the relation between SER and BER—the SER curves for LRA preequalization and LRA precoding from Fig. 4 (SIVP) are combined (top). Again, the superiority of LRA precoding and a packing, shaping, and factorization gain of the Eisenstein lattice [26] is visible. Considering the BER (bottom), the precoding gain expectable from the

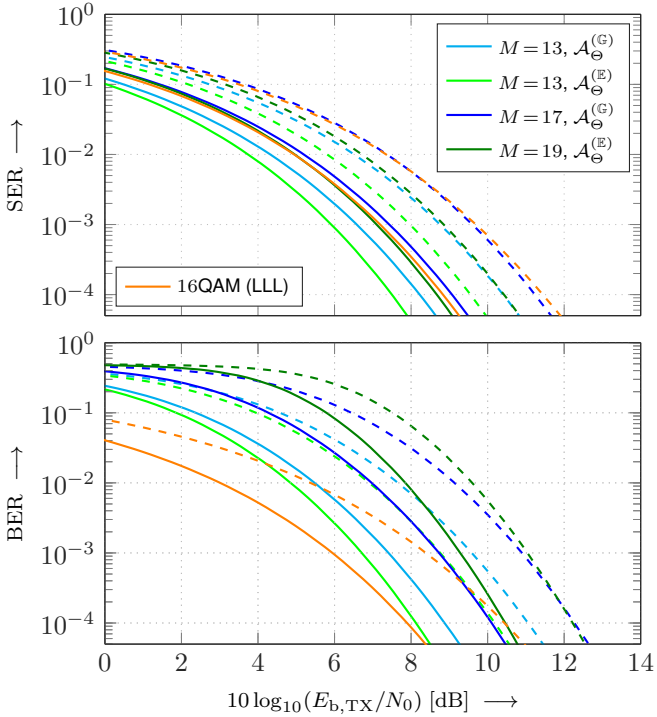


Fig. 5. SER (top) and BER (bottom) over $E_{b,TX}/N_0$ in dB for LRA preequalization (dashed) and LRA precoding (solid) assuming uncoded binary transmission. Parameter: M ; variation of the signal constellation \mathcal{A} and the related signal-point lattice and precoding lattice. Factorization according to the SIVP; $N_T = N_R = 8$. The results for 16QAM (LLL factorization) are shown in addition.

SER curves is present, but all non-QAM constellations suffer from an error propagation in the inverse modulus conversion [26]. Additionally, a direct mapping from bits allows a *Gray labeling*, still more increasing the BER performance. Even the 13-ary Eisenstein constellation only achieves a performance similar to the 16-ary QAM one (high-SNR regime), but with a decrease in modulation rate. An uncoded binary transmission over algebraic constellations is hence not advisable.

B. Coded Transmission

Finally, coded modulation over algebraic signal constellations is assessed. For this purpose, non-binary irregular repeat-accumulate codes as proposed in [27] are employed. A systematic linear encoding and a belief-propagation decoding with a maximum number of 100 iterations is performed.

Fig. 6 illustrates the results for an end-to-end binary coded transmission; the transmission and code parameters are again listed in Table I. To have a fair comparison among the different settings, by adapting the code rate R_c the number of information bits per code block is fixed to achieve the same amount of transmission data. For comparison, a 16-ary square-QAM transmission is studied performing the channel coding in the extension field \mathbb{F}_{16} with the above code construction. Besides, the conventional bit-interleaved coded modulation (BICM) [2] approach is applied, where a respective bit-log-likelihood metric and a well-optimized binary repeat-accumulate parity-

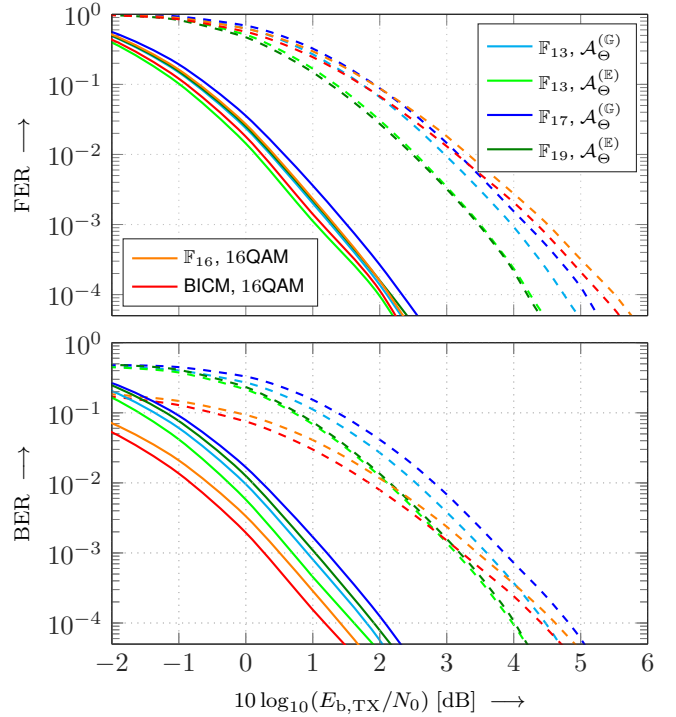


Fig. 6. FER (top) and BER (bottom) over $E_{b,TX}/N_0$ in dB for coded modulation over algebraic signal constellations. LRA preequalization (dashed) or LRA precoding (solid). Variation of the cardinality / order of the field, the signal constellation \mathcal{A} and the related signal-point lattice and precoding lattice. Factorization according to the SIVP; $N_T = N_R = 8$. The results for 16QAM (LLL factorization; BICM or coding over \mathbb{F}_{16}) are shown in addition.

check matrix from the DVB-S2 standard [8] are employed. Noteworthy, for the algebraic constellations the SIVP is again present, for the QAM one the SBP (LLL factorization) instead.

Considering the frame-error rate (FER; i.e., $\hat{m} \neq m$) of LRA preequalization, equivalent results and hence conclusions as in [27] are present: even though the binary code is highly optimized (cf. BICM vs. code over \mathbb{F}_{16}), the algebraic coded-modulation approach shows a superior performance due to the factorization gain induced by the SIVP. Furthermore, the additional gain of the Eisenstein constellations is visible (gain of about 1 dB compared to BICM). Applying LRA precoding, the transmission performance can generally be increased (by about 2.5–3 dB). However, as then a factorization according to the SIVP doesn't achieve a significant gain any more, all algebraic and non-algebraic constellations nearly perform the same. The gain of the Eisenstein lattice is lowered; the 13-ary Eisenstein constellation is nevertheless still performing best. As a maximum-likelihood channel equalization is more approached with LRA precoding, the optimization of the channel code becomes more important for the system performance.

Concerning the BER (Fig. 6 bottom), the negative impact of the error propagation in the inverse modulus conversion (if the block/frame cannot be decoded correctly) is visible, degrading the performance of the non-QAM constellations. However, in the low-BER regime of LRA preequalization, the Eisenstein-based transmissions even perform better than the BICM one

with optimized code. For LRA precoding, in contrast, a small loss is present in comparison to BICM since the factorization gain of the SIVP and the Eisenstein lattice is lowered.

V. SUMMARY AND CONCLUSIONS

In this paper, we have presented an LRA MIMO broadcast channel transmission strategy where the integer interference is eliminated over finite-field constellations. Since non-unimodular integer equalization matrices can be employed in this case, a factorization according to the shortest independent vector problem is enabled. A coded modulation scheme, where the channel coding and the integer channel equalization are performed over the same q -ary arithmetic, has been assessed. For the cancellation of the non-integer interference, both LRA linear preequalization and LRA precoding have been considered, including a discussion on the advantages of precoding.

Numerical results have revealed that a factorization according to the SIVP is only reasonable for LRA preequalization, as the achievable gain for LRA precoding is very low. However—especially on the basis of the Eisenstein lattice—the algebraic coded-modulation approach is promising: in combination with the application of optimized non-binary (q -ary) LDPC codes, e.g. [19], further gains in the coding and thus the transmission performance can be expected. Since non-binary LDPC codes outperform binary ones in BP decoding [4], q -ary coded modulation schemes are suited to achieve a significant increase in performance compared to conventional strategies like BICM.

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