# Multilevel Coding over Eisenstein Integers with Ternary Codes

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***Abstract*—This work introduces new signal constellations based on Eisenstein integers, i.e., the hexagonal lattice. These sets of Eisenstein integers have a cardinality which is an integer power of three. They are proposed as signal constellations for representation in the equivalent complex baseband model, especially for applications like physical-layer network coding or MIMO transmission where the constellation is required to be a subset of a lattice. It is shown that these constellations form additive groups where the addition over the complex plane corresponds to the addition with carry over ternary Galois fields. A ternary set partitioning is derived that enables multilevel coding based on ternary error-correcting codes. In the subsets, this partitioning achieves a gain of 4.77 dB, which results from an increased minimum squared Euclidean distance of the signal points. Furthermore, the constellation-constrained capacities over the AWGN channel and the related level capacities in case of ternary multilevel coding are investigated. Simulation results for multilevel coding based on ternary LDPC codes are presented which show that a performance close to the constellation-constrained capacities can be achieved.**

## I. Introduction

Research on signal constellations for bandwidth-efficient coherent transmission has a long history. Among the large number of different types, some constellations were proposed in order to construct linear or group codes over complex-valued alphabets. This concerns, e.g., phase-shift keying constellations [1], or algebraic constellations based on the Gaussian (complex) integers [2] and the Eisenstein integers [3]. The latter form the hexagonal lattice in the complex plane.

In digital communications, square quadrature-amplitude modulation (QAM) constellations are usually used, which form a special case of Gaussian-integer constellations. However, the packing of the related lattice—the complex-valued (Gaussian) integers—is not optimal. In [4], [5], so-called triangular constellations were proposed in order to achieve a power gain. Thereby, the signal points are drawn from the Eisenstein integers—in combination with square boundaries for the constellation like in square QAM. For these constellations, the exact error probability over the additive-white-Gaussian-noise (AWGN) channel was derived in [6].

Since the Eisenstein integers are the densest packing in two dimensions [7], related modulation schemes were proposed for various applications like multicarrier modulation [8] or hierarchical transmission [9]. They are especially relevant in physical-layer networking coding [10], [11] and multiple-input/multiple-output (MIMO) transmission [12], [13], [14], where the signal points have to be drawn from regular grids, particularly from (periodically extendable) subsets of lattices. However, only a few publications on *coded* modulation over Eisenstein integers are available. A reason for that might be that bit-interleaved coded modulation [15] is not suited since a Gray labeling is not possible [14]. Particularly, in [16], a 243-ary hexagonal constellation with corresponding set partitioning has been given. Besides, an 18-ary constellation for multilevel coding [17] with ternary turbo codes has been studied in [18].

In this contribution, we propose new signal constellations based on Eisenstein integers with cardinalities of the form $3^m$, with $m \in \mathbb{N}$. They are defined by a mapping from ternary vectors onto the Eisenstein integers using a modulo function with hexagonal boundaries. Hence, these constellations have some similarity to the Eisenstein-integer fields introduced in [3], where an isomorphic mapping from the Galois field $\mathrm{GF}(p)$ onto the Eisenstein integers was introduced. Those signal constellations are isomorphic to finite fields. However, the isomorphism in [3] is only defined for primes $p$, where $p-1$ is divisible by 6. For such constellation, no natural set partitioning exists as the number of signal points is a prime.

In order to enable set partitioning and multilevel-coded modulation for Eisenstein integers, multiplicative groups were considered in [13]. Similarly, we present constellations that are based on additive groups. The proposed signal sets are constructed by mapping $m$-tuples of ternary symbols onto the Eisenstein integers. We show that the addition over these sets corresponds to the addition with carry over the related $m$-tuples by analogy with binary carry-based addition and Gaussian-integer sets [19], [20]. Besides, the derived constellations have a natural set partitioning that enables multilevel coding with ternary codes. The squared Euclidean distance increases by a factor of 3, i.e., by $4.77$ dB, with the partition level. Hence, in practice, only the first levels need protection by error-correcting codes to achieve the desired performance.

The paper is organized as follows: In Sec. II, we review the basic notions of Eisenstein integers. The proposed constellations and their partitioning are discussed in Sec. III. The related capacities over the AWGN channel are investigated in Sec. IV. In Sec. V, simulation results for multilevel coding with ternary low-density parity-check (LDPC) codes are presented. Sec. VI gives a brief summary and conclusions.

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## II. Eisenstein Integers and Constellations

The Eisenstein integers

$$\mathbb{Z}[\omega] = \{z = a + b\,\omega \mid a, b \in \mathbb{Z}\} \tag{1}$$

are complex numbers which isomorphically represent the hexagonal lattice ($\boldsymbol{A}_2$) in the complex plane [7]. Their basis element $\omega = \frac{1}{2}(-1 + \mathrm{j}\sqrt{3}) = \mathrm{e}^{\mathrm{j}2\pi/3}$ is a complex root of unity, i.e., $|\omega|^2 = \omega\,\omega^* = 1$, with the complex conjugate $\omega^* = \frac{1}{2}(-1 - \mathrm{j}\sqrt{3})$. They form a quadratic integer ring which is a principle ideal domain and also a Euclidean domain. The norm of an Eisenstein integer reads $N(z) = |z|^2 = z\,z^*$, with the complex conjugate $z^* = a + b\,\omega^*$. For any $\lambda \in \mathbb{Z}[\omega]$ with inverse $\lambda^{-1} = \frac{\lambda^*}{N(\lambda)}$, we can define a modulo function

$$\mathrm{mod}_\lambda\{z\} = z - \mathrm{Q}\left\{z\lambda^{-1}\right\} \cdot \lambda\,, \tag{2}$$

where $\mathrm{Q}\{\cdot\}$ denotes the quantization to the closest Eisenstein integer [3], [7]. In particular, $\mathrm{mod}_\lambda\{z\} \in \mathcal{R}_\mathrm{V}(\lambda\,\mathbb{Z}[\omega])$, i.e., it is located in the hexagonal Voronoi region [7], [21] of the Eisenstein-integer lattice scaled by $\lambda$. The boundaries of the modulo function (i.e., the boundaries of the Voronoi region) have to be handled properly to ensure that a unique remainder is obtained. This can be achieved via Algorithm 1 [11].

**Algorithm 1** Calculate remainder $\mathrm{mod}_\lambda\{z\}$ [11]

i) Calculate the complex number $x = z/\lambda$.

ii) Calculate the nearest Eisenstein integers $\beta_1$ and $\beta_2$ as

$$\beta_1 = \lfloor \mathrm{Re}\{x\} \rceil + \sqrt{-3}\left\lfloor \frac{\mathrm{Im}\{x\}}{\sqrt{3}} \right\rceil \tag{3}$$

$$\beta_2 = \lfloor \mathrm{Re}\{x - \omega\} \rceil + \sqrt{-3}\left\lfloor \frac{\mathrm{Im}\{x - \omega\}}{\sqrt{3}} \right\rceil + \omega \tag{4}$$

where $\lfloor\cdot\rceil$ denotes rounding to real-valued integers.

iii) Calculate $r_1 = z - \lambda\beta_1$ and $r_2 = z - \lambda\beta_2$.

iv) Set

$$\mathrm{mod}_\lambda\{z\} = \begin{cases} r_1, & \text{if } |r_1| < |r_2| \cup \\ & |r_1| = |r_2| \cap \mathrm{Re}\{\beta_1\} < \mathrm{Re}\{\beta_2\} \\ r_2, & \text{else.} \end{cases} \tag{5}$$

We consider finite sets of Eisenstein integers as signal constellations. From lattice theory, it is known that a constellation

$$\mathcal{A} = \boldsymbol{\Lambda}_\mathrm{a} \cap \mathcal{R}_\mathrm{V}(\boldsymbol{\Lambda}_\mathrm{b}) \tag{6}$$

is constructed by the intersection of the signal-point lattice $\boldsymbol{\Lambda}_\mathrm{a}$ and the Voronoi region of the boundary or shaping lattice $\boldsymbol{\Lambda}_\mathrm{b}$. By choosing $\boldsymbol{\Lambda}_\mathrm{a} = \mathbb{Z}[\omega]$ and $\boldsymbol{\Lambda}_\mathrm{b} = \lambda\,\mathbb{Z}[\omega]$, we obtain $M = |\boldsymbol{\Lambda}_\mathrm{a}/\boldsymbol{\Lambda}_\mathrm{b}| = N(\lambda)$ periodically extendable signal points drawn from $\mathbb{Z}[\omega]$ within hexagonal boundaries. Thereby, (2) is the related modulo function. In comparison to square QAM constellations, we asymptotically achieve a *packing gain* of $0.6247$ dB (hexagonal signal points) and a *shaping gain* of $0.1671$ dB (hexagonal boundaries), cf. [7], [21]. Non-uniform signal-point probabilities [21] enable additional shaping gains for both QAM and Eisenstein constellations. That shaping approach is beyond the scope of this paper, though.

In [3], the abovementioned construction is applied to define zero-mean finite sets of Eisenstein integers with cardinality $M = p$, where the prime $p$ has to be chosen such that $p - 1$ is divisible by 6. Then, $p$ can be represented as the product of two conjugate Eisenstein integers $p = \epsilon\,\epsilon^* = N(\epsilon)$ [3], where $\epsilon$ is called an Eisenstein prime that is employed as the scaling factor $\lambda = \epsilon$ in (2) or (6), respectively. For such primes, (2) is an isomorphic mapping from the Galois field $\mathrm{GF}(p)$ onto the hexagonal lattice, i.e., the signal constellations are isomorphic to finite fields. They are formed by

$$\mathcal{E}_p = \{\mathrm{mod}_\epsilon\{\psi(\mathfrak{z})\} \mid \mathfrak{z} \in \mathrm{GF}(p)\} \tag{7}$$

where $\psi(\cdot)$ denotes the *natural* mapping from the $p$ finite-field elements[1] to their corresponding integers $0, 1, \ldots, p-1$, i.e., $\psi(\mathfrak{0}) = 0, \psi(\mathfrak{1}) = 1, \ldots$. However, since the number of field elements is a prime, no regular set partitioning exists for such constellations. In order to enable set partitioning and multilevel-coded modulation for Eisenstein integers, multiplicative groups are considered in [13], which are based on the constellations [3] omitting the zero element. Hence, they have a cardinality $M = N(\epsilon) - 1$ which is always a multiple of 6. The corresponding multilevel codes are based on binary as well as ternary error correcting codes.

In the next section, we will propose new Eisenstein signal constellations with cardinality $M = 3^m$, $m \in \mathbb{N}$. Similar to the construction in [3], the signal points are defined by a mapping from ternary vectors onto the Eisenstein integers using the complex modulo function. We use the following notations: The average energy of a constellation $\mathcal{A}$ with cardinality $M$ is the expected energy when all elements are used with equal probability, i.e., we have $E(\mathcal{A}) = \frac{1}{M}\sum_{z\in\mathcal{A}} N(z)$. The squared Euclidean distance of two signal points $z, y \in \mathcal{A}$ is defined as $d_\mathrm{E}(y, z) = N(z - y)$ and the constellation's minimum squared Euclidean distance as $\delta^2(\mathcal{A}) = \min_{z,y\in\mathcal{A}, z\neq y} d_\mathrm{E}(y, z)$.

## III. Novel Eisenstein Constellations and their Set Partitioning

In the following, we will discuss novel signal constellations and related set partitions over Eisenstein integers. The concept is similar to the multilevel coding scheme and binary address labeling via basis extension discussed in [20]. However, we consider a ternary address labeling. Such an address can be represented as a ternary $m$-tuple $\mathbf{t} = (\mathfrak{t}_{m-1}, \ldots, \mathfrak{t}_1, \mathfrak{t}_0)$ with $\mathfrak{t}_l \in \mathrm{GF}(3)$, $l = 0, \ldots, m-1$. The ternary $m$-tuple is mapped to a corresponding Eisenstein integer via the ternary basis extension w.r.t. the complex-valued basis $\phi = -1 + \omega$, particularly given as

$$\mathcal{M}_m(\mathbf{t}) = \mathrm{mod}_{\phi^m}\Big\{\sum_{l=0}^{m-1} \phi^l\,\psi(\mathfrak{t}_l)\Big\}\,, \tag{8}$$

with the natural mapping $\psi(\mathfrak{0}) = 0$, $\psi(\mathfrak{1}) = 1$, and $\psi(\mathfrak{2}) = 2$. It involves the modulo reduction with parameter $\lambda = \phi^m$ according to (2) in order to map the resulting signal points into $\mathcal{R}_\mathrm{V}(\phi^m\,\mathbb{Z}[\omega])$.

[1] Elements that are drawn from the Galois field $\mathrm{GF}(p)$ and related variables are written in Fraktur font, e.g., $\mathfrak{z}$ or $\mathfrak{t}$.

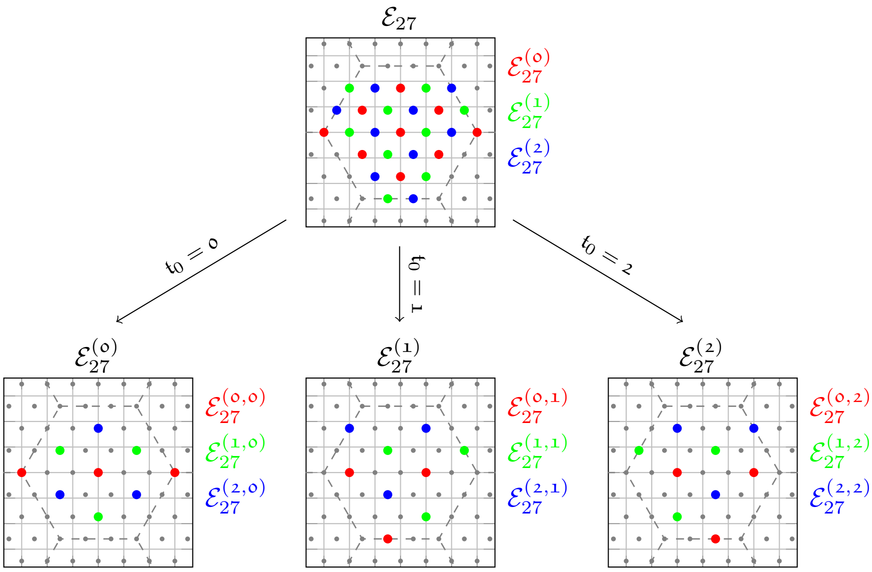


Fig. 1. Eisenstein constellation $\mathcal{E}_{27}$ according to (8), its three subsets of level $l = 1$ denoted as $\mathcal{E}_{27}^{(\mathfrak{t}_0)}$, and the nine subsets in level $l = 2$ denoted as $\mathcal{E}_{27}^{(\mathfrak{t}_1,\mathfrak{t}_0)}$. Points with the same color belong to the same subset. The boundaries of the Voronoi region $\mathcal{R}_\mathrm{V}(\phi^3\,\mathbb{Z}[\omega])$ and the related periodic extensions are shown.

The mapping $\mathcal{M}_m(\mathbf{t})$ of an $m$-tuple $\mathbf{t}$ onto the Eisenstein integers defines an Eisenstein constellation

$$\mathcal{E}_M = \{\mathcal{M}_m(\mathbf{t}) \mid \mathbf{t} = (\mathfrak{t}_{m-1}, \ldots, \mathfrak{t}_1, \mathfrak{t}_0)\ ,\ \ \mathfrak{t}_l \in \mathrm{GF}(3)\} \quad (9)$$

with cardinality $M = N(\phi^m) = |-1+\omega|^{2m} = 3^m$. As an example, in Fig. 1, the constellation $\mathcal{E}_{27}$ including the boundaries of $\mathcal{R}_\mathrm{V}(\phi^3\,\mathbb{Z}[\omega])$ and the related periodic extensions are shown (i.e., for the case $m = 3$).

The resulting Eisenstein constellations are zero-mean and form additive groups w.r.t. addition $\mathrm{mod}_{\phi^m}\{\cdot\}$. The linear combination $\sum_{l=0}^{m-1} \phi^l\,\psi(\mathfrak{t}_l)$ in (8) is a special case of a ternary arithmetic with the complex basis $\phi = -1+\omega$. A similar binary arithmetic for the Gaussian integers is described in [19]. We illustrate this arithmetic in the following examples.

**Example 1.** *We consider the ternary* $2$*-tuple* $\mathbf{t} = (\mathfrak{t}_1, \mathfrak{t}_0)$ *with* $m = 2$ *and* $M = 3^m = 9$*. Hence, we have the basis* $\phi^0 = 1$, $\phi^1 = -1+\omega = -\frac{3}{2} + \mathrm{j}\frac{1}{2}\sqrt{3}$*, and* $\lambda = \phi^2 = 1 + \omega^2 - 2\,\omega = \frac{3}{2} - \mathrm{j}\frac{3}{2}\sqrt{3}$*. All possible linear combinations* $\phi\,\mathfrak{t}_1 + \mathfrak{t}_0$ *and the set* $\mathcal{E}_9$ *resulting from the mapping* $\mathcal{M}_2(\mathbf{t})$ *are provided in Table I. Note that the mapping* $\mathcal{M}_2(\mathbf{t})$ *is an isomorphism with respect to addition. Consider, e.g., the element-wise sum* $(2,1)+(2,2) = (1,0)$*. In the complex plane this corresponds to* $\omega^* - \omega = -2\,\mathrm{j}\,\mathrm{Im}\{\omega\} = -\mathrm{j}\sqrt{3}$ *which is modulo-equivalent to* $\mathrm{mod}_{\phi^2}\{\omega^* - \omega\} = 1 - \omega^*$*.*

For $m > 2$, addition of the ternary $m$-tuples requires addition with carry as described in [19] for Gaussian integers. For Gaussian integers, the equivalent mapping of (9) employs a binary address labeling with $\mathfrak{t}_l \in \mathrm{GF}(2)$ and the complex basis $\phi = -1 + \mathrm{j}$. Then, the carry is equivalent to the binary representation of the Gaussian (or real-valued) integer 2 that corresponds to the binary 4-tuple $(1,1,0,0)$ (see also [20]).

TABLE I
MAPPING FOR THE SET WITH $M = 3^2 = 9$ ELEMENTS.

| $(\mathfrak{t}_1, \mathfrak{t}_0)$ | $\phi\,\psi(\mathfrak{t}_1) + \psi(\mathfrak{t}_0)$ | $\mathcal{M}_2(\mathbf{t})$ |
|---|---|---|
| $(0,0)$ | $0$ | $0$ |
| $(0,1)$ | $1$ | $1$ |
| $(0,2)$ | $2$ | $-1$ |
| $(1,0)$ | $-1+\omega$ | $1-\omega^*$ |
| $(1,1)$ | $\omega$ | $\omega$ |
| $(1,2)$ | $1+\omega$ | $1+\omega$ |
| $(2,0)$ | $-2+2\,\omega$ | $-1+\omega^*$ |
| $(2,1)$ | $-1+2\,\omega$ | $\omega^*$ |
| $(2,2)$ | $2\,\omega$ | $-\omega$ |

For the ternary case with $\mathfrak{t}_l \in \mathrm{GF}(3)$ and basis $\phi = -1+\omega$ the carry corresponds to the ternary representation of the Eisenstein (or real-valued) integer $3 = \phi^3 + 2\,\phi^2$, i.e., the ternary 4-tuple $(1,2,0,0)$. Let $\mathfrak{a}$ and $\mathfrak{b}$ denote two ternary $m$-tuples with complex images $a = \mathcal{M}_m(\mathfrak{a})$ and $b = \mathcal{M}_m(\mathfrak{b})$, respectively. The sum $\mathrm{mod}_{\phi^m}\{a+b\}$ can be calculated as

$$\begin{aligned}\mathrm{mod}_{\phi^m}\{a+b\} = \mathrm{mod}_{\phi^m}\Big\{ &\sum_{l=0}^{m-1} \phi^l\,\psi(\mathfrak{a}_l + \mathfrak{b}_l) \\ &+ \sum_{l=0}^{m-3} (\phi^{3+l} + 2\,\phi^{2+l})\,\psi(\mathfrak{a}_l \odot \mathfrak{b}_l)\Big\}\ ,\end{aligned} \quad (10)$$

where the operator $\odot$ is defined as

$$\mathfrak{a}_l \odot \mathfrak{b}_l = \begin{cases} 0, & \text{if } \psi(\mathfrak{a}_l) + \psi(\mathfrak{b}_l) < 3, \\ 1, & \text{else.} \end{cases} \quad (11)$$

The first sum in (10) corresponds to the element-wise addition of the two ternary $m$-tuples in $\mathrm{GF}(3)$, whereas the second sum represents the carries. We illustrate this addition with carry with a brief example.

**Example 2.** *Using the basis $\phi = -1 + \omega$ and $m = 4$, the complex-plane points $a = \frac{7}{2} - \mathrm{j}\,\frac{\sqrt{3}}{2}$ and $b = \frac{5}{2} - \mathrm{j}\,\frac{\sqrt{3}}{2}$ have the sum $c = \mathrm{mod}_{\phi^4}\{a + b\} = -3 - \mathrm{j}\,\sqrt{3}$. Addition of the two 4-tuples $\mathfrak{a} = (1, 0, 2, 2)$ and $\mathfrak{b} = (1, 0, 2, 1)$ is calculated as:*

$$\begin{array}{cc|l} & (0,1,0,2,2) & \mathfrak{a} \\ + & (0,1,0,2,1) & \mathfrak{b} \\ \hline + & (0,1,2,0,0) & \mathrm{carry}_{l=0} \\ + & (1,2,0,0,0) & \mathrm{carry}_{l=1} \\ \hline\hline = & (1,2,2,1,0) & \tilde{\mathfrak{c}} \end{array}$$

*Note that the sum in the least significant position ($l = 0$) leads to the carry $(0, 1, 2, 0, 0)$ which represents the Eisenstein integer 3. The sum in the second position ($l = 1$) leads to the carry $(1, 2, 0, 0, 0)$, which is obtained by left-shifting the carry $(0, 1, 2, 0, 0)$. Only the $m$ least significant positions are incorporated into the final result due to $\mathrm{mod}_{\phi^m}\{\cdot\}$ in (8). In particular, the modulo leads to a truncation of the elements above $\tilde{\mathfrak{c}}_{m-1} = \tilde{\mathfrak{c}}_3$ in ternary representation. Hence, the leading position in $\tilde{\mathfrak{c}}$ is omitted, resulting in the sum $\mathfrak{c} = (2, 2, 1, 0)$ that corresponds to the complex image $c = -3 - \mathrm{j}\,\sqrt{3}$.*

The minimum squared Euclidean distance in the Eisenstein-integer constellations is $\delta^2(\mathcal{A}) = 1$. The proposed construction enables a natural partitioning into additive subgroups and their cosets where the minimum distance in the subsets increases. In the following proposition we demonstrate that this construction also simplifies the search for the minimum squared Euclidean distance in the subsets.

**Proposition 1.** *Let $\mathcal{A}$ be an additive group w.r.t. the addition $\mathrm{mod}_\lambda\{\cdot\}$. Furthermore, let $\mathcal{A}'$ be any coset of $\mathcal{A}$, i.e., $\mathcal{A}' = \mathcal{A} + c$, $c \in \mathbb{Z}[\omega] \setminus \mathcal{A}$, then the minimum squared Euclidean distances $\delta^2(\mathcal{A})$, $\delta^2(\mathcal{A}')$ satisfy*

$$\delta^2(\mathcal{A}) = \delta^2(\mathcal{A}') = \min_{x \in \mathcal{A}, x \neq 0} N(x). \tag{12}$$

*Proof.* Consider two elements $z$, $z'$ from the set $\mathcal{A}$. The squared Euclidean distance is $d_\mathrm{E}(z, z') = N(z - z')$. Now, consider $z'' = \mu_\lambda(z - z')$. Due to Algorithm 1, we can ensure that $N(z - z') \geq N(z'')$ [11]. Hence, we have

$$\delta^2(\mathcal{A}) = \min_{z,z' \in \mathcal{A}, z \neq z'} d_\mathrm{E}(z, z') \geq \min_{z'' \in \mathcal{A}, z'' \neq 0} N(z''). \tag{13}$$

However, $d_\mathrm{E}(0, z'') = N(z'')$ and hence equality holds.

Next, we consider the distance in the cosets, e.g., consider the points $x, x' \in \mathcal{A}'$, i.e., $x = z + c$ and $x' = z' + c$ with $z, z' \in \mathcal{A}$. All cosets have the same minimum squared Euclidean distance as $d_\mathrm{E}(x, x') = N(x - x') = N(z - z')$. □

**Example 3.** *We demonstrate the set partitioning with $\mathcal{E}_9$ from Example 1. We have the subset $\mathcal{E}_9^{(0)} = \{0, 1 - \omega^*, -1 + \omega^*\}$ which is an additive subgroup corresponding to the tuples $(0, 0)$, $(1, 0)$, and $(2, 0)$. The minimum squared Euclidean distance is $\delta^2(\mathcal{E}_9^{(0)}) = N(1 - \omega^*) = N(-1 + \omega^*) = 3$. Hence, also the cosets $\mathcal{E}_9^{(1)} = \{1, \omega, \omega^*\}$ and $\mathcal{E}_9^{(2)} = \{-1, 1 + \omega, -\omega\}$ have minimum squared Euclidean distance $\delta^2 = 3$. The elements $\mathfrak{t}_0$ and $\mathfrak{t}_1$ of the ternary tuples provide an address labeling, where $\mathfrak{t}_0$ determines the subset and $\mathfrak{t}_1$ the element in the subset.*

This concept can be generalized to larger sets and more partitioning levels. The set partitioning for the constellation with $M = 2^3 = 27$ elements is depicted in Fig. 1, where the three subsets of level $l = 1$ are presented in the lower part of the figure. The minimum squared Euclidean distances are $\delta^2 = 1$ in the complete constellation, $\delta^2 = 3$ in the subsets with nine elements at level $l = 1$, and $\delta^2 = 9$ in the subsets with three elements at level $l = 2$.

## IV. Constellation-Constrained Capacity and Multilevel Coding

The constellation-constrained capacity is an important performance measure in order to compare different signal constellations. Specifically, we consider transmission of the symbols $x \in \mathcal{E}_M$ over the (complex-valued) AWGN channel. In each modulation step, a symbol $y = x + n$ is received, where the noise term $n$ is a complex-valued zero-mean Gaussian random variable with variance $\sigma_n^2 = \mathbb{E}\{|n^2|\}$. The constellation-constrained capacity $C$ (in bit per symbol) depends on the signal-to-noise ratio (SNR) and can be obtained by numerical integration [22]. We define the SNR as the energy per complex-valued transmit symbol $x$ over the noise-power spectral density, i.e., as $E_\mathrm{s}/N_0 = \frac{E(\mathcal{E}_M)}{\sigma_n^2}$.

It is well known that $C$ may be achieved via multilevel coding in combination with multistage decoding [17] as illustrated in Fig. 2. To that end, a set partitioning is required, which is inherently defined by (8). Applying the chain rule of information theory, the capacity $C$ of the (partitioned) constellation can be split into its level capacities $C_l$, where $C = \sum_l C_l$. Ternary multistage decoding can achieve the overall capacity $C$ as long as the relative rates of the component codes $R_{\mathrm{c},l} \in [0, 1]$, $l = 0, \ldots, m-1$, satisfy $R_{\mathrm{c},l} = \frac{K_{\mathrm{c},l}}{N_\mathrm{c}} = \frac{C_l}{\log_2(3)}$, where $K_{\mathrm{c},l}$ denotes the dimensions and $N_\mathrm{c}$ an identical code length.

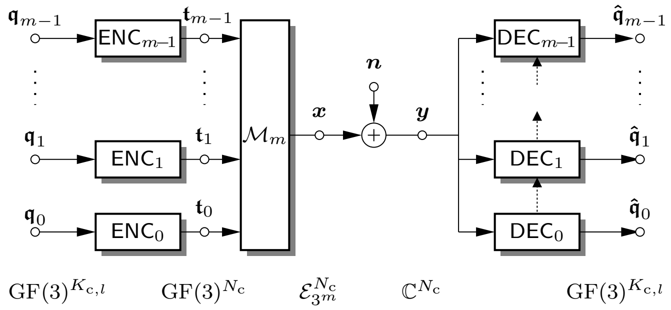


Fig. 2. Block diagram of ternary multilevel-coded modulation, i.e., with source symbols and encoded symbols drawn from GF(3) that are mapped to a signal constellation with cardinality $3^m$. The stream of ternary source symbols is parallelized into blocks $\mathfrak{q}_l$, $l = 0, \ldots, m-1$, of length (dimension) $K_{\mathrm{c},l}$ which are encoded to codewords $\mathfrak{t}_l$ of same length $N_\mathrm{c}$ (component codes with code rate $R_\mathrm{c} = K_{\mathrm{c},l}/N_\mathrm{c}$). In each modulation step $\kappa = 1, \ldots, N_\mathrm{c}$, one encoded finite-field symbol of each level is taken to form the $m$-tuple $\mathfrak{t}[\kappa] = (\mathfrak{t}_{m-1}[\kappa], \ldots, \mathfrak{t}_0[\kappa])$ which is subsequently mapped onto signal points of the Eisenstein constellation $\mathcal{E}_M$ according to (9). This results in the block of transmit symbols $\boldsymbol{x}$ (length $N_\mathrm{c}$), which is transmitted over the AWGN channel $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{n}$, where $\boldsymbol{y}$ denotes the block of receive symbols and $\boldsymbol{n}$ the block of i.i.d. zero-mean complex-valued white-Gaussian-noise samples. At the receiver, the $m$ component codes are individually decoded taking the results from the previous levels into account (multistage decoding [17]). We finally obtain the blocks of estimated source symbols $\hat{\mathfrak{q}}_l$.

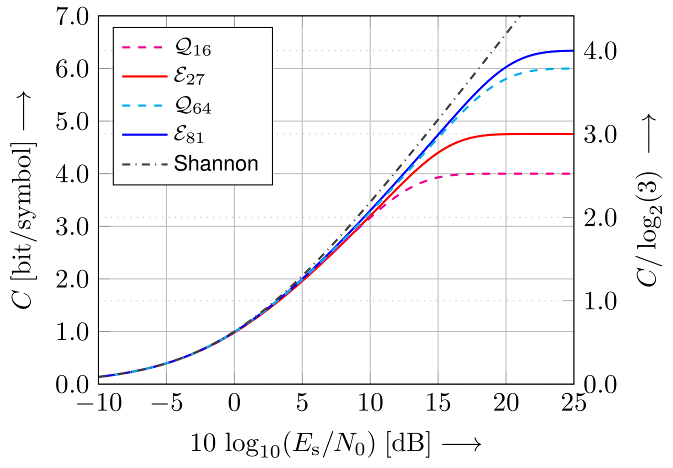


Fig. 3. Constellation-constrained capacity $C$ (left axis; in bit per symbol) and normalized capacity $C/\log_2(3)$ (right axis) for transmission over the AWGN channel vs. the SNR in dB. The capacity is shown for Eisenstein ($\mathcal{E}_M$) and QAM ($\mathcal{Q}_M$) constellations with different cardinality $M$. For comparison, the Shannon capacity $\log_2(1 + E_s/N_0)$ is illustrated.

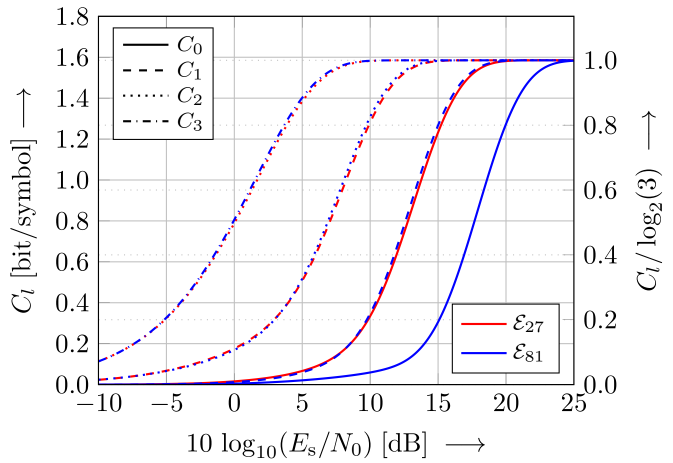


Fig. 4. Level capacities $C_l$ (left axis; in bit per symbol) and normalized level capacities $C_l/\log_2(3)$ (right axis) of the Eisenstein constellations $\mathcal{E}_M$ with the cardinalities $M = 27$ and $M = 81$. They are partitioned according to (8) into three and four levels, respectively. In each level, the maximum capacity of $C_l \to \log_2(3) \approx 1.585$ can be obtained.

Fig. 3 shows the (total) capacity $C$ over the SNR in dB for square QAM and Eisenstein constellations with different cardinalities. The depicted Eisenstein ones approach the capacities $C = \log_2(27) \approx 4.75$ and $C = \log_2(81) \approx 6.34$, respectively. When keeping $C$ fixed, they enable an SNR gain especially in the regions where the close-by QAM ones are already approaching the maximum capacity of 4 or 6 bit/symbol, respectively. This benefit is not only caused by the increased cardinality, but also from the packing and shaping gain of the Eisenstein lattice, cf. Sec. II.

Fig. 4 illustrates the related level capacities $C_l$ for the Eisenstein constellations. The 27-ary can be partitioned into three levels, whereas the 81-ary is partitioned into four levels. As can be seen from Fig. 4, the distance gain achieved with the partitioning results in an SNR gain from level to level which is around $10\log_{10}(3) \approx 4.77$ dB (cf. Sec. III). Due to the large distance and SNR gains, protection with codes may only be required in the lower levels. Alternatively, high-rate algebraic codes may be used in one or several upper levels.

## V. Numerical Results

In this section, we present results obtained from Monte-Carlo simulations. We assess the average symbol error ratio (SER) over the AWGN channel w.r.t. the estimated source symbols after decoding (cf. Fig. 2; bits or ternary symbols, respectively). Different scenarios are considered that are listed in Table II. Thereby, the number of (equivalent) information bits per symbols is varied, denoted as (coded) modulation rate $R_m = \chi \cdot \sum_{l=0}^{m-1} R_{c,l}$, where $\chi = 1$ for the binary and $\chi = \log_2(3)$ for the ternary case. Multilevel coding over QAM constellations ($\mathcal{Q}_M$) using the basis expansion approach in [20] is contrasted to the proposed Eisenstein scheme ($\mathcal{E}_M$).

TABLE II
Rates of the component codes $R_{c,l}$ for multilevel coding over QAM ($\mathcal{Q}_M$) and Eisenstein ($\mathcal{E}_M$) constellations depending on the modulation rate $R_m$. Levels are uncoded if $R_{c,l} = 1$.

| Scenario | | $R_{c,0}$ | $R_{c,1}$ | $R_{c,2}$ | $R_{c,3}$ | $R_{c,4}$ | $R_{c,5}$ |
|---|---|---|---|---|---|---|---|
| $R_m = 3.5$ | $\mathcal{Q}_{16}$ | 0.5745 | 0.9284 | 0.9971 | 1 | – | – |
| | $\mathcal{E}_{27}$ | 0.3126 | 0.8957 | 1 | – | – | – |
| $R_m = 4.6$ | $\mathcal{E}_{27}$ | 0.8392 | 1 | 1 | – | – | – |
| $R_m = 5.5$ | $\mathcal{Q}_{64}$ | 0.5753 | 0.9272 | 0.9975 | 1 | 1 | 1 |
| | $\mathcal{E}_{81}$ | 0.5008 | 0.9693 | 1 | 1 | – | – |

All coded levels in Table II were protected with binary or ternary LDPC codes, respectively. To this end, a semi-random construction based on irregular repeat-accumulate codes [23] has been applied, where the left (arbitrary) part of the parity-check matrix has randomly been chosen according to a given degree distribution (weight 4 is present in 10 % of the columns; weight 3 in the rest). The code length is $N_c = 64800$ for the binary case and $N_c = 64800/\log_2(3) = 40884$ for the ternary case so that the same number of transmit symbols is present per codeword.[2] For the binary codes, belief-propagation decoding in log-likelihood domain has been performed. For the ternary case, non-binary belief-propagation in probability-domain [24] has been applied. In both cases, all constellations points of the given (sub)sets were considered for metric calculation; 50 iterations have been performed.

Fig. 5 shows the SERs for the scenarios listed in Table II. The constellation-constrained capacities are included (vertical lines). For a fair comparison, the energy per (equivalent) information bit is now incorporated in $E_b/N_0 = (E_s/N_0)/R_m$. Considering the first scenario with $R_m = 4.5$, we see that the 27-ary Eisenstein constellation enables a gain of about 0.2 dB over the 16-ary QAM one, even though only two instead of three levels are protected. However, the SNR gain is a little

[2]The large code length ensures that even for the code rates $R_{c,2} > 0.997$ in case of QAM about 200 redundancy bits are still present. However, in practice, algebraic codes may rather be convenient to handle such rates.

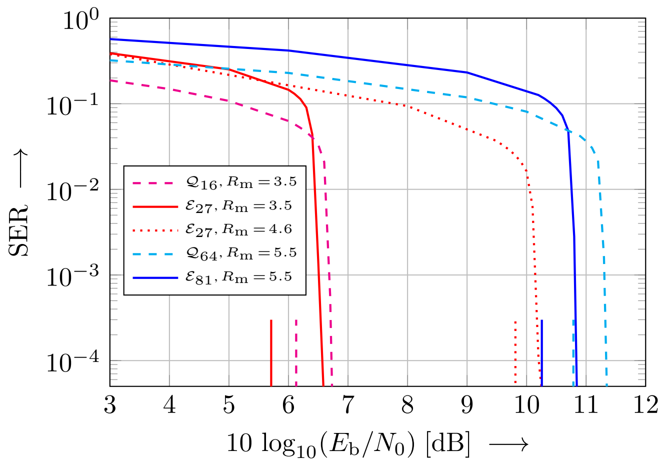


Fig. 5. SERs of the estimated source symbols over the SNR in dB for multilevel-coded modulation with QAM constellations ($\mathcal{Q}_M$) and Eisenstein constellations ($\mathcal{E}_M$) with cardinality $M$ for the scenarios listed in Table II. The related SNRs of the constellation-constrained capacities for the given coded-modulation rates $R_\mathrm{m}$ are illustrated as vertical lines.

less than we can expect from the capacities; the loss is caused since the constructed LDPC code does not perform very well for the low rate $R_{\mathrm{c},0} = 0.3126$. Going over to the scenario with $R_\mathrm{m} = 4.6$ where we consider the same 27-ary Eisenstein constellation with only the first level being protected, we see that a performance close to the capacity is possible. However, in this scenario, it is appropriate to protect the first uncoded level ($l = 1$) with a high-rate algebraic code in order to avoid error floors below $10^{-5}$. Finally, considering the third scenario with $R_\mathrm{m} = 5.5$, the 81-ary Eisenstein constellation shows an SNR gain over the 64-ary QAM one of around $0.4$ dB, which roughly corresponds to the gap w.r.t. the capacities. Here, only the first two levels (instead of three) have to be protected.

## VI. Summary and Conclusions

In this work, we have introduced signal constellations based on Eisenstein integers with cardinality $M = 3^m$. Addition over the related signal points corresponds to addition with carry over ternary number fields. Hence, these constellations are suited for schemes where linear combinations of codewords have to be decoded, cf. the multilevel-coded scheme for lattice-reduction-aided equalization over Gaussian integers in [20].

The capacity curves for the proposed constellations complement the curves for binary ones, i.e., constellations with $M = 2^m$. The proposed set partitioning enables multilevel coding and achieves large distance and related SNR gains in the subsets. For high spectral efficiencies this can be an advantage because only the first levels may need protection with channel codes to achieve the desired performance.

A drawback of the proposed Eisenstein constellation might be that the number of signal points is not a power of two. However, for the proposed multilevel codes this issue can be resolved by a simple mapping procedure from binary to ternary vectors (modulus conversion) as shown in [25], [21], [26].

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