# Advantages of Shared Data Structures for Sequences of Balanced Parentheses

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# Outline

- Basic Definitions
- Our DS for RMQ
- Geary et al.'s DS for balanced parentheses
- Our result
    - Computing RMQs with $2n + o(n)$ bits
    - Computing LCA with $2n + o(n)$ bits
- Experimental study
    - Comparison of RMQ data structures
    - Comparison of CST implementations
- Conclusion

# Range Minimum Queries

## Definition

Given an array $A$ of $n$ values. A range minimum query (RMQ) $rmq_A(i,j)$ with $i \leq j$ returns index $k$ and $A[k] = \min\{A[\ell] | i \leq \ell \leq j\}$.

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## Solution

- Preprocess a RMQ data structure $R$ for $A$
- $R$ answers a RMQ then in constant time
- Two versions of the problem
  - Systematic: $R$ needs $A$ to answer RMQs
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## Our solution

- Non-systematic
- $2n + o(n)$ bits ($3n$ bits in practice)
- $3n + o(n)$ bits for construction in linear time

# Balanced Parentheses Sequences (BPS)

- Sequence S over the alphabet '(' and ')'
- Each prefix of S contains more '('s than ')'s
- Fundamental operations on S:
  - $rank_{(}(S, i)$
  - $select_{(}(S, i)$
  - $excess(S, i) = rank_{(}(i) - rank_{)}(i)$
  - $find\_close(S, i)$ and $find\_open(S, i)$
  - $enclose(S, i)$

**Example**

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$rank_{(}(S, 5) = 4$

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**Example**

(()(()(()()()()()()))(())

$select_{(}(S, 2) = 1$

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**Example**

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$find\_close(S, 3) = 20$ and $find\_open(S, 20) = 3$

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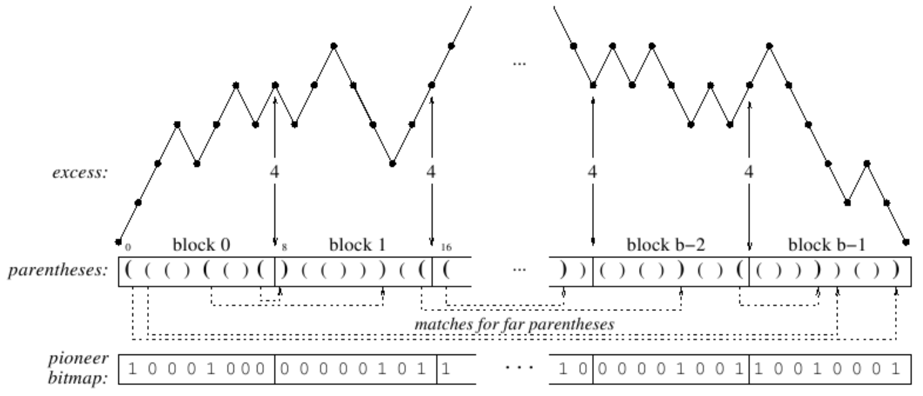
**Time and space**

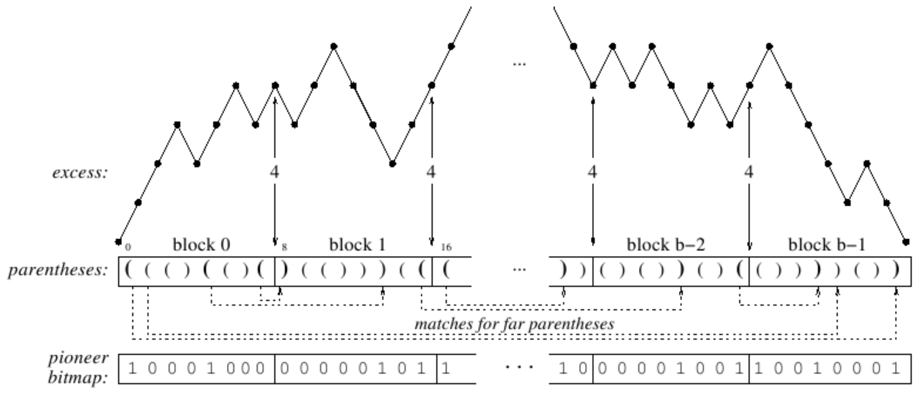
Geary et al.'s data structure of size $o(n)$ supports all operations in constant time

# Geary et al.'s support structure for BP




- Partition BPS into $b$ blocks of length $\mathcal{O}(\log n)$
- Calculate far parentheses/pioneers to answer *find_close*, *find_open*, *enclose*
- pioneer bitmap takes $\mathcal{O}(\frac{n \log \log n}{\log n})$ bits

# Geary et al.'s support structure for BP




- set of pioneers is again a BPS (of length $n_1 < 4b - 6$ )
- recusively build data structure for pioneers
- store answers explicitly on the second level

# The range restricted enclose method

# Construction of the BP of the SCT

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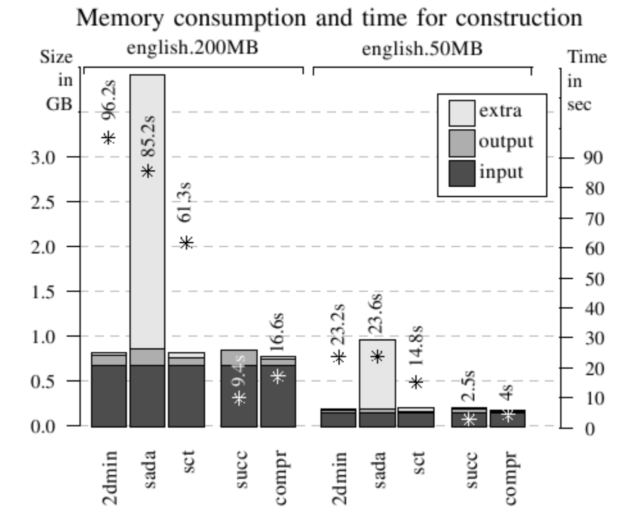
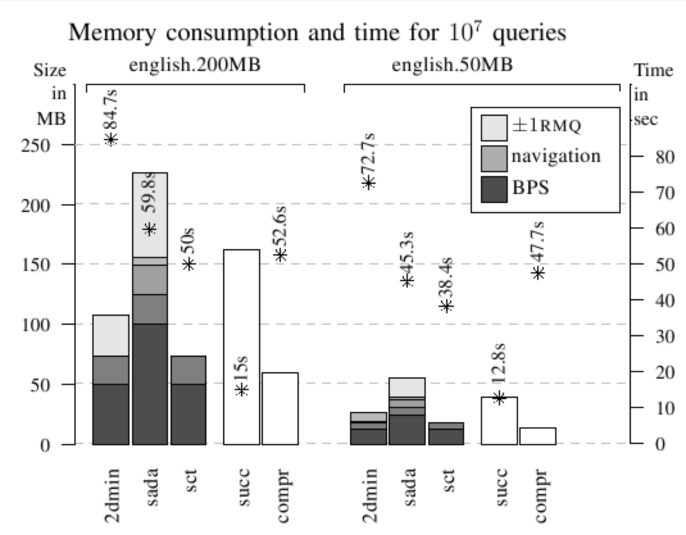
| 1 | 2 | | 3 | 4 | | 5 | 6 | | | | 7 | 8 | | 9 | 10 | | | 11 | | |

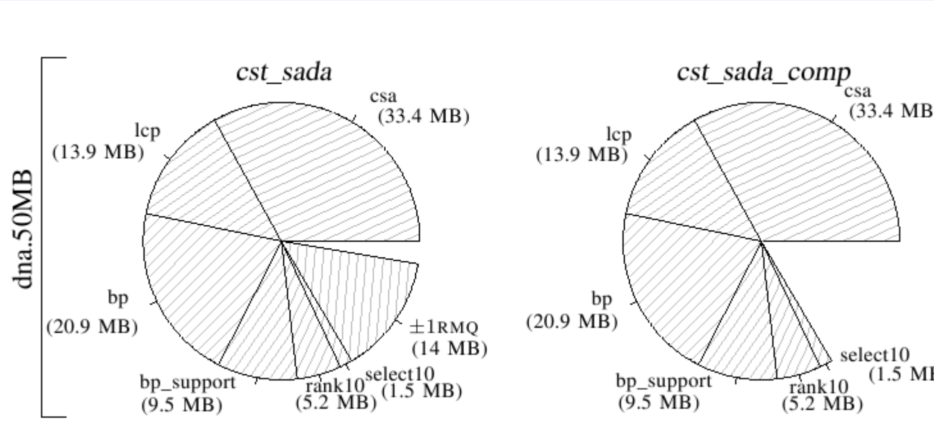
# RMQ: Peak memory consumption at construction



# RMQ: Final memory consumption and query time



# Compressed Suffix Trees: Memory



# Other data structures for RMQs

## Non-systematic solutions

- sada: BPS of the extended Caresian Tree ($4n + o(n)$ bits) by Sadakane (JDA 2007)
- 2dmin: BPS of the 2d-Min-Heap ($2n + o(n)$) by Fischer (2009)

## Systematic solutions

- succ: Succinct solution ($7n$ bits+size of input array) by Fischer ()
- compr: Compressed solution ($\approx 3n$ bits + size of input array) by Fischer et al. (DCC 2008)

# Experimental results

Any Questions?