Constrained Ordering

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Abstract

We investigate the problem of finding a total order of a finite set that satisfies various local ordering constraints. Depending on the admitted constraints, we provide an efficient algorithm or prove NP-completeness. To this end, we define a reduction technique and discuss its properties.

Key words: total ordering, NP-completeness, computational complexity, betweenness, cyclic ordering, topological sorting.

1 Introduction

An instance of the betweenness problem is given by a finite set A and a collection C of triples from A, and one has to decide if there is a total order $<$ of A such that for each $(a, b, c) \in C$, either $a < b < c$ or $c < b < a$ [GJ79, problem MS1]. The betweenness problem is NP-complete [Opa79]. Applications arise, for example, in the design of circuits and in computational biology [Opa79, CS98].

Similarly, the cyclic ordering problem asks for a total order $<$ of A such that for each $(a, b, c) \in C$, either $a < b < c$ or $b < c < a$ or $c < a < b$ [GJ79, problem MS2]. The cyclic ordering problem, too, is NP-complete [GM77]. Applications arise, for example, in qualitative spatial reasoning [IC00].

On the other hand, if $a < b < c$ or $a < c < b$ is allowed, the problem can be solved with linear time complexity by topological sorting [Knu97, Section 2.2.3].

Yet another choice, namely $c < a$ or $c < b$, is needed to model an object-relational mapping problem described in Section 2. The reader is invited to think about the time complexity of this problem before reading the solution.

Starting with Section 3, several kinds of generalisations to these problems are explored with respect to their time complexity and interdependence. The main instrument, a reduction technique, and its properties are presented in Section 4. Applications of this method—manual and automatic—are discussed in Section 5. The conclusion discusses related work, generalisations, and open problems. Some of the proofs are presented in the appendix.

2 Motivation

We consider the part of an object-oriented model of a system specified by the UML class diagram shown in Figure 1. The classes L and M are related to

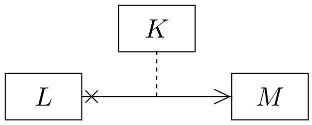

Figure 1: UML class diagram with association class

each other, and the association class K details this relationship. Note that the association from L to M is directed which means that objects of the class M cannot access those of K and L [Obj05].

From time to time, a software that implements this model needs to make the instances that have been accumulated in memory persistent to a database. The representations in memory using pointers and in a relational database clash, however, resulting in object-relational mapping problems [Fow02]. For our special problem, the following approach is appropriate.

There should be one database table for each of the classes K, L, and M, into which objects of the respective classes save themselves, with unique identifiers being generated upon storage. To hold the instances of the associations, the so-called links, another table is devised that keeps the identifiers of related objects. For efficiency reasons, one of the three objects that participate in a link should make the entry into the association table. Since all three identifiers are needed for this, only the last of the three objects of each link is in the right position to do this. Moreover, because of the restricted visibility in the model, this must not be the object of class M for it cannot access the other identifiers.

To summarise, for each triple (a, b, c) of objects from classes (K, L, M) that constitute such a link, a or b must be stored after c. This is the reason for the requirement $c < a$ or $c < b$ given above for the total order.

In practice, an UML class diagram may also have directed associations without a detailing association class. Such a pair (d, e) of objects would have the requirement $d < e$ modelling that d must be stored before e. We therefore state the decision problem of this, more general version.

- INSTANCE: Finite set A, collection B of pairs from A, collection C of triples from A.
- QUESTION: Is there a bijection $f : A \to \{1, 2, \ldots, |A|\}$ such that $f(a_1) < f(a_2)$ for each $(a_1, a_2) \in B$, and $f(a_3) < f(a_1)$ or $f(a_3) < f(a_2)$ for each $(a_1, a_2, a_3) \in C$?

We prove it is efficiently decidable by the algorithm shown in Figure 2, an extension of topological sorting [Knu97, Section 2.2.3]. Algorithm T maintains working sets $E \subseteq A$, $F \subseteq B$, and $G \subseteq C$.

Assume algorithm T proposes an order. That order is a permutation of A since during every iteration one element e is removed from E and prepended to the order. To see that the constraints specified by B are satisfied note that each $(a_1, a_2) \in B$ remains in F until the iteration where $a_2 = e$, thus a_2 is prepended

input finite set A, collection of pairs B, collection of triples C
output total order of A such that the first element of each pair in B precedes the second, and the third element of each triple in C precedes the first or the second

method $(E, F, G) \leftarrow (A, B, C)$
Order $\leftarrow$ empty sequence
while $E \neq \emptyset$ **do**
  find $e \in E$ such that $\forall x, y \in E : (e, y) \notin F \wedge (x, y, e) \notin G$
  if such an e exists **then**
    $G \leftarrow \{(x, y, z) \in G \mid x \neq e \wedge y \neq e\}$
    $F \leftarrow \{(x, y) \in F \mid y \neq e\}$
    $E \leftarrow E \setminus \{e\}$
    prepend e to Order
  else
    output "there is no order"
    halt
  end
end
output Order

Figure 2: Algorithm T

to the order. While $(a_1, a_2) \in F$, however, the chosen element e cannot be a_1, hence a_1 precedes a_2 in the order.

Similarly, to see that the constraints specified by C are satisfied note that each $(a_1, a_2, a_3) \in C$ remains in G until the iteration where $a_1 = e \vee a_2 = e$, thus a_1 or a_2 is prepended to the order. While $(a_1, a_2, a_3) \in G$, however, the chosen element e cannot be a_3, hence a_3 precedes a_1 or a_2 in the order.

Assume algorithm T fails to find an order. In this case, there is a non-empty subset $E \subseteq A$ such that no $e \in E$ satisfies the required property. Thus, for each $e \in E$ either $(e, y) \in F$ for some $y \in E$ or $(x, y, e) \in G$ for some $x, y \in E$. Since $F \subseteq B$ and $G \subseteq C$ each $e \in E$ must precede some $e' \in E$ in a total order. There is no such order of finite sets.

The time complexity of algorithm T is polynomial in the size of the input. Implemented very carefully one can even achieve a linear time complexity.

Taking $C = \emptyset$ and requiring B to be a (strict) partial order over A demonstrates that algorithm T is indeed a generalisation of topological sorting. Since we do not assume that the elements of a triple in C are distinct, one may even entirely dispose of B by adding a triple (a_2, a_2, a_1) to C for each pair $(a_1, a_2) \in B$. While this procedure works for the problem at hand, it might fail for other types of problems discussed in Section 3.

3 Generalisation

We explore different kinds of generalising the betweenness, cyclic ordering, and topological sorting problems introduced above. By $\mathfrak{S}_k$ we denote the symmetric group of size k.

3.1 Constraints over three elements

The first generalisation still assumes that a collection C of triples is given but abstracts from the constraint $P \subseteq \mathfrak{S}_3$ specifying the relative order of the elements of each triple. We therefore have a family of problems, one for each P.

- INSTANCE: Finite set A, collection C of triples (a_1, a_2, a_3) of distinct elements from A.
- QUESTION: Is there a bijection $f : A \to \{1, 2, \ldots, |A|\}$ such that for each $(a_1, a_2, a_3) \in C$ there is a $p \in P$ with $f(a_{p(1)}) < f(a_{p(2)}) < f(a_{p(3)})$?

Choose $P = \{(123), (321)\}$ for betweenness, $P = \{(123), (231), (312)\}$ for cyclic ordering, and $P = \mathfrak{S}_3 \setminus \{(123), (213)\}$ to get the problem discussed in Section 2. The distinctness condition $a_1 \neq a_2 \neq a_3 \neq a_1$ is obviously easy to check and further elaborated in Appendix A.

The total number of problems in this family is $2^{|\mathfrak{S}_3|} = 2^{3!} = 2^6 = 64$. Already from the small sample just presented it is clear that some of these problems are tractable while others are NP-complete. Thus the task arises to classify the remaining problems. All of them are in NP, since a nondeterministic algorithm can guess the order and check in polynomial time that the constraints specified by C are satisfied with respect to the chosen P. This remark applies to all problems discussed in this paper.

To reduce the number of problems that must be investigated, the following symmetry consideration applies. Intuitively, a systematic permutation of the elements of each triple can be compensated by adjusting the constraints to access the triples at the permuted positions. Precisely, symmetry is exploited by permuting the elements of each triple and applying the *inverse* permutation to all constraints. It follows that two problems P_1 and P_2 that differ just by consistently renaming the elements of their permutations, that is $P_1 = \pi \circ P_2$ for some $\pi \in \mathfrak{S}_3$, have the same time complexity. For example, $\{(123), (213)\}$ and $\{(321), (231)\}$ are two such problems.

The problems can be classified as shown in Figure 3. It displays the reduction graph for our family of problems with its strongly connected components collapsed, and symmetrical problems pooled. The reduction graph, in turn, has the problems as its vertices and an edge from P_1 to P_2 if, by the method described in Section 4, P_1 is many-one reducible in polynomial time to P_2. The problems surrounded by double boxes are NP-complete, and the others are tractable. The latter can be solved trivially, by topological sorting, or by algorithm T from Section 2.

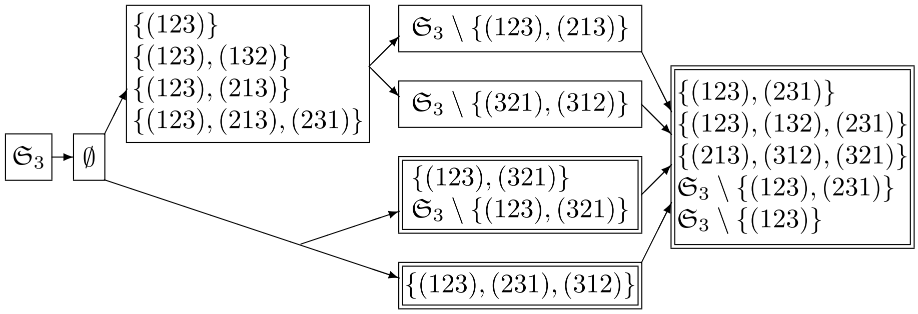

Figure 3: Reducibility among the problems $\subseteq \mathfrak{S}_3$

Let us mention another kind of symmetry that is not captured by the method presented in Section 4. Intuitively, reversing each constraint can be compensated by transposing the resulting total order. Precisely, a partial order can be extended to a total order if and only if its transpose can be extended—just take the transpose of the total order. It follows that two problems P_1 and P_2 that differ just by reversing their permutations, that is $P_1 = P_2 \circ (321)$, have the same time complexity. For example, $\mathfrak{S}_3 \setminus \{(123), (213)\}$ and $\mathfrak{S}_3 \setminus \{(321), (312)\}$ are two such problems.

3.2 Constraints over additional pairs

The second generalisation has already been touched in Section 2, where the collection of triples was joined by a collection of pairs. For that special instance, the additional constraint pairs have no impact on the complexity of algorithm T since they could also be replaced by triples. In general, however, this is not the case. For example, whatever additional betweenness triples are devised to replace a pair (a_1, a_2) that requires a_1 to precede a_2, they are also satisfied by transposing the resulting total order. There simply is no way to express absolute direction in the betweenness problem. We therefore have another family of 64 problems, again indexed by $P \subseteq \mathfrak{S}_3$.

- INSTANCE: Finite set A, collection B of pairs from A, collection C of triples (a_1, a_2, a_3) of distinct elements from A.
- QUESTION: Is there a bijection $f : A \to \{1, 2, \ldots, |A|\}$ such that $f(a_1) < f(a_2)$ for each $(a_1, a_2) \in B$, and for each $(a_1, a_2, a_3) \in C$ there is a $p \in P$ with $f(a_{p(1)}) < f(a_{p(2)}) < f(a_{p(3)})$?

With the results of Section 3.1 in place, the complexity of each problem in the new family can easily be derived. Taking $B = \emptyset$ demonstrates that the new problems are indeed generalisations. All NP-complete problems of Section 3.1 thus remain NP-complete. On the other hand, all tractable problems are—directly or by reduction—solvable using algorithm T that already accepts an additional collection of constraining pairs. The classification therefore remains unchanged.

Symmetry by reversing each constraint can be extended to this, more general case by transposing the relation B to accommodate the reversed order.

3.3 Constraints over disjoint triples

The third variation takes advantage of the expressivity gained by the pairs introduced in Section 3.2. It is rather a specialisation of those problems where one assumes that any two triples in the collection C are pairwise disjoint when viewed as sets. This family of problems is also indexed by $P \subseteq \mathfrak{S}_3$.

- INSTANCE: Finite set A, collection B of pairs from A, collection C of pairwise disjoint triples (a_1, a_2, a_3) of distinct elements from A.
- QUESTION: Is there a bijection $f : A \to \{1, 2, \ldots, |A|\}$ such that $f(a_1) < f(a_2)$ for each $(a_1, a_2) \in B$, and for each $(a_1, a_2, a_3) \in C$ there is a $p \in P$ with $f(a_{p(1)}) < f(a_{p(2)}) < f(a_{p(3)})$?

Since the new problems are restrictions of those of Section 3.2, algorithm T can still be applied to solve the tractable problems. The question remains whether some of the NP-complete problems become more easy. The negative answer to this question is given in Appendix B. Here, we only provide a summary of that proof.

By symmetry, the eight NP-complete problems shown in Figure 3 remain. One of them can be eliminated by the reverse symmetry mentioned in Sections 3.1 and 3.2. The problem $P = \{(123), (231)\}$, called *intermezzo*, is singled out and its NP-completeness is proved by reduction from 3SAT in Appendix B.1. It is then further reduced to the betweenness problem $\{(123), (321)\}$ in Appendix B.2. Finally, intermezzo is reduced to each of the remaining five problems by an easier construction described in Appendix B.3. Note that the existing NP-completeness proofs for betweenness and cyclic ordering do not apply because both use non-disjoint triples [GM77, Opa79].

Our first three generalisations can be summarised as shown in Figure 4. These variants have the same class structure. The picture is completed by the variant that requires disjoint triples but does not permit pairs—this variant is trivially solvable.

3.4 Constraints over tuples

Our final generalisation abstracts from the number of elements of the constraints. We have started with triples in Section 3.1, added pairs in Section 3.2, and now assume the number of elements is given by the positive integer k—to be held constant for a single problem. For each value of k we have a family of problems indexed by $P \subseteq \mathfrak{S}_k$.

- INSTANCE: Finite set A, collection C of k-tuples $(a_1, \ldots, a_k)$ of distinct elements from A.

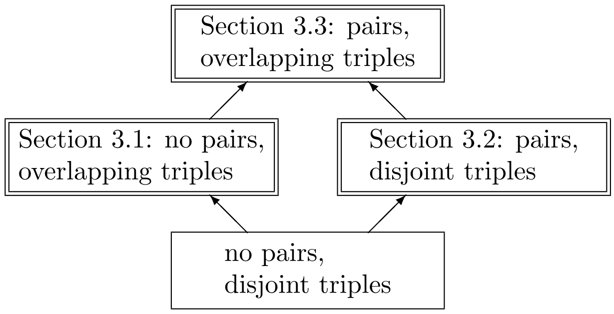

Figure 4: Variants of problems with triples

- QUESTION: Is there a bijection $f : A \to \{1, 2, \ldots, |A|\}$ such that for each $(a_1, \ldots, a_k) \in C$ there is a $p \in P$ with $f(a_{p(i)}) < f(a_{p(j)})$ for all $1 \leq i < j \leq k$?

The total number of problems for a fixed k is $2^{|\mathfrak{S}_k|} = 2^{k!}$. A concrete problem will be referred to as the *constrained ordering problem* (k, P).

This is the wording of the problem that will be used in Section 4. We will introduce a method there by which the constrained ordering problems (k_1, P_1) and (k_2, P_2) may be compared, and investigate its properties. This method has been, among other things, fully applied to the case $k = 4$ which we describe here.

Just as with triples, all 16777216 problems with quadruples as constraints are either tractable or NP-complete. They can be classified as shown by the simplified reduction graph in Figure 5. See Section 3.1 for its interpretation. For want of space, only one or two problems from each class are provided. The classes correspond to those of Figure 3, only the dihedral class is new.

A few remarks on the classes are appropriate. The swap class contains the problems $P = P \circ (4321)$ that are closed under reversion—except for those in the dihedral class and the trivial ones. The cyclic class contains, with the same exceptions, the problems $P = P \circ (2341)$ that are closed under rotation. The dihedral class contains the problems $P = P \circ (4321) = P \circ (2341)$ that are closed under both kinds of symmetries, except for $\mathfrak{S}_4$ and $\emptyset$ again. This class is new with the quadruples since for the triples there are no problems besides $\mathfrak{S}_3$ and $\emptyset$ sharing both symmetries. In Section 5.2 we will further discuss the canonical problem from the dihedral class shown in Figure 5, called 4-separation.

The partial order class contains all problems where the constraints exactly specify a partial order that must be satisfied, except for $\mathfrak{S}_4$ that specifies the discrete order. They are solved by topological sorting. The problems in the classes labelled with algorithm T—they are symmetric under reversion of each other—can be reduced to problems solvable by that algorithm. Finally, the exclusion class contains the remaining problems and is by far the largest.

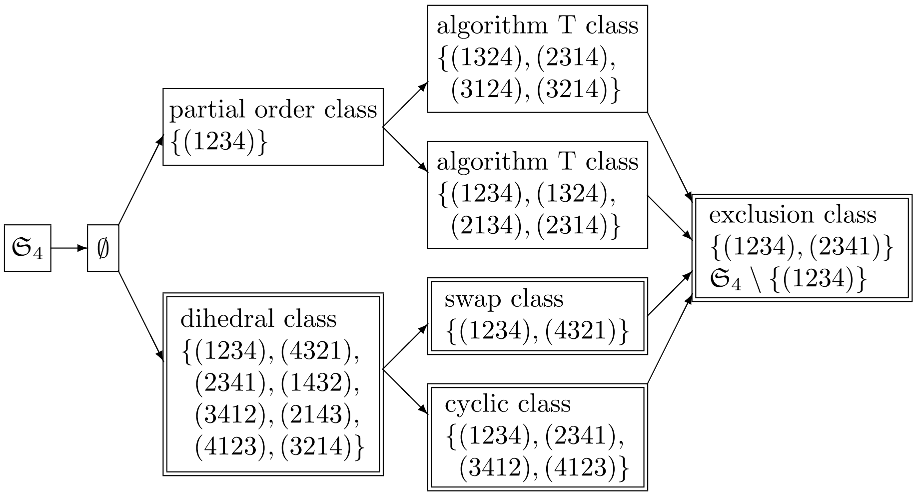

Figure 5: Reducibility among the problems $\subseteq \mathfrak{S}_4$

Table 1 compares the number of problems in each class for different values of k. We do not know the general values for the exclusion and algorithm T classes. We also do not know if the exclusion class splits for larger values of k. No closed formula is known for the size of the partial order class [Slo05].

Let us conclude this section by remarking that two further kinds of generalisations are discussed in Section A.

4 Reduction

In this section we explore a method that can be used for reductions between different constrained ordering problems. Let k be a positive integer and $P \subseteq \mathfrak{S}_k$. Recall the problem statement for the constrained ordering problem (k, P) from Section 3.4.

- INSTANCE: Finite set A, collection C of k-tuples $(a_1, \ldots, a_k)$ of distinct elements from A.
- QUESTION: Is there a bijection $f : A \to \{1, 2, \ldots, |A|\}$ such that for each $(a_1, \ldots, a_k) \in C$ there is a $p \in P$ with $f(a_{p(i)}) < f(a_{p(j)})$ for all $1 \leq i < j \leq k$?

We need some general definitions concerning finite sequences. For $k \in \mathbf{N}$ the set $\mathbf{N}_k = \{1, \ldots, k\}$ denotes the first k positive integers. A k-tuple from a set S, or a (finite) sequence of length k in S, is a function with type $\mathbf{N}_k \to S$. The k-tuples from S without repetition are the injective functions with type $\mathbf{N}_k \to S$, denoted by $\binom{S}{k}$. In particular, a permutation $\pi \in \mathfrak{S}_k$ is such a k-tuple.

class	$k=1$	$k=2$	$k=3$	$k=4$	$k \geq 3$
discrete	1	1	1	1	1
empty	1	1	1	1	1
partial order	–	2	18	218	A001035(k) $-$ 1
algorithm T	–	–	3	266	?
algorithm T reverse	–	–	3	266	?
dihedral	–	–	–	6	$2^{k!/2k} - 2$
cyclic	–	–	2	56	$2^{k!/k} - 2^{k!/2k}$
swap	–	–	6	4088	$2^{k!/2} - 2^{k!/2k}$
exclusion	–	–	30	16772314	?
total	2	4	64	16777216	$2^{k!}$

Table 1: Class sizes for $k = 1, 2, 3, 4$, and the general case

The k-tuple $s : \mathbf{N}_k \to S$ is a subsequence of the l-tuple $t : \mathbf{N}_l \to S$, in symbols $s \sqsubseteq t$, if and only if there is a strictly increasing $u : \mathbf{N}_k \to \mathbf{N}_l$ such that $s = t \circ u$. In that case t is called a supersequence of s. The relation $\sqsubseteq$ partially orders the finite sequences in S.

Given these definitions, we can transform the problem statement as follows. Let $c = (a_1, \dots, a_k)$, then $a_x = c(x)$ for $1 \leq x \leq k$, hence $f(a_{p(i)}) < f(a_{p(j)})$ is equivalent to $(f \circ c \circ p)(i) < (f \circ c \circ p)(j)$. This holds for all $1 \leq i < j \leq k$ if and only if $u = f \circ c \circ p : \mathbf{N}_k \to \mathbf{N}_{|A|}$ is strictly increasing. Since f is a bijection and $c \circ p = f^{-1} \circ u$, this is equivalent to $c \circ p \sqsubseteq f^{-1}$. Instead of looking for a bijection f we therefore might as well seek the corresponding sequence $w = f^{-1} : \mathbf{N}_{|A|} \to A$. The instance (A, C) of the constrained ordering problem (k, P) can thus be restated as

$$\exists w \in \binom{A}{|A|} : \forall c \in C : \exists p \in P : c \circ p \sqsubseteq w.$$

This form will be used for the proof in Section 4.2.

4.1 CO-reduction

The reduction technique introduces fresh elements on a per-clause basis. Let $k_1, k_2 \in \mathbf{N}$ and $P_1 \subseteq \mathfrak{S}_{k_1}$, $P_2 \subseteq \mathfrak{S}_{k_2}$. The constrained ordering problem (k_1, P_1) is *CO-reducible* to the constrained ordering problem (k_2, P_2) if and only if there are $b \in \mathbf{N}$ and $R \subseteq \binom{\mathbf{N}_{k_1+b}}{k_2}$ such that

$$\forall t_1 \in \mathfrak{S}_{k_1} : t_1 \in P_1 \iff \exists t_2 \in \mathfrak{S}_{k_1+b} : t_1 \sqsubseteq t_2 \land \forall r \in R : \exists p \in P_2 : r \circ p \sqsubseteq t_2.$$

We then write $(k_1, P_1) \leq_{\mathrm{CO}} (k_2, P_2)$. Furthermore we abbreviate $\leq_{\mathrm{CO}} \cap \leq_{\mathrm{CO}}^{\smile}$ as $=_{\mathrm{CO}}$ where $\leq_{\mathrm{CO}}^{\smile}$ is the converse of $\leq_{\mathrm{CO}}$.

The intuition is that b new elements are introduced for each tuple in an instance of (k_1, P_1). That tuple is simulated according to R by several k_2-tuples that may use the new elements in addition to the k_1 elements already available.

The new elements are needed to express the constraints in P_1 with the new means given by P_2.

In the sequel we will prove that this *local* criterion can be lifted to the *global* requirement that the constructed set can be totally ordered. Section 4.3 then proves transitivity that is of assistance in the application of our reduction. A few specialisations of the method are investigated in Section 4.4.

4.2 CO-reducibility implies polynomial reducibility

Let $\leq_{\mathrm{m}}^{\mathrm{P}}$ denote many-one reducibility in polynomial time. We will now prove the main result $\leq_{\mathrm{CO}} \subseteq \leq_{\mathrm{m}}^{\mathrm{P}}$.

To this end, let $(k_1, P_1) \leq_{\mathrm{CO}} (k_2, P_2)$ be witnessed by b and R according to the definition. Let (A_1, C_1) specify an instance of (k_1, P_1), where we assume that $C_1 \subseteq \mathbf{N}_{k_1} \to A_1$ is ordered arbitrarily as per $C_1 = \{c_1^{(i)} \mid 1 \leq i \leq |C_1|\}$. For $1 \leq i \leq |C_1|$ the sets $B^{(i)} = \{b_j^{(i)} \mid 1 \leq j \leq b\}$ contain, respectively, b new elements (distinct among each other and those in A_1). For $c_1^{(i)} = (a_1, \ldots, a_{k_1})$ let

- $c_2^{(i)} = (a_1, \ldots, a_{k_1}, b_1^{(i)}, \ldots, b_b^{(i)})$,
- $A_1^{(i)} = \{a_1, \ldots, a_{k_1}\}$, and
- $A_2^{(i)} = \{a_1, \ldots, a_{k_1}, b_1^{(i)}, \ldots, b_b^{(i)}\} = A_1^{(i)} \cup B^{(i)}$.

Note that $A_1^{(i)}$ does not depend on i, but is defined for notational convenience. Finally, let

$$A_2^{(\leq l)} = A_1 \cup \bigcup_{1 \leq i \leq l} B^{(i)} \quad \text{and} \quad C_2^{(\leq l)} = \bigcup_{1 \leq i \leq l} \{c_2^{(i)} \circ r \mid r \in R\}.$$

Construct the instance (A_2, C_2) of (k_2, P_2) such that $A_2 = A_2^{(\leq |C_1|)}$ and $C_2 = C_2^{(\leq |C_1|)}$. This is possible in polynomial time since b and R are constants. We will now show that (A_1, C_1) is solvable if and only if (A_2, C_2) is.

For the backward direction let the ordering w_2 be a solution of (A_2, C_2), and let w_1 be the subsequence of A_1-elements of w_2. Let $v_{1/2}^{(i)}$ denote the subsequence of $A_{1/2}^{(i)}$-elements of $w_{1/2}$; it follows that $v_1^{(i)} \sqsubseteq v_2^{(i)}$. Let $t_{1/2}^{(i)}$ be the arrangement of $c_{1/2}^{(i)}$ in $v_{1/2}^{(i)}$, that is, $c_{1/2}^{(i)} \circ t_{1/2}^{(i)} = v_{1/2}^{(i)}$; it follows that $t_1^{(i)} \sqsubseteq t_2^{(i)}$. For $r \in R$ we have $c_2^{(i)} \circ r \in C_2$, so, given that w_2 is a solution, there is $p_2 \in P_2$ such that $c_2^{(i)} \circ r \circ p_2 \sqsubseteq w_2$. Since $c_2^{(i)} \circ r \circ p_2$ contains but $A_2^{(i)}$-elements, actually $c_2^{(i)} \circ r \circ p_2 \sqsubseteq v_2^{(i)} = c_2^{(i)} \circ t_2^{(i)}$; it follows that $r \circ p_2 \sqsubseteq t_2^{(i)}$. Given CO-reducibility we have $t_1^{(i)} \in P_1$ and w_1 solves (A_1, C_1) since $c_1^{(i)} \circ t_1^{(i)} = v_1^{(i)} \sqsubseteq w_1$.

For the forward direction let the ordering w_1 be a solution of (A_1, C_1). Let $v_1^{(i)}$ denote the subsequence of $A_1^{(i)}$-elements of w_1, and let $t_1^{(i)}$ be the arrangement of $c_1^{(i)}$ in $v_1^{(i)}$, that is, $c_1^{(i)} \circ t_1^{(i)} = v_1^{(i)}$. Given that w_1 is a solution, there are $p_1^{(i)} \in P_1$ such that $c_1^{(i)} \circ p_1^{(i)} \sqsubseteq w_1$. Since $c_1^{(i)} \circ p_1^{(i)}$ contains exactly the $A_1^{(i)}$-elements, actually $c_1^{(i)} \circ p_1^{(i)} = v_1^{(i)} = c_1^{(i)} \circ t_1^{(i)}$; it follows that $t_1^{(i)} = p_1^{(i)} \in P_1$. Given CO-reducibility we have $t_2^{(i)}$ such that $t_1^{(i)} \sqsubseteq t_2^{(i)}$ and $\forall r \in R : \exists p \in P_2 : r \circ p \sqsubseteq t_2^{(i)}$. Let $v_2^{(i)} = c_2^{(i)} \circ t_2^{(i)}$; it follows that $v_1^{(i)} \sqsubseteq v_2^{(i)}$.

We complete the proof by inductively defining solutions $w_2^{(l)}$ of the instances $(A_2^{(\leq l)}, C_2^{(\leq l)})$ of (k_2, P_2) that also satisfy $w_1 \sqsubseteq w_2^{(l)}$. It follows that $w_2^{(|C_1|)}$ solves (A_2, C_2). For the base case $l = 0$ we define $w_2^{(0)} = w_1$, a supersequence of w_1 solving $(A_2^{(\leq 0)}, C_2^{(\leq 0)}) = (A_1, \emptyset)$. For the inductive case l assume that $w_2^{(l-1)}$ is already defined and choose $w_2^{(l)}$ as some supersequence of $w_2^{(l-1)}$ and $v_2^{(l)}$, that is, $w_2^{(l-1)} \sqsubseteq w_2^{(l)}$ and $v_2^{(l)} \sqsubseteq w_2^{(l)}$. This is possible because

- the $A_1^{(l)}$-elements of $v_2^{(l)}$ are arranged as $v_1^{(l)} \sqsubseteq v_2^{(l)}$ and $v_1^{(l)} \sqsubseteq w_1 \sqsubseteq w_2^{(l-1)}$ by the induction hypothesis, and
- all other elements of $v_2^{(l)}$ are in $B^{(l)}$, hence not in $A_2^{(\leq l-1)}$ and not in $w_2^{(l-1)}$.

We have $w_1 \sqsubseteq w_2^{(l-1)} \sqsubseteq w_2^{(l)}$. To see that $w_2^{(l)}$ solves the instance $(A_2^{(\leq l)}, C_2^{(\leq l)})$, note that the tuples from $C_2^{(\leq l-1)}$ are already solved by its subsequence $w_2^{(l-1)}$. For $r \in R$ there is, given CO-reducibility, $p \in P_2$ such that $r \circ p \sqsubseteq t_2^{(l)}$; it follows that $c_2^{(l)} \circ r \circ p \sqsubseteq c_2^{(l)} \circ t_2^{(l)} = v_2^{(l)} \sqsubseteq w_2^{(l)}$. ὅπερ ἔδει δεῖξαι

4.3 CO-reducibility is transitive

Let $\leq_{\text{CO}}^*$ denote the reflexive and transitive closure of $\leq_{\text{CO}}$. We will now prove that $\leq_{\text{CO}} = \leq_{\text{CO}}^*$.

Reflexivity is easily seen by choosing $b = 0$ and $R = \{1_{\mathfrak{S}_k}\}$ where $1_{\mathfrak{S}_k}$ is the identity permutation of the group $\mathfrak{S}_k$.

For transitivity, let (k_1, P_1), (k_2, P_2) and (k_3, P_3) be three constrained ordering problems such that

- $(k_1, P_1) \leq_{\text{CO}} (k_2, P_2)$ by b_1 and $R_1 \subseteq \binom{\mathbf{N}_{k_1+b_1}}{k_2}$, and
- $(k_2, P_2) \leq_{\text{CO}} (k_3, P_3)$ by b_2 and $R_2 \subseteq \binom{\mathbf{N}_{k_2+b_2}}{k_3}$

according to the definition of CO-reducibility. We assume that R_1 is ordered arbitrarily as per $R_1 = \{r_1^{(i)} \mid 0 \leq i \leq |R_1| - 1\}$. Let $b_3 = b_1 + |R_1| \cdot b_2$.

For $r_1^{(i)} \in R_1$ define $\tilde{r}_1^{(i)} \in \binom{\mathbf{N}_{k_1+b_1+b_2}}{k_2+b_2}$ as

$$\tilde{r}_1^{(i)}(x) = \begin{cases} r_1^{(i)}(x), & \text{if } 1 \leq x \leq k_2, \\ k_1 + b_1 + x - k_2, & \text{if } k_2 + 1 \leq x \leq k_2 + b_2. \end{cases}$$

Again for $r_1^{(i)} \in R_1$ define $q^{(i)} \in \binom{\mathbf{N}_{k_1+b_3}}{k_1+b_1+b_2}$ as

$$q^{(i)}(x) = \begin{cases} x, & \text{if } 1 \leq x \leq k_1 + b_1, \\ x + i \cdot b_2, & \text{if } k_1 + b_1 + 1 \leq x \leq k_1 + b_1 + b_2. \end{cases}$$

Let $R_3 = \{q^{(i)} \circ \tilde{r}_1^{(i)} \circ r_2 \mid 0 \le i \le |R_1| - 1 \wedge r_2 \in R_2\} \subseteq \binom{\mathbf{N}_{k_1+b_3}}{k_3}$. We will now show that for all $t_1 \in \mathfrak{S}_{k_1}$,

$$t_1 \in P_1 \iff \exists t_3 \in \mathfrak{S}_{k_1+b_3} : t_1 \sqsubseteq t_3 \wedge \forall r_3 \in R_3 : \exists p_3 \in P_3 : r_3 \circ p_3 \sqsubseteq t_3.$$

For the forward direction, let $t_1 \in P_1$. By definition of CO-reducibility,

$$\exists t_2 \in \mathfrak{S}_{k_1+b_1} : t_1 \sqsubseteq t_2 \wedge \forall r_1^{(i)} \in R_1 : \exists p_2^{(i)} \in P_2 : r_1^{(i)} \circ p_2^{(i)} \sqsubseteq t_2.$$

We fix that supersequence t_2 of t_1. For each $p_2^{(i)}$, again by definition of CO-reducibility,

$$\exists t_3^{(i)} \in \mathfrak{S}_{k_2+b_2} : p_2^{(i)} \sqsubseteq t_3^{(i)} \wedge \forall r_2 \in R_2 : \exists p_3 \in P_3 : r_2 \circ p_3 \sqsubseteq t_3^{(i)}.$$

Since composition from the left is monotonic with respect to $\sqsubseteq$ it follows that

$$\tilde{r}_1^{(i)} \circ p_2^{(i)} \sqsubseteq \tilde{r}_1^{(i)} \circ t_3^{(i)} \wedge \forall r_2 \in R_2 : \exists p_3 \in P_3 : \tilde{r}_1^{(i)} \circ r_2 \circ p_3 \sqsubseteq \tilde{r}_1^{(i)} \circ t_3^{(i)},$$

or, abbreviating $\tilde{t}_3^{(i)} = \tilde{r}_1^{(i)} \circ t_3^{(i)} \in \binom{\mathbf{N}_{k_1+b_1+b_2}}{k_2+b_2}$,

$$r_1^{(i)} \circ p_2^{(i)} \sqsubseteq \tilde{r}_1^{(i)} \circ p_2^{(i)} \sqsubseteq \tilde{t}_3^{(i)} \wedge \forall r_2 \in R_2 : \exists p_3 \in P_3 : \tilde{r}_1^{(i)} \circ r_2 \circ p_3 \sqsubseteq \tilde{t}_3^{(i)}.$$

Next, we combine the $\tilde{t}_3^{(i)}$ by using different new elements for each i. Technically, this is achieved by composing the helper functions $q^{(i)}$ from the left. Note that $q^{(i)} \circ r_1^{(i)} \circ p_2^{(i)} = r_1^{(i)} \circ p_2^{(i)}$, hence

$$r_1^{(i)} \circ p_2^{(i)} \sqsubseteq q^{(i)} \circ \tilde{t}_3^{(i)} \wedge \forall r_2 \in R_2 : \exists p_3 \in P_3 : q^{(i)} \circ \tilde{r}_1^{(i)} \circ r_2 \circ p_3 \sqsubseteq q^{(i)} \circ \tilde{t}_3^{(i)}.$$

Choose some $t_3 \in \mathfrak{S}_{k_1+b_3}$ such that $t_2 \sqsubseteq t_3$ and $q^{(i)} \circ \tilde{t}_3^{(i)} \sqsubseteq t_3$ for all i. This is possible for the following reasons.

- We have already shown that $r_1^{(i)} \circ p_2^{(i)}$ is a subsequence of both t_2 and $q^{(i)} \circ \tilde{t}_3^{(i)}$ for each $r_1^{(i)} \in R_1$. The sequence $\tilde{t}_3^{(i)}$ contains $k_2 + b_2$ elements, exactly b_2 of which are in $\{k_1 + b_1 + 1, \ldots, k_1 + b_1 + b_2\}$. They are mapped by $q^{(i)}$ to values greater than $k_1 + b_1$ which do neither occur in t_2 nor in $r_1^{(i)} \circ p_2^{(i)}$. Therefore, the remaining k_2 elements of $q^{(i)} \circ \tilde{t}_3^{(i)}$ must be exactly those in $r_1^{(i)} \circ p_2^{(i)}$ that contains just k_2 elements. Hence, the only common elements of t_2 and $q^{(i)} \circ \tilde{t}_3^{(i)}$ are those in their common subsequence $r_1^{(i)} \circ p_2^{(i)}$.

- Of the $k_2 + b_2$ elements of $q^{(i)} \circ \tilde{t}_3^{(i)}$, exactly k_2 are in $\{1, \ldots, k_1 + b_1\}$, the other b_2 elements being mapped to disjoint ranges for different values of i. Again, these k_2 elements are those in $r_1^{(i)} \circ p_2^{(i)}$ that is a subsequence of t_2 for each $r_1^{(i)} \in R_1$. Hence, all $q^{(i)} \circ \tilde{t}_3^{(i)}$ can be merged.

Let $r_3 \in R_3$, that is, $r_3 = q^{(i)} \circ \tilde{r}_1^{(i)} \circ r_2$ for some i and $r_2 \in R_2$. Hence, there exists $p_3 \in P_3$ such that $r_3 \circ p_3 = q^{(i)} \circ \tilde{r}_1^{(i)} \circ r_2 \circ p_3 \sqsubseteq q^{(i)} \circ \tilde{t}_3^{(i)} \sqsubseteq t_3$. The forward direction is complete by remarking that $t_1 \sqsubseteq t_2 \sqsubseteq t_3$.

For the backward direction, let $t_3 \in \mathfrak{S}_{k_1+b_3}$ such that $t_1 \sqsubseteq t_3$ and $\forall r_3 \in R_3 : \exists p_3 \in P_3 : r_3 \circ p_3 \sqsubseteq t_3$. Let $\tilde{s}^{(i)} \in \mathfrak{S}_{k_2+b_2}$ be the unique permutation such that $q^{(i)} \circ \tilde{r}_1^{(i)} \circ \tilde{s}^{(i)} \sqsubseteq t_3$ and let $s^{(i)}$ be its unique subsequence in $\mathfrak{S}_{k_2}$.

Let $r_1^{(i)} \in R_1$ and $r_2 \in R_2$, then for $r_3 = q^{(i)} \circ \tilde{r}_1^{(i)} \circ r_2$ there is a $p_3 \in P_3$ such that $r_3 \circ p_3 \sqsubseteq t_3$. Hence, $q^{(i)} \circ \tilde{r}_1^{(i)} \circ r_2 \circ p_3 \sqsubseteq t_3$, from which $r_2 \circ p_3 \sqsubseteq \tilde{s}^{(i)}$ follows since $\tilde{s}^{(i)}$ describes the ordering of all elements of $q^{(i)} \circ \tilde{r}_1^{(i)}$ in t_3. We thus have

$$\exists \tilde{s}^{(i)} \in \mathfrak{S}_{k_2+b_2} : s^{(i)} \sqsubseteq \tilde{s}^{(i)} \wedge \forall r_2 \in R_2 : \exists p_3 \in P_3 : r_2 \circ p_3 \sqsubseteq \tilde{s}^{(i)},$$

and conclude by CO-reducibility that $s^{(i)} \in P_2$.

Choose some $t_2 \in \mathfrak{S}_{k_1+b_1}$ such that $t_1 \sqsubseteq t_2$ and $r_1^{(i)} \circ s^{(i)} \sqsubseteq t_2$ for all i. This is possible since $t_1 \sqsubseteq t_3$ and

$$r_1^{(i)} \circ s^{(i)} = \tilde{r}_1^{(i)} \circ s^{(i)} = q^{(i)} \circ \tilde{r}_1^{(i)} \circ s^{(i)} \sqsubseteq q^{(i)} \circ \tilde{r}_1^{(i)} \circ \tilde{s}^{(i)} \sqsubseteq t_3,$$

and thus t_3 is a common supersequence. By CO-reducibility, $t_1 \in P_1$.

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4.4 Special cases of CO-reduction

The symmetry argument carried out in Section 3.1 is a special instance of CO-reducibility. We show that $(k, P) =_{\text{CO}} (k, \pi \circ P)$ for any $\pi \in \mathfrak{S}_k$.

To this end, verify the instance of CO-reducibility where $k_1 = k_2 = k$, $P_1 = P$, $P_2 = \pi \circ P_1$, $b = 0$, and $R = \{\pi^{-1}\}$. The reverse direction follows too, since π is chosen arbitrarily. As the special case where π is the identity, we again obtain reflexivity.

A slightly more general argument is useful, for example, to reduce betweenness to the “non-betweenness” problem $(3, \mathfrak{S}_3 \setminus \{(123), (321)\})$. By choosing $b = 0$ and $R = \{(213), (231)\}$ we represent each betweenness triple (a_1, a_2, a_3) by the two non-betweenness triples (a_2, a_1, a_3) and (a_2, a_3, a_1). Intuitively, if neither the first nor the third element of a triple must be arranged between the other two, only the second element remains for that position.

We therefore discuss when $(k, P_1) \leq_{\text{CO}} (k, P_2)$ using $b = 0$ and some $R \subseteq \mathfrak{S}_k$. Then CO-reducibility simplifies to

$$\forall t \in \mathfrak{S}_k : t \in P_1 \iff \forall r \in R : \exists p \in P_2 : r \circ p = t,$$

and further, denoting $P_2^{-1} = \{p^{-1} \mid p \in P_2\}$, to

$$\forall t \in \mathfrak{S}_k : t \in P_1 \iff R \subseteq t \circ P_2^{-1}.$$

The forward direction requires $R \subseteq t \circ P_2^{-1}$ for all $t \in P_1$ which is equivalent to

$$R \subseteq \bigcap_{t \in P_1} t \circ P_2^{-1}.$$

On the other hand if $t \notin P_1$ we should have $R \not\subseteq t \circ P_2^{-1}$ which is the more easy to satisfy the larger R is. If we therefore choose

$$R = \bigcap_{t \in P_1} t \circ P_2^{-1},$$

CO-reducibility is available if and only if $\forall t \in \mathfrak{S}_k \setminus P_1 : R \not\subseteq t \circ P_2^{-1}$.

5 Application

The reduction method introduced in Section 4 is now applied to interesting classes of problems. In the first three parts, we take a look at several problems (k, P) that differ in their values of k but have resembling constraints P. Section 5.4 then thoroughly discusses the special case $(4, P)$ in the context of automatic calculation.

5.1 The exclusion problems

By k-exclusion, we identify the constrained ordering problem $(k, \mathfrak{S}_k \setminus \{1_{\mathfrak{S}_k}\})$. For fixed k, it is at least as difficult as every other problem with the same value of k. Indeed, for $P \subseteq \mathfrak{S}_k$, $(k, P) \leq_{\mathrm{CO}}$ k-exclusion by choosing $b = 0$ and $R = \mathfrak{S}_k \setminus P$. Intuitively, one simulates every other problem by prohibiting all unwanted tuples one by one. We will now prove that the exclusion problems are equally hard among each other, that is, 3-exclusion $=_{\mathrm{CO}}$ k-exclusion for all $k \geq 3$.

We will firstly show that 3-exclusion $\leq_{\mathrm{CO}}$ k-exclusion for all $k \geq 3$. To this end, define $b = k - 3$ and $R = \{t' \in \mathfrak{S}_k \mid (1, 2, 3) \sqsubseteq t'\}$. In this special case, CO-reducibility simplifies to

$$\forall t \in \mathfrak{S}_3 : t \neq (1, 2, 3) \iff \exists t' \in \mathfrak{S}_k : t \sqsubseteq t' \wedge t' \notin R.$$

If $t \neq (1, 2, 3)$ choose as t' any supersequence of t, and it will not be in R. On the other hand, if $t = (1, 2, 3)$ all supersequences of t are in R by definition.

We will secondly show that k-exclusion $\leq_{\mathrm{CO}}$ 3-exclusion for all $k \geq 3$. The case $k = 3$ follows by reflexivity. For $k \geq 4$ we will define appropriate b and R for CO-reduction. Since in the special case of 3-exclusion $\exists p \in \mathfrak{S}_3 \setminus \{1_{\mathfrak{S}_3}\} : r \circ p \sqsubseteq t'$ is equivalent to $r \not\sqsubseteq t'$, it then remains to show

$$\forall t \in \mathfrak{S}_k : t \in \mathfrak{S}_k \setminus \{1_{\mathfrak{S}_k}\} \Leftrightarrow \exists t' \in \mathfrak{S}_{3k-6} : t \sqsubseteq t' \wedge \forall r \in R : r \not\sqsubseteq t'.$$

For $k = 4$ define $b = 2$ and $R = \{(1, 2, 6), (2, 3, 5), (5, 6, 4), (6, 5, 4)\}$. For the forward direction let $t \neq (1, 2, 3, 4)$.

- If $(2, 1) \sqsubseteq t$ define $t' = (5, t_1, \ldots, t_4, 6)$, otherwise
- if $(3, 2) \sqsubseteq t$ define $t' = (6, t_1, \ldots, t_4, 5)$, otherwise
- $(4, 3) \sqsubseteq t$, define $t' = (6, \tilde{t}_1, \ldots, \tilde{t}_5)$ where $t \sqsubseteq \tilde{t} \in \mathfrak{S}_5$ with $(4, 5, 3) \sqsubseteq \tilde{t}$.

In all cases $t \sqsubseteq t' \wedge \forall r \in R : r \not\sqsubseteq t'$.

For the backward direction let $t = (1,2,3,4) \sqsubseteq t' \in \mathfrak{S}_6$. Assume that $r \not\sqsubseteq t'$ for all $r \in R$.

- From $(1,2,6) \not\sqsubseteq t'$ it follows that $(6,2) \sqsubseteq t'$, hence $(6,4) \sqsubseteq t'$.
- From $(2,3,5) \not\sqsubseteq t'$ it follows that $(5,3) \sqsubseteq t'$, hence $(5,4) \sqsubseteq t'$.

Therefore, either $(5,6,4) \sqsubseteq t'$ or $(6,5,4) \sqsubseteq t'$, a contradiction.

For $k \geq 5$ define $b = 2k-6$ and $R = W \cup X \cup Y \cup Z$, where

$$
\begin{aligned}
W &= \{(i, i+1, 2k-1-i) \mid 1 \leq i \leq k-2\} \\
X &= \{(2k-1-i, 2k-4+i, 2k-3+i) \mid 2 \leq i \leq k-3\} \\
Y &= \{(2k-4+i, 2k-1-i, 2k-3+i) \mid 2 \leq i \leq k-3\} \\
Z &= \{(k+1, 3k-6, k), (3k-6, k+1, k)\}
\end{aligned}
$$

For the forward direction let $t \neq (1,\ldots,k)$, then i_0+1 precedes i_0 in t for some $i_0 \in \{1,\ldots,k-1\}$. We distinguish three cases, in which we will define t' such that $t \sqsubseteq t' \wedge \forall r \in R : r \not\sqsubseteq t'$.

- If $1 < i_0 < k-1$, define $t' = (2k-4+i_0, \ldots, 2k-i_0, 2k-2-i_0, \ldots, k+1, t_1, \ldots, t_k, 3k-6, \ldots, 2k-3+i_0, 2k-1-i_0)$.
  - For $r \in W$ note that $2k-1-i$ precedes i in t' if $i \neq i_0$, and $i+1$ precedes i if $i = i_0$.
  - For $r \in X$ and $r \in Y$ note that $2k-3+i$ precedes $2k-4+i$ in t' if $i \neq i_0$, and $2k-1-i$ is the last element of t' if $i = i_0$.
  - For $r \in Z$ note that k precedes $3k-6$.
- If $i_0 = 1$, define $t' = (2k-3, \ldots, k+1, t_1, \ldots, t_k, 3k-6, \ldots, 2k-2)$. The argument runs similar to the previous case.
- If $i_0 = k-1$, define $t' = (3k-6, \ldots, k+2, \tilde{t}_1, \ldots, \tilde{t}_{k+1})$, where $t \sqsubseteq \tilde{t} \in \mathfrak{S}_{k+1}$ with $(k, k+1, k-1) \sqsubseteq \tilde{t}$.
  - For $r \in W$ note that $2k-1-i$ precedes $i+1$ in t'.
  - For $r \in X$ and $r \in Y$ note that $2k-3+i$ precedes $2k-4+i$ in t'.
  - For $r \in Z$ note that k precedes $k+1$ in t'.

For the backward direction let $t = (1,\ldots,k) \sqsubseteq t' \in \mathfrak{S}_{3k-6}$. Assume that $r \not\sqsubseteq t'$ for all $r \in R$. According to W, all values $k+1, \ldots, 2k-2$ precede k in t'. Combining X and Y, we show by induction that $2k-1, \ldots, 3k-6$ precede k in t', too. The induction starts with $i = 2$ where $2k-3+i$ precedes $2k-1-i$ or $2k-4+i$, which both precede k. For $3 \leq i \leq k-3$ we assume inductively that $2k-4+i$ precedes k and conclude that $2k-3+i$ precedes k since $2k-1-i$ precedes k again. It follows that both $k+1$ and $3k+6$ precede k in t'. This is a contradiction to the assumption that $r \not\sqsubseteq t'$ for both $r \in Z$. ὅπερ ἔδει δεῖξαι

5.2 The symmetric problems

Given a positive integer k and tuple $t : \mathbf{N}_k \to S$, we will use the following notations.

$$\begin{array}{rcl@{\qquad\qquad}rcl} \overline{k} & = & (k, \ldots, 1) & \overline{t} & = & t \circ \overline{k} \\ \overleftarrow{k} & = & (2, \ldots, k, 1) & \overleftarrow{t} & = & t \circ \overleftarrow{k} \\ \overrightarrow{k} & = & (k, 1, \ldots, k-1) & \overrightarrow{t} & = & t \circ \overrightarrow{k} \end{array}$$

Generalising betweenness, cyclic ordering, and their combination, we identify the following constrained ordering problems.

$$\begin{array}{rcl} k\text{-betweenness} & = & (k, \{1_{\mathfrak{S}_k}, \overline{k}\}) \\ k\text{-cyclic ordering} & = & (k, \{\overleftarrow{k}^i \mid i \in \mathbf{N}\}) \\ k\text{-separation} & = & (k, \{\overleftarrow{k}^i \circ \overline{k}^j \mid i, j \in \mathbf{N}\}) \end{array}$$

Appendix C shows that problems with different kinds of symmetries are not mutually CO-reducible. Precisely, from $(k_1, P_1) \leq_{\mathrm{CO}} (k_2, P_2)$ one can infer that

$$\begin{array}{rcl} P_2 = P_2 \circ \overline{k_2} & \Rightarrow & P_1 = P_1 \circ \overline{k_1}, \\ P_2 = P_2 \circ \overleftarrow{k_2} & \Rightarrow & P_1 = P_1 \circ \overleftarrow{k_1}. \end{array}$$

Just as we did with the exclusion problems in Section 5.1, however, we will prove that the problems in each of the three families are mutually CO-reducible.

Assuming $k \geq 3$, the instance of CO-reduction for $k_1 = 3$, $k_2 = k$, $b = k - 3$, and $R = \{1_{\mathfrak{S}_k}\}$ simplifies to

$$\forall t_1 \in \mathfrak{S}_3 : t_1 \in P_1 \iff \exists t_2 \in P_2 : t_1 \sqsubseteq t_2.$$

This is readily verified for betweenness and cyclic ordering. Assuming $k \geq 4$, the similar instance for $k_1 = 4$, $k_2 = k$, $b = k - 4$, and $R = \{1_{\mathfrak{S}_k}\}$ can be checked for separation. Altogether,

$$\begin{array}{lll} \text{3-betweenness} & \leq_{\mathrm{CO}} & k\text{-betweenness}, \\ \text{3-cyclic ordering} & \leq_{\mathrm{CO}} & k\text{-cyclic ordering, and} \\ \text{4-separation} & \leq_{\mathrm{CO}} & k\text{-separation } (k \geq 4). \end{array}$$

Assuming $k \geq 3$, the instance of CO-reduction for $k_1 = k$, $k_2 = 3$, and $b = 0$ simplifies to

$$\forall t_1 \in \mathfrak{S}_k : t_1 \in P_1 \iff \forall r \in R : \exists p \in P_2 : r \circ p \sqsubseteq t_1.$$

Choose $R = \{r \in \binom{\mathbf{N}_k}{3} \mid r \sqsubseteq 1_{\mathfrak{S}_k}\}$ for both betweenness and cyclic ordering. Assuming $k \geq 4$, check the similar instance for $k_1 = k$, $k_2 = 4$, $b = 0$, and $R = \{r \in \binom{\mathbf{N}_k}{4} \mid r \sqsubseteq 1_{\mathfrak{S}_k}\}$ for separation.

There is a related definition of k-element clauses for the cyclic order in terms of 3-element clauses, and k-element clauses for separation in terms of 4-element ones [JZ05].

5.3 Reductions for partial ordering problems

In this section, we prove some necessary conditions for CO-reductions involving partial ordering problems.

Assuming $(k, P) \leq_{\text{CO}} (2, \{1_{\mathfrak{S}_2}\})$ leads to

$$\forall t_1 \in \mathfrak{S}_k : t_1 \in P \iff \exists t_2 \in \mathfrak{S}_{k+b} : t_1 \sqsubseteq t_2 \wedge \forall r \in R : r \sqsubseteq t_2.$$

In this case, R is a relation of type $\mathbf{N}_{k+b} \leftrightarrow \mathbf{N}_{k+b}$. We may replace R by its transitive closure R^+ in the above formula without affecting its validity since

$$(a_1, a_2) \sqsubseteq t_2 \wedge (a_2, a_3) \sqsubseteq t_2 \quad \Rightarrow \quad (a_1, a_3) \sqsubseteq t_2.$$

If R contains a cycle—in other words, R^+ is not irreflexive—the left hand side is false and we obtain the special case $(k, \emptyset) \leq_{\text{CO}} (2, \{1_{\mathfrak{S}_2}\})$. Otherwise, R^+ is a strict partial order, and exactly the arrangements t_2 obtained by topological sorting satisfy $\forall r \in R^+ : r \sqsubseteq t_2$. One of them has to be a supersequence of t_1, and this is the case if and only if t_1 is a total ordering of the restriction of R^+ to $\mathbf{N}_k \leftrightarrow \mathbf{N}_k$. Therefore, P must contain precisely all such total orderings for CO-reducibility to hold.

Moreover, we see that it is unnecessary for R to refer to elements other than in $\mathbf{N}_k$, hence $b = 0$ can be assumed without loss of generality. By the construction in the proof that $\leq_{\text{CO}}$ implies polynomial reducibility, an instance of (k, P) can thus be solved with topological sorting. This is the reason why we call (k, P) a partial ordering problem, if P consists of all total orderings that extend some partial order. In this case, $(k, P) \leq_{\text{CO}} (2, \{1_{\mathfrak{S}_2}\})$ by the above argument.

Let us now investigate the converse, to see when $(2, \{1_{\mathfrak{S}_2}\}) \leq_{\text{CO}} (k, P)$ holds for a partial ordering problem (k, P). Choosing $b = k - 2$ and $R = \{1_{\mathfrak{S}_k}\}$ this simplifies to

$$(\exists t_2 \in P : (1, 2) \sqsubseteq t_2) \ \wedge \ (\neg \exists t_2 \in P : (2, 1) \sqsubseteq t_2).$$

By definition there is some partial order that P extends. If this is the discrete order, $P = \mathfrak{S}_k$ and CO-reducibility fails. Otherwise, the partial order contains (a_1, a_2) for some a_1 and a_2, hence $\forall t_2 \in P : (a_1, a_2) \sqsubseteq t_2$. Using the symmetry by renaming introduced in Section 3.1, we assume $a_1 = 1$ and $a_2 = 2$ without loss of generality. Therefore, $\forall t_2 \in P : (1, 2) \sqsubseteq t_2$, which implies both conjuncts of the above formula.

Altogether, we obtain $(k, P) =_{\text{CO}} (2, \{1_{\mathfrak{S}_2}\})$ for all partial ordering problems (k, P) such that $P \neq \mathfrak{S}_k$.

5.4 Calculating the quadruples

The definition of CO-reducibility qualifies for automatic evaluation. Once the values of b and R are fixed, the corresponding equivalence can be verified by expanding the $\forall$ quantifier to finite conjunction and the $\exists$ quantifier to finite disjunction. A rough estimate on the number of elementary operations to be then performed is $k_1! \cdot (k_1+b)! \cdot \binom{k_1+b}{k_2} \cdot k_2 \cdot (k_1+b)$, which is feasible for small values of k_1, k_2, and b.

If only b is fixed, additionally the $2^{\binom{k_1+b}{k_2}}$ possible values of R have to be tried—this is a very time consuming task even for small values involved. It becomes entirely impossible in the most general case where all possible values of b, namely the non-negative integers, must be considered.

The approach actually implemented to completely resolve the case $k = 4$ is to fix $b = 1$, use an optimised backtracking search for R, and employ transitivity of CO-reduction. We will now describe the procedure in more detail.

1. In the first step, symmetrical problems as described in Section 3.1 are identified. It turns out that the 2^{24} problems of the form $(4, P)$ can be grouped into 700688 classes this way.

2. We have observed in Section 5.1 that the exclusion problems are among the most difficult. By proving 4-exclusion $\leq_{\mathrm{CO}} (4, P)$ we thus know that both problems belong to the same class. Using transitivity, a series of such reductions is performed, where it is possible to choose even $b = 1$ for each reduction. The longest reduction chain contains 18 transitive steps. It turns out that of the remaining 700688 classes all but 285 belong to the same as 4-exclusion, leaving us with 286 classes—one of which is quite large.

3. In the third step, further reductions are applied to the remaining classes. Successively, two classes are considered and merged if they contain problems $(4, P_1)$ and $(4, P_2)$, respectively, such that $(4, P_1) =_{\mathrm{CO}} (4, P_2)$. This leaves us with those 9 classes shown in Figure 5. Appendix D describes why they are tractable or NP-complete, respectively.

The classes are held in a disjoint-set forest [CLR90]. Backtracking is used to search for a suitable value of R, the necessary calculations being optimised by a large amount of pre-calculation. The total calculation is performed in less than $5 \cdot 10^{14}$ CPU cycles using less than $5 \cdot 10^9$ bits of memory. To ensure correctness, the reductions are made explicit and can thus be verified independently [CO05].

We have thus identified a conservative approximation to CO-reducibility among $(4, P)$. Let us now state the conditions under which this approximation may be strengthened, that is, the possibility of further CO-reductions. The following discussion applies not only to the quadruples but to all $k \geq 3$.

- By definition of CO-reducibility, the only problem reducible to $(k, \mathfrak{S}_k)$ is $(k, \mathfrak{S}_k)$ itself. It follows that it is unique in its class.

- Again by definition of CO-reducibility, there are exactly two problems reducible to $(k, \emptyset)$. By choosing $R = \emptyset$ we can take $(k, \mathfrak{S}_k)$ as the source of the reduction. By choosing $R \neq \emptyset$ we obtain the reflexive case. It follows that $(k, \emptyset)$ is unique in its class, too.

- In Section 5.3 we have proved that the partial ordering problems form their own class.
- By the reasoning in Appendix C, those classes that exhibit symmetry are not mutually reducible.
- Finally, if P $\neq$ NP, no NP-complete problem is CO-reducible to some tractable problem.

For the cases $k = 3$ and $k = 4$ only the question of CO-reducibility between the two, mutually reverse algorithm T classes is open. Finally, some classes could split for larger values of k.

6 Conclusion

Let us summarise the contributions of this paper. In Section 2 we have presented an efficient algorithm—a generalisation of topological sorting—that solves an object-relational mapping problem. Several generalisations of this and other known tractable and NP-complete ordering problems have been explored in Section 3. We have introduced and investigated a reduction method for such problems in Section 4. It was then applied to prove NP-completeness and tractability results for large classes of these problems in Section 5, both manually and by machine.

6.1 Related work

The problems discussed in this paper also arise in the context of qualitative spatial reasoning [IC00]. The algebraic treatment in that area originates in qualitative temporal reasoning, notably with Allen's interval algebra [All83]. All subclasses of Allen's interval algebra have been classified as being either NP-complete or tractable [NB95, KJJ03]. We conjecture that the problems described in this paper enjoy the same dichotomy.

Note that a simple translation from Allen's interval algebra to our formalism fails for two reasons. First, the relative positions of intervals use not only $<$ but also the $\leq$, $=$, and $\neq$ relations—see also the discussion in Appendix A. Second, there may be different disjunctions in effect between different pairs of intervals. This could be simulated with the exclusion problem, but that is already NP-complete.

Conversely, a simple translation from our formalism to Allen's interval algebra fails also for two reasons. First, the start and end points of intervals are correlated, whereas no such restrictions apply for constrained ordering problems. Second, there is only one clause for each pair of intervals, but our set C models arbitrary conjunctions.

6.2 Open problems

The most important open question is about the decidability of CO-reduction. This would be answered in the affirmative if, for example, one could give an upper bound for the number of new elements b such that no expressiveness is gained by introducing more new elements. Another way to attack this problem is to try to prove that if $(k_1, P_1) \leq_{\mathrm{CO}} (k_3, P_3)$ using $b \geq 2$ new elements, there is always an intermediate (k_2, P_2) such that the reductions from (k_1, P_1) to (k_2, P_2) and from (k_2, P_2) to (k_3, P_3) need $\leq b - 1$ new elements, respectively. So far, we have no counterexample where $b = 1$ and transitivity is not enough.

Another important open question concerns the class structure induced by $\leq_{\mathrm{CO}}$ for $k > 4$. The discussion at the end of Section 5.4 goes some way but, for example, we do not know if all problems that exhibit the same kind of symmetries remain in the same class. Yet more obscure is the situation of the algorithm T classes and the exclusion class, where we have no exact characterisation. Making the class structure transparent is a prerequisite for engaging with the dichotomy conjecture, or could even provide its solution. Further classes of constraints that exhibit special forms could also be investigated separately.

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A Further generalisations

We augment Section 3 by addressing two further kinds of generalisations.

In contrast to Section 3.2, where the constraint triples were extended by pairs, we entirely replaced them by k-tuples in Section 3.4. There is no reason, however, why one should not consider, for example, constraint quadruples extended by pairs or by both triples and pairs. In general, one could allow arbitrary tuples with up to k elements. This family of problems is indexed by k constraints $P_l \subseteq \mathfrak{S}_l$ for $1 \leq l \leq k$.

- INSTANCE: Finite set A, collections C_l for $1 \leq l \leq k$ such that each C_l contains l-tuples $(a_1, \ldots, a_l)$ of distinct elements from A.
- QUESTION: Is there a bijection $f : A \to \{1, 2, \ldots, |A|\}$ such that for each C_l and each $(a_1, \ldots, a_l) \in C_l$ there is a $p \in P_l$ with $f(a_{p(i)}) < f(a_{p(j)})$ for all $1 \leq i < j \leq l$?

The number of problems for each k is $\prod_{l=1}^{k} 2^{l!} = 2^{\sum_{l=1}^{k} l!}$.

We have seen in Section 2 that it sometimes makes sense to have the same element occur more than once in a tuple. The problem statement of Section 3.4, however, is not suited for that since the same elements cannot be arranged in a strict order. The difference is illustrated by the fact that

$$(c < a) \vee (c < b) \iff (a < c < b) \vee (b < c < a) \vee (c < a < b) \vee (c < b < a)$$

holds only if the distinctness condition $a \neq b \neq c \neq a$ is true. We therefore need to allow for a weak order. Thus,

$$(c < a) \vee (c < b) \iff (a \leq c < b) \vee (b \leq c < a) \vee (c < a \leq b) \vee (c < b \leq a)$$

holds without further restriction. More basic building blocks are reached by replacing $\leq$ with a disjunction of $<$ and $=$, as in

$$(a \leq c < b) \iff (a < c < b) \vee (a = c < b).$$

One way to model this is to require that a constraint in P is no longer just a permutation of k elements, but an ordered partition of $\{1, 2, \ldots, k\}$. This way, $(c < a) \vee (c < b)$ could be expressed by the triple (a, b, c) and the constraint

$$\begin{aligned} P \;=\; & \{(\{1\},\{3\},\{2\}), (\{2\},\{3\},\{1\}), (\{3\},\{1\},\{2\}), (\{3\},\{2\},\{1\}), \\ & (\{1,3\},\{2\}), (\{2,3\},\{1\}), (\{3\},\{1,2\})\}. \end{aligned}$$

The number of problems for each k is $2^{\sum_{0 \leq i < l \leq k} (-1)^i \binom{l}{i} (l-i)^k} = 2^{\sum_{l=1}^{k} l! \cdot S_2(k,l)}$ where $S_2(k,l) = \frac{1}{l!} \cdot \sum_{i=0}^{l-1} (-1)^i \binom{l}{i} (l-i)^k$ are the Stirling numbers of the second kind.

Both generalisations just discussed may of course be combined. The ultimate generalisation would allow to annotate each tuple with its own constraint.

B Reductions for disjoint triples

We prove the NP-completeness of some of the problems from the following family indexed by $P \subseteq \mathfrak{S}_3$ introduced in Section 3.3.

- INSTANCE: Finite set A, collection B of pairs from A, collection C of pairwise disjoint triples (a_1, a_2, a_3) of distinct elements from A.
- QUESTION: Is there a bijection $f : A \to \{1, 2, \dots, |A|\}$ such that $f(a_1) < f(a_2)$ for each $(a_1, a_2) \in B$, and for each $(a_1, a_2, a_3) \in C$ there is a $p \in P$ with $f(a_{p(1)}) < f(a_{p(2)}) < f(a_{p(3)})$?

B.1 Intermezzo

We call the problem $P = \{(123), (231)\}$ the intermezzo problem. The requirement for the triples in $(a_1, a_2, a_3) \in C$ therefore reads $f(a_1) < f(a_2) < f(a_3)$ or $f(a_2) < f(a_3) < f(a_1)$. We prove its NP-completeness by reduction from 3SAT. The component design technique is described in [GJ79, Section 3.2.3].

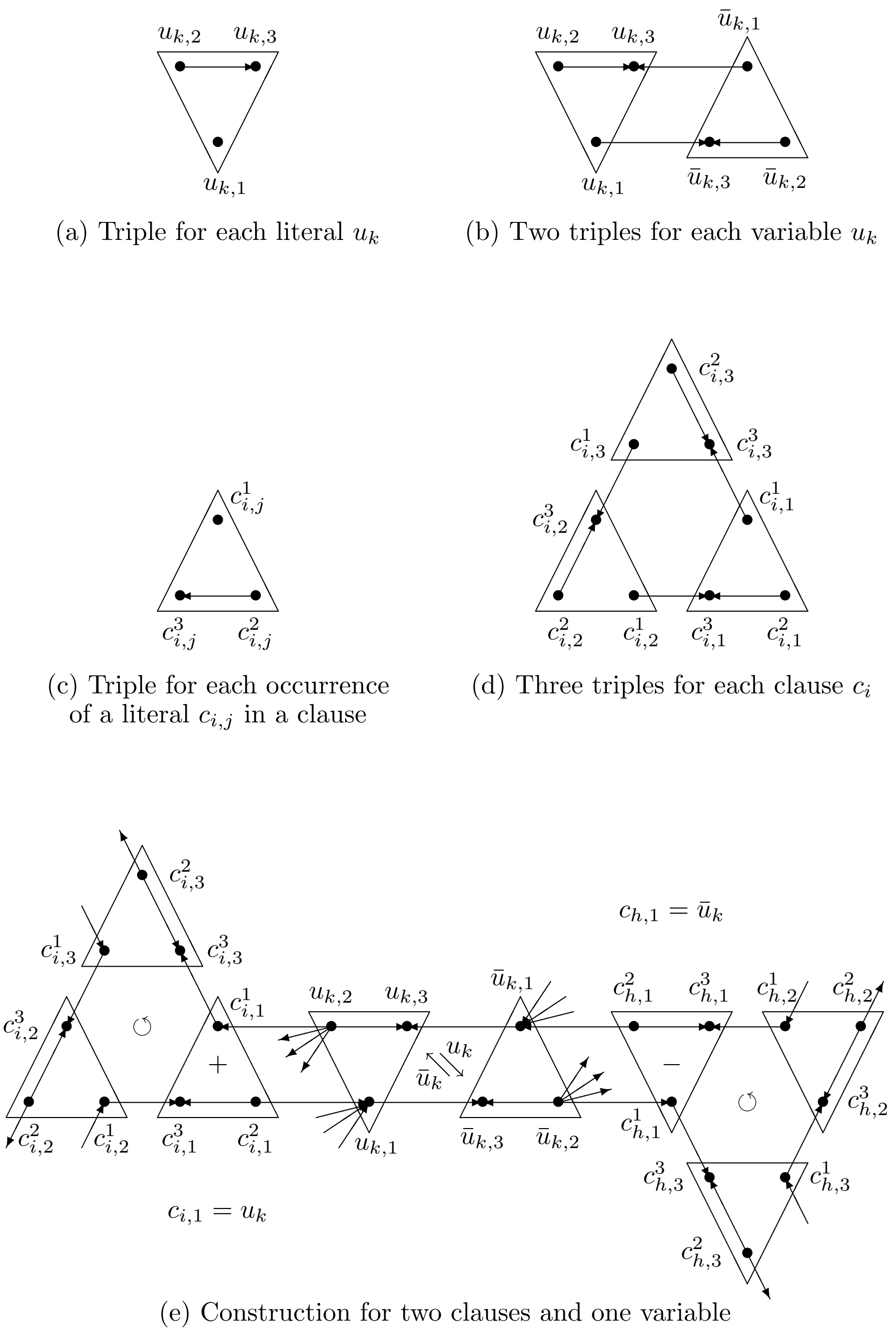
Let an instance of 3SAT be characterised by the set of variables $U = \{u_1, \dots, u_n\}$ and the set of clauses $C' = \{(c_{1,1} \vee c_{1,2} \vee c_{1,3}), \dots, (c_{m,1} \vee c_{m,2} \vee c_{m,3})\}$, where $c_{i,j} = u_k$ or $c_{i,j} = \bar{u}_k$ for some k. Let $\bar{\bar{u}}_k = u_k$, and let $a \oplus b$ denote the number $c \in \{1, 2, 3\}$ such that $a + b \equiv c \pmod 3$. Construct the instance of intermezzo where

$$
\begin{aligned}
A &= \{u_{k,l}, \bar{u}_{k,l} \mid 1 \le k \le n \wedge 1 \le l \le 3\} \cup \\
&\quad\; \{c_{i,j}^l \mid 1 \le i \le m \wedge 1 \le j \le 3 \wedge 1 \le l \le 3\} \\
B &= \{(u_{k,1}, \bar{u}_{k,3}), (\bar{u}_{k,1}, u_{k,3}) \mid 1 \le k \le n\} \cup \\
&\quad\; \{(c_{i,j,2}, c_{i,j}^1), (c_{i,j}^2, c_{i,j,1}) \mid 1 \le i \le m \wedge 1 \le j \le 3\} \cup \\
&\quad\; \{(c_{i,j\oplus 1}^1, c_{i,j}^3) \mid 1 \le i \le m \wedge 1 \le j \le 3\} \\
C &= \{(u_{k,1}, u_{k,2}, u_{k,3}), (\bar{u}_{k,1}, \bar{u}_{k,2}, \bar{u}_{k,3}) \mid 1 \le k \le n\} \cup \\
&\quad\; \{(c_{i,j}^1, c_{i,j}^2, c_{i,j}^3) \mid 1 \le i \le m \wedge 1 \le j \le 3\}
\end{aligned}
$$

The notation $c_{i,j,l}$ is an abbreviation of $u_{k,l}$ where $u_k = c_{i,j}$. We will now describe the construction that is illustrated in Figure 6 in more detail.

For each literal u_k we construct three elements $u_{k,l}$ that are grouped in the triple $(u_{k,1}, u_{k,2}, u_{k,3})$ as shown in Figure 6(a). The same construction is applied for each literal $\bar{u}_k$. For each variable u_k we thus have two such triples, and we connect them by two edges $(u_{k,1}, \bar{u}_{k,3})$ and $(\bar{u}_{k,1}, u_{k,3})$ as shown in Figure 6(b). The subgraph for each variable therefore consists of 6 nodes, 2 edges, and 2 triples.

For each occurrence of a literal $c_{i,j}$ in a clause c_i we construct three elements $c_{i,j}^l$ that are grouped in the triple $(c_{i,j}^1, c_{i,j}^2, c_{i,j}^3)$ as shown in Figure 6(c). For each clause c_i we thus have three such triples, and we connect them pairwise by edges $(c_{i,j\oplus 1}^1, c_{i,j}^3)$ as shown in Figure 6(d). The subgraph for each clause therefore consists of 9 nodes, 3 edges, and 3 triples.



(a) Triple for each literal u_k

(b) Two triples for each variable u_k

(c) Triple for each occurrence of a literal $c_{i,j}$ in a clause

(d) Three triples for each clause c_i

(e) Construction for two clauses and one variable

Figure 6: Graph constructed for the reduction to intermezzo

The connection between the subgraphs for the variables and those for the clauses is obtained by constructing two edges $(c_{i,j,2}, c^1_{i,j})$ and $(c^2_{i,j}, c_{i,j,1})$ for each occurrence of a literal $c_{i,j}$ in a clause. Note that $c_{i,j,l} = u_{k,l}$ for positive literals $c_{i,j} = u_k$, and $c_{i,j,l} = \bar{u}_{k,l}$ for negative literals $c_{i,j} = \bar{u}_k$. Figure 6(e) shows this construction for the occurrences of the positive literal $c_{i,1} = u_k$ and the negative literal $c_{h,1} = \bar{u}_k$ in two different clauses c_i and c_h. Further connections are suggested by arrows attached to one node only.

The complete graph consists of $|A| = 6n + 9m$ nodes, $|B| = 2n + 9m$ edges, and $|C| = 2n + 3m$ triples. We will now prove that this instance of intermezzo is solvable if and only if the corresponding instance of 3SAT is satisfiable.

Let f be an ordering function as required by the specification of intermezzo. Define the truth assignment $t(u_k) = f(u_{k,3}) < f(\bar{u}_{k,3})$. Assume that t does not satisfy C' and let $(c_{i,1} \vee c_{i,2} \vee c_{i,3})$ be a clause such that $\neg t(c_{i,j})$ for $1 \le j \le 3$.

1. By definition of t we have $f(\bar{c}_{i,j,3}) < f(c_{i,j,3})$.
2. Since $(c_{i,j,1}, \bar{c}_{i,j,3}) \in B$ we have $f(c_{i,j,1}) < f(c_{i,j,3})$.
3. Since $(c_{i,j,1}, c_{i,j,2}, c_{i,j,3}) \in C$ we have $f(c_{i,j,1}) < f(c_{i,j,2})$.
4. Since $(c^2_{i,j}, c_{i,j,1}), (c_{i,j,2}, c^1_{i,j}) \in B$ we have $f(c^2_{i,j}) < f(c^1_{i,j})$.
5. Since $(c^1_{i,j}, c^2_{i,j}, c^3_{i,j}) \in C$ we have $f(c^3_{i,j}) < f(c^1_{i,j})$.
6. Since $(c^1_{i,j\oplus 1}, c^3_{i,j}) \in B$ we have $f(c^1_{i,j\oplus 1}) < f(c^1_{i,j})$.
7. Therefore we have $f(c^1_{i,j}) = f(c^1_{i,j\oplus 3}) < f(c^1_{i,j\oplus 2}) < f(c^1_{i,j\oplus 1}) < f(c^1_{i,j})$, a contradiction.

Let t be a truth assignment that satisfies C'. For $1 \le k \le n$ let $t_k = u_k$ if $t(u_k)$ and $t_k = \bar{u}_k$ if $\neg t(u_k)$. For $1 \le i \le m$ let l_i be such that $t(c_{i,l_i})$. Define the mapping $g : A \to \mathbf{N}$ such that

$$\begin{array}{rclll}
g(c^2_{i,j}) & = & 3i + j & \text{for } 1 \le i \le m \wedge 1 \le j \le 3 \wedge \neg t(c_{i,j}) \\
g(\bar{t}_{k,1}) & = & D + k & \text{for } 1 \le k \le n \\
g(\bar{t}_{k,2}) & = & 2D + k & \text{for } 1 \le k \le n \\
g(t_{k,2}) & = & 3D + k & \text{for } 1 \le k \le n \\
g(c^1_{i,j}) & = & 4D + 3i + j & \text{for } 1 \le i \le m \wedge 1 \le j \le 3 \wedge t(c_{i,j}) \\
g(c^2_{i,j}) & = & 5D + 3i + j & \text{for } 1 \le i \le m \wedge 1 \le j \le 3 \wedge t(c_{i,j}) \\
g(c^3_{i,j}) & = & 6D + 3i + j & \text{for } 1 \le i \le m \wedge j = l_i \oplus 2 \wedge \neg t(c_{i,j}) \\
g(c^1_{i,j}) & = & 7D + 3i + j & \text{for } 1 \le i \le m \wedge j = l_i \oplus 2 \wedge \neg t(c_{i,j}) \\
g(c^3_{i,j}) & = & 8D + 3i + j & \text{for } 1 \le i \le m \wedge j = l_i \oplus 1 \wedge \neg t(c_{i,j}) \\
g(c^1_{i,j}) & = & 9D + 3i + j & \text{for } 1 \le i \le m \wedge j = l_i \oplus 1 \wedge \neg t(c_{i,j}) \\
g(c^3_{i,j}) & = & 10D + 3i + j & \text{for } 1 \le i \le m \wedge 1 \le j \le 3 \wedge t(c_{i,j}) \\
g(t_{k,3}) & = & 11D + k & \text{for } 1 \le k \le n \\
g(t_{k,1}) & = & 12D + k & \text{for } 1 \le k \le n \\
g(\bar{t}_{k,3}) & = & 13D + k & \text{for } 1 \le k \le n
\end{array}$$

where D is large enough to keep the definitions separate, for instance choose $D = 2n + 4m + 4$. g satisfies the constraints specified by B since

1. $g(t_{k,1}) < 13D < g(\bar{t}_{k,3})$ and $g(\bar{t}_{k,1}) < 2D < 11D < g(t_{k,3})$.
2. $g(c_{i,j,2}) < 4D < g(c^1_{i,j})$ and $g(c^2_{i,j}) < 6D < 12D < g(c_{i,j,1})$ if $t(c_{i,j})$, and $g(c_{i,j,2}) < 3D < 4D < g(c^1_{i,j})$ and $g(c^2_{i,j}) < D < g(c_{i,j,1})$ if $\neg t(c_{i,j})$.
3. $g(c^1_{i,l_i}) < 5D < 6D < g(c^3_{i,l_i \oplus 2})$ and $g(c^1_{i,l_i \oplus 2}) < 8D < g(c^3_{i,l_i \oplus 1})$ and $g(c^1_{i,l_i \oplus 1}) < 10D < g(c^3_{i,l_i})$.

g satisfies the constraints specified by C since

4. $g(\bar{t}_{k,1}) < 2D < g(\bar{t}_{k,2}) < 3D < 13D < g(\bar{t}_{k,3})$.
5. $g(t_{k,2}) < 4D < 11D < g(t_{k,3}) < 12D < g(t_{k,1})$.
6. $g(c^2_{i,l_i \oplus 2}) < D < 6D < g(c^3_{i,l_i \oplus 2}) < 7D < g(c^1_{i,l_i \oplus 2})$ if $\neg t(c_{i,l_i \oplus 2})$, and $g(c^2_{i,l_i \oplus 1}) < D < 8D < g(c^3_{i,l_i \oplus 1}) < 9D < g(c^1_{i,l_i \oplus 1})$ if $\neg t(c_{i,l_i \oplus 1})$, and $g(c^1_{i,j}) < 5D < g(c^2_{i,j}) < 6D < 10D < g(c^3_{i,j})$ if $t(c_{i,j})$.

The function $f(e) = |\{a \in A \mid g(a) \leq g(e)\}|$ is one-to-one, and satisfies the constraints specified by B and C. ὅπερ ἔδει δεῖξαι

B.2 Betweenness

The requirement for the triples in $(a_1, a_2, a_3) \in C$ reads $f(a_1) < f(a_2) < f(a_3)$ or $f(a_3) < f(a_2) < f(a_1)$ for the betweenness problem. We prove it is NP-complete by reduction from intermezzo.

Let A', B', and C' characterise an instance of intermezzo. Construct the instance of betweenness where A extends A' by three new elements a'_1, a'_2, a'_3 for each $(a_1, a_2, a_3) \in C'$. Note that there are $3|C'|$ distinct new elements since the triples in C' are pairwise disjoint. Moreover, C consists of two triples (a_1, a'_3, a_3), (a'_1, a'_2, a_2) for each $(a_1, a_2, a_3) \in C'$. Finally, for each $(a_1, a_2, a_3) \in C'$, B extends B' by inserting three new pairs (a'_1, a_1), (a'_3, a'_2), (a_2, a_3) and, for each pair (a, a_1), one new pair (a, a'_1). Intuitively, an element a_1 is split into two elements a_1 and a'_1 such that a'_1 immediately precedes a_1.

Assume there is a total order $\prec'$ of the instance of intermezzo. The order $\prec$ modifies $\prec'$ by replacing, for each $(a_1, a_2, a_3) \in C'$, the occurrence of a_1 in $\prec'$ with

- either $a'_1 \prec a_1 \prec a'_3 \prec a'_2$ if $a_1 \prec' a_2 \prec' a_3$,
- or $a'_3 \prec a'_2 \prec a'_1 \prec a_1$ if $a_2 \prec' a_3 \prec' a_1$,

such that these four elements succeed without a gap. By definition of intermezzo exactly one of the two cases applies for each triple, thus $\prec$ is a total order of A.

The order $\prec$ satisfies each triple (a_1, a'_3, a_3) since $a_1 \prec a'_3 \prec a_3$ in the first case and $a_3 \prec a'_3 \prec a_1$ in the second case. The order $\prec$ satisfies each triple (a'_1, a'_2, a_2) since $a'_1 \prec a'_2 \prec a_2$ in the first case and $a_2 \prec a'_2 \prec a'_1$ in the second case. The order $\prec$, being an extension of $\prec'$, satisfies B'. In both cases $a'_1 \prec a_1$, $a'_3 \prec a'_2$, and $a_2 \prec a_3$ for each triple $(a_1, a_2, a_3) \in C'$, and, since a'_1 and a_1

succeed without a gap, $a \prec a_1'$ whenever $a \prec' a_1$. Hence, $\prec$ is a total order of the constructed instance.

Assume there is a total order $\prec$ of the constructed instance. The order $\prec'$ is the restriction of $\prec$ to A. For each triple $(a_1, a_2, a_3) \in C'$, $a_2 \prec' a_3$ since $(a_2, a_3) \in B$. Assume that $a_2 \prec' a_1 \prec' a_3$ for some such triple.

1. By definition of $\prec'$ we also have $a_2 \prec a_1 \prec a_3$.
2. Since $(a_1, a_3', a_3) \in C$ we have $a_1 \prec a_3' \prec a_3$.
3. Since $(a_1', a_1) \in B$ and $(a_3', a_2') \in B$ we have $a_1' \prec a_1 \prec a_3' \prec a_2'$.
4. Since $(a_1', a_2', a_2) \in C$ we have $a_1' \prec a_1 \prec a_3' \prec a_2' \prec a_2$.
5. Therefore $a_1 \prec a_2$ and $a_2 \prec a_1$, a contradiction.

Thus, $a_1 \prec' a_2 \prec' a_3$ or $a_2 \prec' a_3 \prec' a_1$, so $\prec'$ satisfies all triples in C'. Finally, $(a_1, a_2) \in B' \Rightarrow (a_1, a_2) \in B \Rightarrow a_1 \prec a_2 \Rightarrow a_1 \prec' a_2$, so $\prec'$ satisfies B'. Hence, $\prec'$ is a total order of the instance of intermezzo. ὅπερ ἔδει δεῖξαι

B.3 The remaining NP-complete problems

Five classes of problems are left, each class containing several problems that are symmetric as described in Section 3.1. Choose one specific problem from each class, namely $\{(123), (231), (312)\}$, $\{(123), (231), (321)\}$, $\mathfrak{S}_3 \setminus \{(213), (312)\}$, $\mathfrak{S}_3 \setminus \{(213), (132)\}$, and $\mathfrak{S}_3 \setminus \{(213)\}$. The following argument shows that each of these problems is NP-complete by reduction from intermezzo.

Let A', B', and C' characterise an instance of intermezzo. Construct the instance for any of the five problems where $A = A'$, $C = C'$, and $B = B' \cup \{(a_2, a_3) \mid (a_1, a_2, a_3) \in C'\}$. If the instance of intermezzo has a solution, it also solves the constructed instance since each of the five problems are supersets of $\{(123), (231)\}$. If the constructed instance has a solution, it also solves the instance of intermezzo since by the extension of the pairs a_2 must precede a_3 for each triple $(a_1, a_2, a_3) \in C'$ and none of the five problems contains (213). Hence, the constructed instance has a solution if and only if the corresponding instance of intermezzo has one.

C Reductions between symmetric problems

In this section, we prove some necessary conditions for CO-reductions involving symmetric problems. We use the notations introduced in Section 5.2.

C.1 Betweenness

Let $s : \mathbf{N}_k \to S$ and $t : \mathbf{N}_l \to S$. We prove that $s \sqsubseteq t \Rightarrow \overline{s} \sqsubseteq \overline{t}$. By definition of $\sqsubseteq$, let $u : \mathbf{N}_k \to \mathbf{N}_l$ be strictly increasing such that $s = t \circ u$. Define $v = \overline{l} \circ u \circ \overline{k}$, that is strictly increasing since both $\overline{k}$ and $\overline{l}$ are strictly decreasing. Thus, $\overline{s} \sqsubseteq \overline{t}$, since

$$\overline{s} = s \circ \overline{k} = t \circ u \circ \overline{k} = t \circ \overline{l} \circ \overline{l} \circ u \circ \overline{k} = t \circ \overline{l} \circ v = \overline{t} \circ v.$$

Let $(k_1, P_1) \leq_{\mathrm{CO}} (k_2, P_2)$. We prove that $P_2 = P_2 \circ \overline{k_2} \Rightarrow P_1 = P_1 \circ \overline{k_1}$. Let $t_1 \in P_1$, then by definition of $\leq_{\mathrm{CO}}$ there are b, R, and $t_2 \in \mathfrak{S}_{k_1+b}$ such that

$$t_1 \sqsubseteq t_2 \wedge \forall r \in R : \exists p \in P_2 : r \circ p \sqsubseteq t_2.$$

By the above lemma, $\overline{t_1} \sqsubseteq \overline{t_2}$ and $\forall r \in R : \exists p \in P_2 : r \circ \overline{p} = \overline{r \circ p} \sqsubseteq \overline{t_2}$. By the assumption, $p \in P_2 \Rightarrow \overline{p} \in P_2$, hence also $\forall r \in R : \exists p \in P_2 : r \circ p \sqsubseteq \overline{t_2}$. By CO-reduction, $\overline{t_1} \in P_1$.

C.2 Cyclic ordering

Let $s : \mathbf{N}_k \to S$ and $t : \mathbf{N}_l \to S$ such that $s \sqsubseteq t$. We prove that

$$\forall i \in \mathbf{N} : \exists j \in \mathbf{N} : s \circ \overleftarrow{k}^{i} \sqsubseteq t \circ \overleftarrow{l}^{j} \quad \wedge \quad \forall i \in \mathbf{N} : \exists j \in \mathbf{N} : s \circ \overleftarrow{k}^{j} \sqsubseteq t \circ \overleftarrow{l}^{i}.$$

It suffices to deal with the case $i = 1$ because for larger values, the process may be applied repeatedly since rotations are closed under composition. It further suffices to deal with the case $k + 1 = l$, since the case $k = l$ is clear by choosing $j = i$, while for larger differences of k and l, one chooses for $i \in \{k+1, \ldots, l-1\}$ tuples $t_i : \mathbf{N}_i \to S$ such that $s \sqsubseteq t_{k+1} \sqsubseteq \ldots \sqsubseteq t_{l-1} \sqsubseteq t$ and reasons step by step.

Therefore, without loss of generality, $s = t \circ (1, \ldots, h-1, h+1, \ldots, l)$ for some $h \in \mathbf{N}_l$. If $h = 1$,

$$s = t \circ (2, \ldots, l) = \overleftarrow{t} \circ \overrightarrow{l} \circ (2, \ldots, l) = \overleftarrow{t} \circ (1, \ldots, l-1),$$
$$\overleftarrow{s} = \overset{\leftleftarrows}{t} \circ \overset{\rightrightarrows}{l} \circ \overleftarrow{(2, \ldots, l)} = \overset{\leftleftarrows}{t} \circ \overset{\rightrightarrows}{l} \circ (3, \ldots, l, 2) = \overset{\leftleftarrows}{t} \circ (1, \ldots, l-2, l),$$

so that $\overleftarrow{s} \sqsubseteq \overset{\leftleftarrows}{t}$ and $s \sqsubseteq \overleftarrow{t}$. If $h > 1$,

$$\overleftarrow{s} = \overleftarrow{t} \circ \overrightarrow{l} \circ \overleftarrow{(1, \ldots, h-1, h+1, \ldots, l)} = \overleftarrow{t} \circ (1, \ldots, h-2, h, \ldots, l),$$

so that $\overleftarrow{s} \sqsubseteq \overleftarrow{t}$.

Let $(k_1, P_1) \leq_{\mathrm{CO}} (k_2, P_2)$. We prove that $P_2 = P_2 \circ \overleftarrow{k_2} \Rightarrow P_1 = P_1 \circ \overleftarrow{k_1}$. Let $t_1 \in P_1$, then by definition of $\leq_{\mathrm{CO}}$ there are b, R, and $t_2 \in \mathfrak{S}_{k_1+b}$ such that

$$t_1 \sqsubseteq t_2 \wedge \forall r \in R : \exists p \in P_2 : r \circ p \sqsubseteq t_2.$$

By the first part of the above lemma, $\overleftarrow{t_1} \sqsubseteq t_2 \circ \overleftarrow{k_1 + b}^{i}$ for some i. By its second part, $\forall r \in R : \exists p \in P_2 : r \circ p \circ \overleftarrow{k_2}^{j} \sqsubseteq t_2 \circ \overleftarrow{k_1 + b}^{i}$ for some j. By the assumption, $p \in P_2 \Rightarrow p \circ \overleftarrow{k_2}^{j} \in P_2$, hence also $\forall r \in R : \exists p \in P_2 : r \circ p \sqsubseteq t_2 \circ \overleftarrow{k_1 + b}^{i}$. By CO-reduction, $\overleftarrow{t_1} \in P_1$.

D Reductions between triples and quadruples

After performing several reductions between constrained ordering problems with $k = 4$, we are left with 9 classes. Let us discuss why the remaining 9 classes are tractable or NP-complete, respectively, as shown in Figure 5.

Let A' and C' characterise an instance of betweenness. Construct the instance of the 4-separation problem—see Section 5.2—where $A = A' \cup \{n\}$ for some $n \notin A'$ and $C = \{(n, a_1, a_2, a_3) \mid (a_1, a_2, a_3) \in C'\}$. We prove that this instance has a solution if and only if the corresponding instance of betweenness has one.

If there is an ordering of A' that satisfies C', prepend the element n to get an ordering of A that satisfies C, since $(1, 2, 3, 4)$ and $(1, 4, 3, 2)$ are valid permutations.

If there is an ordering of A that satisfies C, rotate it until the element n is on the first position. The result still satisfies C but uses only the permutations $(1, 2, 3, 4)$ and $(1, 4, 3, 2)$ since n is the first element of each quadruple in C. Removing the element n yields an ordering of A' that satisfies C'.

By transitivity, all problems surrounded by double boxes are NP-complete. Moreover, $(4, \{(1324), (2314), (3124), (3214)\}) \leq_{\mathrm{CO}} (3, \mathfrak{S}_3 \setminus \{(123), (213)\})$ using $b = 0$ and $R = \{(1, 2, 3), (1, 4, 2), (2, 4, 1)\}$. Intuitively, the last two tuples of R force the fourth element of the quadruple to follow the first and the second. By transitivity and reverse symmetry, all problems surrounded by simple boxes indeed are tractable.

τέλος