Variations on an Ordering Theme with Constraints

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Introduction

INSTANCE: Finite set A, collection C of ordered triples from A^3.

QUESTION: Is there a total order $<$ of A, such that for all $(a, b, c) \in C$,

- $a < b < c$ or $c < b < a$
  (Betweenness)
- neither $a < b < c$ nor $c < b < a$
  (Nonbetweenness)
- $a < b < c$ or $b < c < a$ or $c < a < b$
  (Cyclic Ordering)
- $b < a$ or $c < a$

Extended topological sorting

input finite set A, partial order $\lessdot$, collection of disjoint triples C

output ordering of A that extends $\lessdot$ such that the first element of each triple in C comes after the second or the third

method

$(S, P, T) \leftarrow (A, \lessdot, C)$
Ordering $\leftarrow$ empty sequence
while $S \neq \emptyset$ **do**
  find $e \in S$ such that $\forall y, z \in S : (y, e) \notin P \land (e, y, z) \notin T$
  if such an e exists **then**
    $T \leftarrow \{(x, y, z) \in T \mid y \neq e \land z \neq e\}$
    $P \leftarrow \{(x, y) \in P \mid x \neq e\}$
    $S \leftarrow S \setminus \{e\}$
    append e to Ordering
  else
    output "there is no ordering"; **halt**
end
output Ordering

Correctness and completeness

An output is correct, because

- (x, y) stays in P until x is appended.
- if $(x, y) \in P$, then x is appended to the ordering before y.
- (x, y, z) stays in T until y or z is appended.
- if $(x, y, z) \in T$, then y or z is appended to the ordering before x.

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If a solution exists, it is found.

Assume there is no output. Then

$$\text{find } e \in S \text{ such that } \forall y, z \in S : (y, e) \notin P \land (e, y, z) \notin T$$

fails to find an e.
$\Rightarrow$ A subset $S \subseteq A$ exists where no minimum can be identified.

Generalisation

INSTANCE: Finite set A, collection C of ordered triples from A^3, constraint $P \subseteq \mathcal{S}_3$.
QUESTION: Is there a one-to-one function $f : A \to \{1, \ldots, |A|\}$, such that for all $(t_1, t_2, t_3) \in C$, there exists a $p \in P$ with $f(t_{p(1)}) < f(t_{p(2)}) < f(t_{p(3)})$.

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Example 1

$P = \{123, 321\}$ specifies the problem Betweenness.

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Example 2

$P = \{213, 231, 132, 312\}$ specifies the problem Nonbetweenness.

Variations

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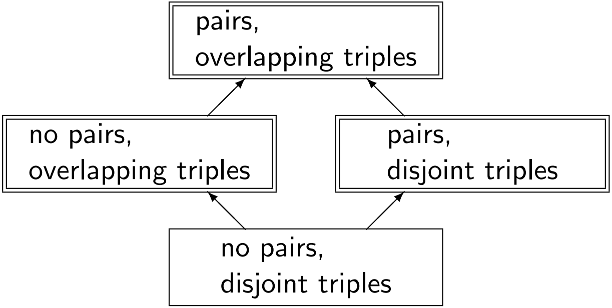

From triples to k-tuples

For $k \in \mathbb{N}^+$ and $P \subseteq \mathcal{S}_k$ the problem (k, P)-CO is:

- INSTANCE: Finite set A, collection C of k-tuples $(t_1, \ldots, t_k)$ of distinct elements from A.
- QUESTION: Is there a one-to-one function $f : A \rightarrow \{1, 2, \ldots, |A|\}$ such that for each $(t_1, \ldots, t_k) \in C$, there is a $p \in P$ with $f(t_{p(i)}) < f(t_{p(j)})$ for all $1 \leq i < j \leq k$?

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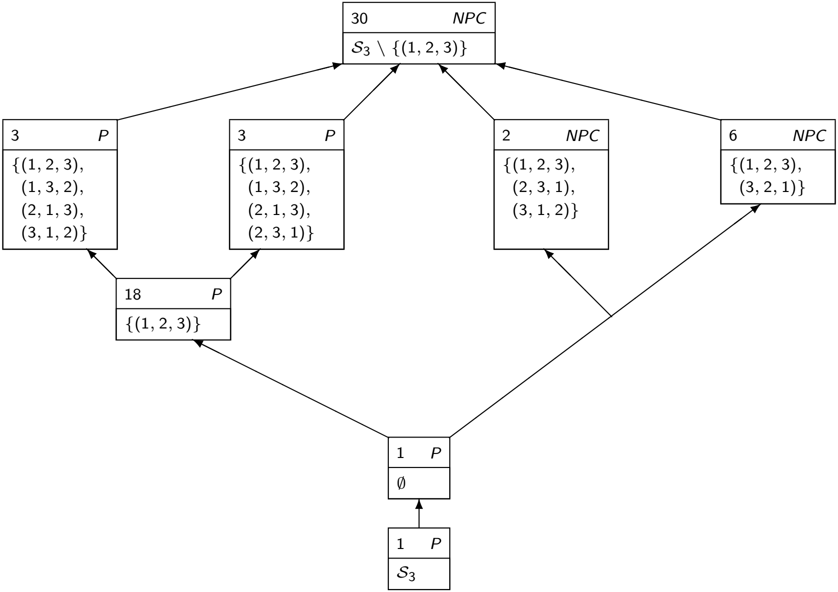
$P = \{1234, 4321\}$ specifies the generalisation of the problem Betweenness to quadruples.

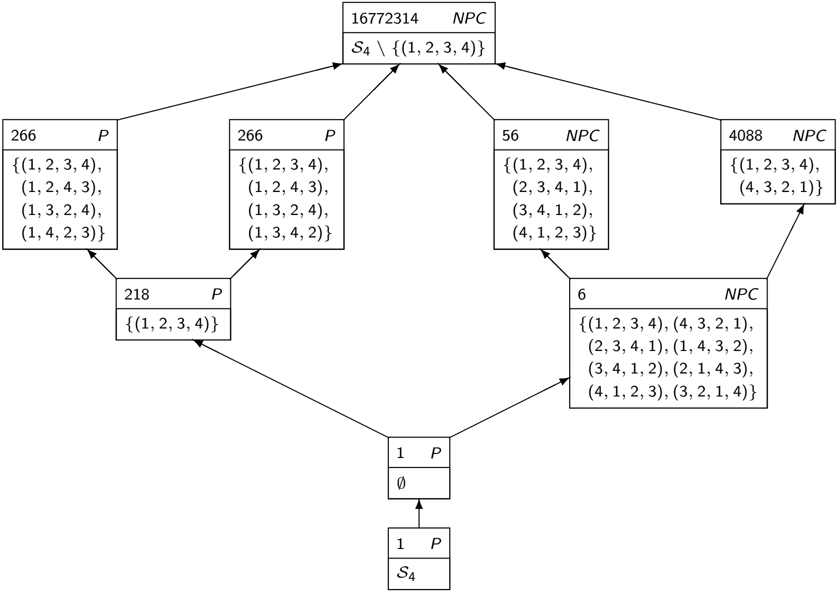
Example 4

$P = \{1234, 2341, 3412, 4123\}$ specifies the problem Cyclic Ordering generalised to quadruples.

Class structure among $(3, \cdot)$-CO



Class structure among $(4, \cdot)$-CO



Conclusion

- for $k \leq 4$, clear “NP-complete” vs. “linear time solvable” dichotomy
- similar class structures for $k = 3$ and $k = 4$
- one new class for $k = 4$

Open problems:

- Can we decide if a given problem A can be expressed in terms of another problem B?
- class structure for $(k, \cdot)$-CO with $k \geq 5$
- weak order instead of strict order