Reversible Pebble Games and the Relation Between Tree-Like and General Resolution Space

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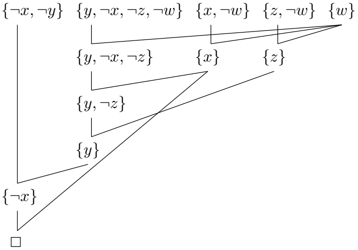
Universität Ulm

Dagstuhl Seminar „*SAT and Interactions*“
February 6, 2020

Just to Check We Are on the Same Page...

Proof π



- The proof system Resolution has only one derivation rule:

$$\frac{B \vee x \quad C \vee \overline{x}}{B \vee C}$$

- Complexity Measures:
  - Length of π = # of clauses in π
  - Clause Space of $\pi = \max \#$ of clauses in memory simultaneously during π
  - $\mu(F \vdash \Box) := \min_{\pi : F \vdash \Box} \mu(\pi)$.
  - Prefix "Tree-" before a complexity measure indicates tree-like resolution.

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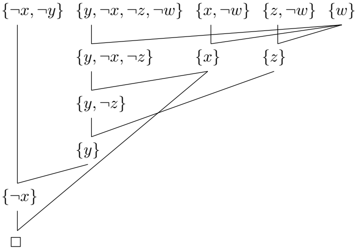

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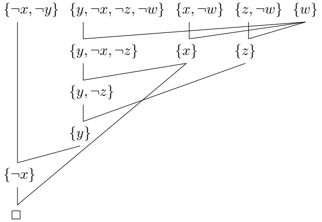
Tree-like vs. General Resolution Refutations

If a clause is needed more than once in a refutation, it has to be rederived each time.



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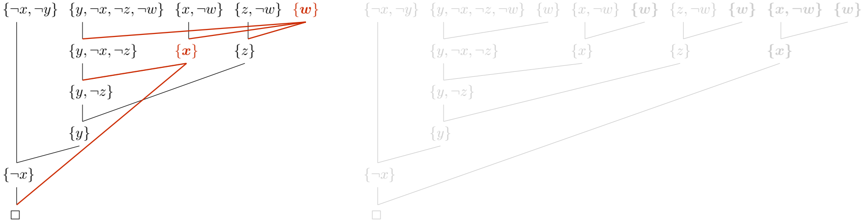
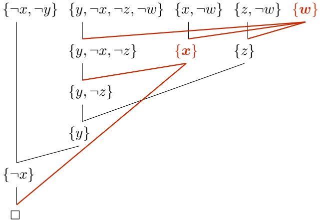
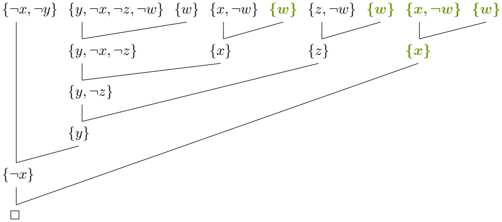


Tree-like vs. General Resolution Refutations

If a clause is needed more than once in a refutation, it has to be rederived each time. ($\rightarrow$ refutation DAG G_π becomes a tree)





After We've Set the Stage: Motivation of This Talk

Thanks to [Ben-Sasson, Impagliazzo, Wigderson '04: Near optimal separation...] we know an almost optimal separation between general and tree-like resolution w. r. t. length:

$\exists$ a family $(F_n)_{n\in\mathbb{N}}$ of unsatsfiable formulas in $\mathrm{O}(n)$ variables with

- resolution refutations of length L (linear in n),
- **but** any tree-like resolution refutation requires length $\exp\left(\Omega(\frac{L}{\log L})\right)$.

Matching upper bound of $\exp\left(\mathrm{O}\left(\frac{L\log\log L}{\log L}\right)\right)$ for tree-like resolution length of any formula that can be refuted in length L by general resolution.

¿What can we say about space?

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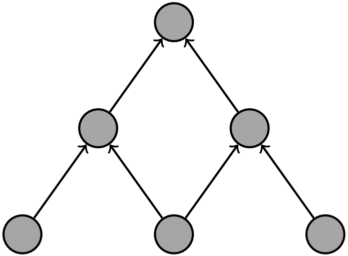
Part I

Separations for Pebbling Formulas

Pebble Games (games played on graphs)

The Black Pebble Game

Goal: Get a single black pebble on the sink of the graph.



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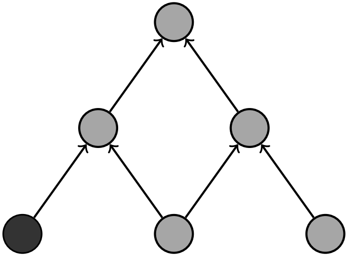

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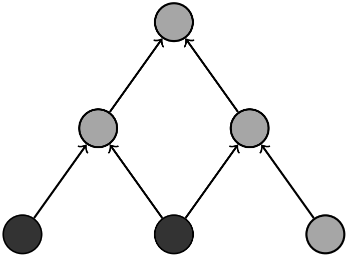

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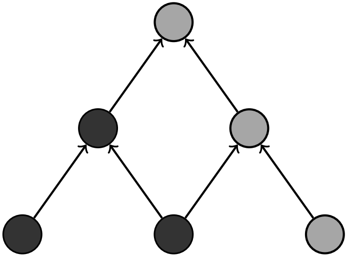

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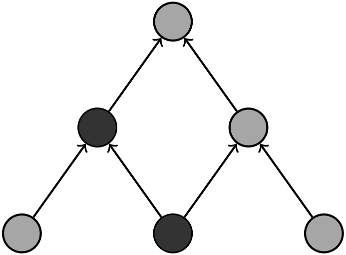

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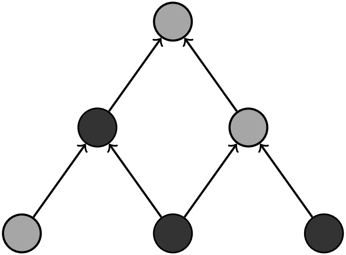

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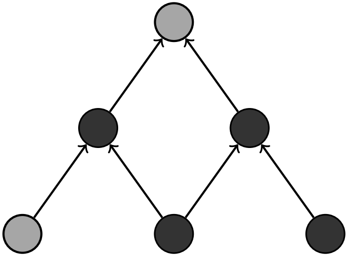

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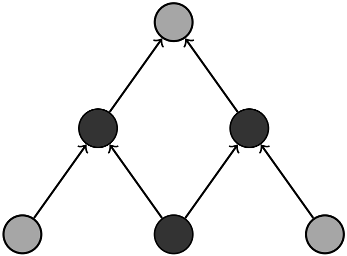

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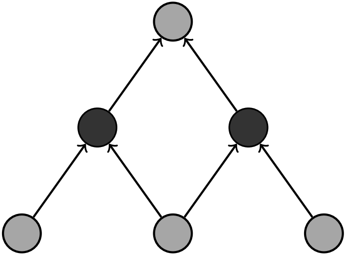

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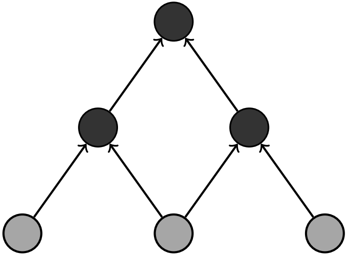

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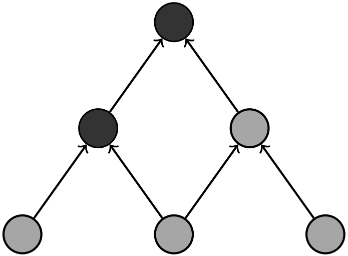

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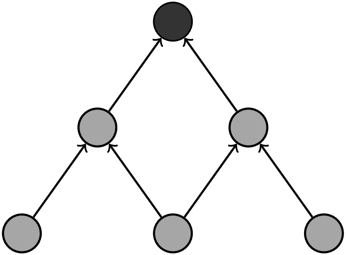

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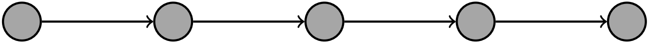

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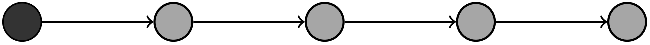

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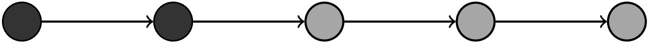

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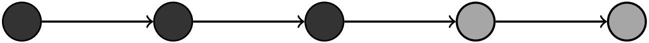

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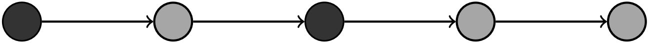

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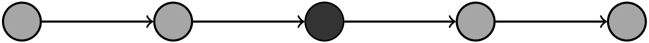

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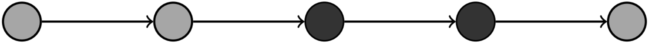

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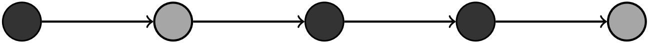

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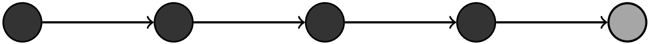

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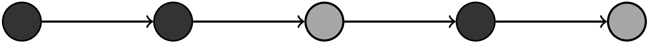

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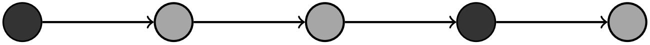

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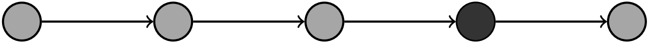

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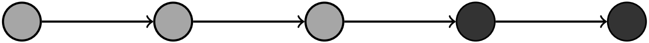

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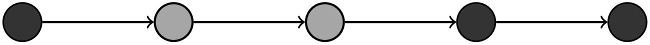

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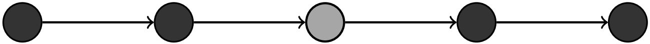

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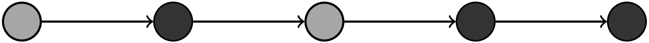

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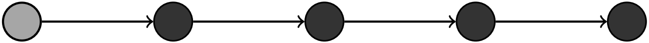

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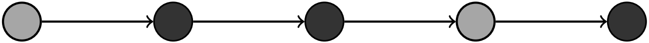

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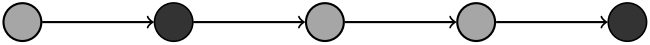

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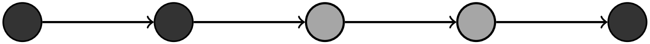

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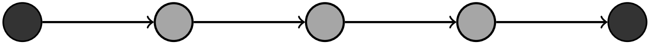

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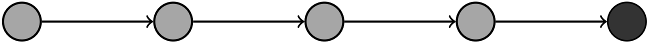

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Complexity Measures for the Pebble Games

$$\text{Black}(G) := \min_{\text{black pebblings } \mathcal{P}} \Big(\max \ \# \text{ of pebbles used at any point in } \mathcal{P} \Big)$$

$$\text{Rev}(G) := \min_{\text{rev. pebblings } \mathcal{P}} \Big(\max \ \# \text{ of pebbles used at any point in } \mathcal{P} \Big)$$

Why even care about these pebbling prices?

$\rightsquigarrow$ Plethora connections to resolution! [Nordström '15: New Wine . . .]

$$\mathrm{CS}(F \vdash \Box) = \min_{\pi : F \vdash \Box} \text{Black}(G_\pi)$$ [Esteban, Torán '01: Space bdds. for res.]

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Reversible Pebbling meets Tree-CS in the Special Case of Pebbling Formulas

Pebbling Formula

Clauses of Peb_G:

u

v

w

$(u \land v) \rightarrow x = \overline{u} \lor \overline{v} \lor x$

$(v \land w) \rightarrow y = \overline{v} \lor \overline{w} \lor y$

$(x \land y) \rightarrow z = \overline{x} \lor \overline{y} \lor z$

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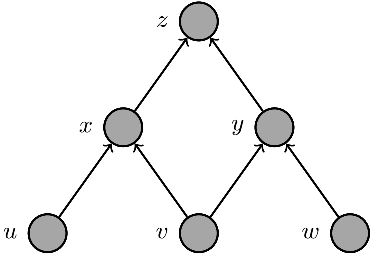

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- source vertices are true
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- but the sink vertex is false

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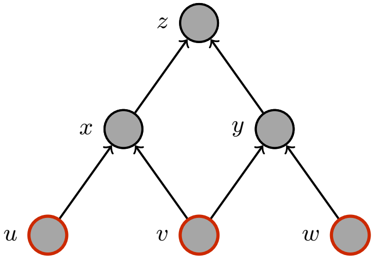

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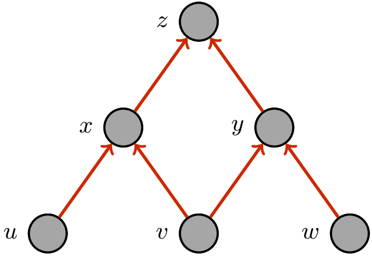

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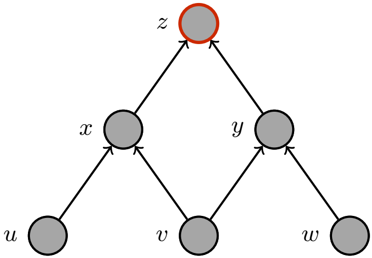

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Reversible Pebbling meets Tree-CS

Theorem

For all DAGs G with a unique sink:

$$\mathsf{Rev}(G) + 2 \leq \text{Tree-CS}\left(\text{Peb}_G[\oplus_2] \vdash \Box\right) \leq 2 \cdot \mathsf{Rev}(G) + 2.$$

Obtaining Space-Separations with Pebble games (1/3)

Idea:

- $\mathrm{CS}(\mathrm{Peb}_G[\oplus_2] \vdash \Box) = \mathrm{O}\big(\mathsf{Black}(G)\big)$
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$\Longrightarrow$ Construct a graph family with a gap between its black and reversible pebbling price

Example: Path graphs P_n of length n

- $\mathsf{Black}(P_n) = \mathrm{O}(1)\ \forall n \in \mathbb{N}$
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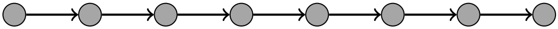

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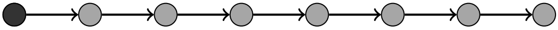

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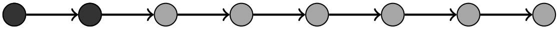

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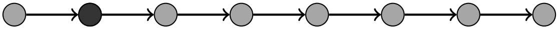

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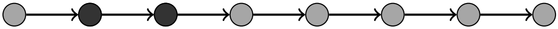

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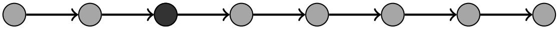

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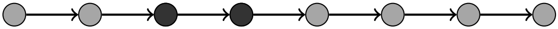

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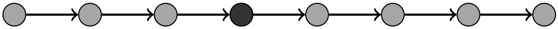

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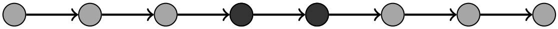

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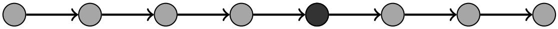

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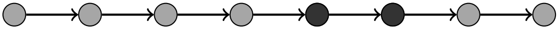

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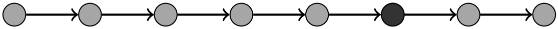

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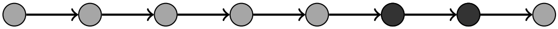

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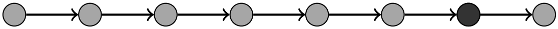

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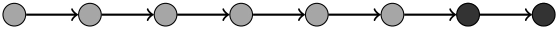

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  [Bennett '89: Time/space trade-offs for reversible computation; Li, Vitányi '96: Reversibility and adiabatic computation: Trading time and space for energy]

Obtaining Space-Separations with Pebble games (1/3)

Idea:

- $\mathrm{CS}(\mathrm{Peb}_G[\oplus_2] \vdash \Box) = \mathrm{O}\big(\mathsf{Black}(G)\big)$
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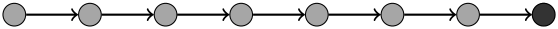
$\Longrightarrow$ Construct a graph family with a gap between its black and reversible pebbling price

Example: Path graphs P_n of length n



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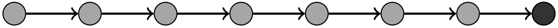

- $\mathsf{Black}(P_n) = \mathrm{O}(1)\ \forall n \in \mathbb{N}$ $\quad\exists$ Results for non-const. space?
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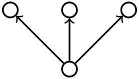
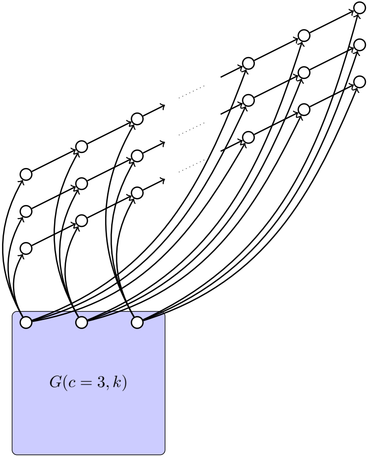
Obtaining Space-Separations with Pebble games (2/3)

Non-constant black pebbling number and Black-Rev-separation:





Obtaining Space-Separations with Pebble games (3/3)

Conclusion: The best known separation

For any "slowly enough" growing space function $s(n)$ there is a family of pebbling formulas $\left(\mathrm{Peb}_{G_n}[\oplus_2]\right)_{n=1}^{\infty}$ with $\Theta(n)$ variables such that

- $\mathrm{CS}(\mathrm{Peb}_{G_n}[\oplus_2] \vdash \Box) = \mathrm{O}\big(s(n)\big)$
- $\text{Tree-CS}(\mathrm{Peb}_{G_n}[\oplus_2] \vdash \Box) = \Omega\big(s(n)\log n\big)$.

¿Can we do any better?

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Part II

Upper Bounds for Tree-CS for General Formulas

An upper bound for Tree-CS

How large can the gap between CS and Tree-CS grow?

Theorem

For any unsatisfiable formula F with n variables it holds (2nd ineq. is tight)

$$\text{Tree-CS}(F \vdash \Box) \leq \min_{\pi : F \vdash \Box} \mathsf{Rev}(G_\pi) + 2, \quad \text{and}$$

$$\min_{\pi : F \vdash \Box} \mathsf{Rev}(G_\pi) \leq \text{Tree-CS}(F \vdash \Box)\Big(\lceil \log n \rceil + 1\Big).$$

Note, that the minimum in the theorem is taken over all possible refutations of F, not only over the tree-like ones.

Recall: $\boxed{\text{CS}(F \vdash \Box) = \min_{\pi : F \vdash \Box} \mathsf{Black}(G_\pi)}$ [ET01] $\rightsquigarrow$ Similarity to Thm!

We will now prove the inequality of the Theorem...

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A combinatorial characterization of Tree-CS (by a game played on formulas)

The Prover-Delayer Game

[Pudlák, Impagliazzo '00: A lower bound for DLL algorithms for k-SAT]

Given: An unsatisfiable CNF formula F

Two players take rounds... until Game Over...

Score of Delayer = # of $*$'s

Prover	**Delayer**
• Wants to falisify $C \in F$ (then Game Over)	
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A Combinatorial Characterisation for $\mathrm{Tree\text{-}CS}$

Definition (Game value of the Prover-Delayer game)

Let F be an unsatisfiable CNF formula.

$\mathsf{PD}(F) := \max$ pts. of Delayer on F against optimal strategy of Prover.

Theorem ([Esteban, Torán '03: A combinatorial char. of treelike res. space])

Let F be an unsatisfiable CNF formula. Then

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The equivalence of Rev and R-Mc

Reversible pebbling is hard to analyse

Raz–McKenzie Game to the help

Given: A single sink DAG G

Two players take rounds... until Game Over...

Pebbler	**Colourer**
• Places pebble on sink	
	• Colours it with red $\widehat{=} 0$
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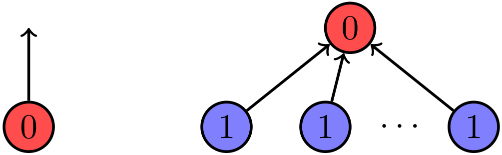

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R-Mc(G) := smallest r s. th. Pebbler wins in $\leq r$ rounds
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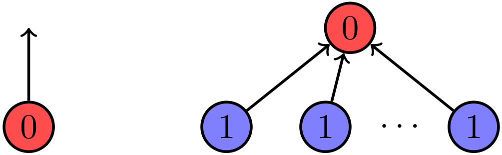

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$\mathsf{Rev}(G) = \mathsf{R\text{-}Mc}(G)$

Theorem ([Chan '13: Just a pebble game])

For any single-sink DAG G:

$$\mathsf{Rev}(G) = \mathsf{R\text{-}Mc}(G).$$

The Actual Proof

An upper bound for Tree-CS

Proof sketch of $\mathrm{Tree\text{-}CS}(F \vdash \Box) \leq \min_{\pi:F\vdash\Box} \mathsf{Rev}(G_\pi) + 2$

Given: a res. refutation π of F with a ref.-graph G_π and $\mathsf{Rev}(G_\pi) =: k$.

AIM: Give a strategy for Prover in the PD-game under which he has to pay at most k points.

Idea: Simulate the strategy of Pebbler in the Raz–McKenzie game
$\rightarrow$ a falsifying part. assignment α of init. clause will be produced

Stages of the game: Pebbler chooses $C \longrightarrow$ Prover queries vars. in C not yet assigned by α (& extends with Delayer's answers) until either

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After at most k stages the Raz–McKenzie game finished
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Only left to show: At the end of the game a clause of F is fals. by α.

When Raz–McKenzie finishes:

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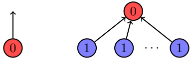

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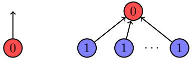


An upper bound for Tree-CS

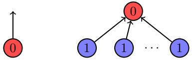
Proof sketch of $\text{Tree-CS}(F \vdash \Box) \leq \min_{\pi : F \vdash \Box} \mathsf{Rev}(G_\pi) + 2$

After at most k stages the Raz–McKenzie game finished
$\Rightarrow$ Delayer can score at most k points.

Only left to show: At the end of the game a clause of F is fals. by α.

When Raz–McKenzie finishes:

1. either a source vertex in G_π is assigned colour 0 by Colourer,
   $\rightarrow$ since α defines Colourer's answer: α fals. a clause in F.
2. or a vertex with all its direct predecessors being coloured 1 is coloured 0.
   $\rightarrow$ not possible, since no α can sat'y two parent clauses in a resolution proof, while falsifying their resolvent! $\Box$



An upper bound for Tree-CS in terms of CS^*

[Razborov '18: On space and depth in resolution] introduced amortised clause space:

$$\mathrm{CS}^*(F \vdash \Box) := \min_{\pi : F \vdash \Box} \big(\mathrm{CS}(\pi) \cdot \log \mathrm{L}(\pi)\big)$$

Corollary

$\text{Tree-CS}(F \vdash \Box) \leq \mathrm{CS}^*(F \vdash \Box) + 2.$

Proof.

- [Královič '04: Time and Space Complexity of Reversible Pebbling] $\mathsf{Rev}(G_\pi) + 2 \leq \min_{\mathcal{P}} \big(\mathsf{space}(\mathcal{P}) \cdot \log \mathsf{time}(\mathcal{P})\big) + 2$, where the minimum is taken over all black pebblings $\mathcal{P}$ of G_π.
- Every black pebbling $\mathcal{P}$ of G_π defines a configurational refutation of F with clause space equal to $\mathsf{space}(\mathcal{P})$ and length $\mathsf{time}(\mathcal{P})$. □

An upper bound for $\mathrm{Tree\text{-}CS}$ in terms of CS^*

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Part III

The Tseitin Formula Case

Better Bounds for Tseitin Formulas

Tseitin Formulas: "Sum of degrees of vertices in a graph is even"

A quick Recap: We have just seen

$$\text{Tree-CS}(F \vdash \Box) \lesssim \min_{\pi : F \vdash \Box} \Big(\text{CS}(\pi) \cdot \boxed{\log \text{L}(\pi)}\Big).$$

Theorem: Matching Upper and Lower Bounds for Tseitin Formulas

- For any connected graph G with n vertices and odd marking χ:

$$\text{Tree-CS}\big(\text{Ts}(G,\chi) \vdash \Box\big) \lesssim \text{CS}\big(\text{Ts}(G,\chi) \vdash \Box\big) \cdot \boxed{\log n}.$$

- $\exists$ a family of Tseitin formulas $\big(\text{Ts}(G_n,\chi_n)\big)_{n=1}^{\infty}$ s. th. $\forall n \in \mathbb{N}$:

$$\text{Tree-CS}\big(\text{Ts}(G_n,\chi_n) \vdash \Box\big) = \Omega\Big(\text{CS}\big(\text{Ts}(G_n,\chi_n) \vdash \Box\big) \cdot \log n\Big).$$

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Open Questions and Conjectures

Is it possible to improve

$$\text{Tree-CS}(F \vdash \Box) \lesssim \min_{\pi : F \vdash \Box} \Big(\mathrm{CS}(\pi) \cdot \log \mathrm{L}(\pi) \Big).$$

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where $n = \#$ vertices / formula size?

Conjecture / Strong gut feeling

Yes (and we know: this is the only room for improvement).

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Related Question in the World of Pebbling

Can [Královič]'s bound

$$\mathsf{Rev}(G) \leq \min_{\mathcal{P}} \Big(\mathsf{space}(\mathcal{P}) \cdot \log \mathsf{time}(\mathcal{P})\Big)$$

be improved to

$$\mathsf{Rev}(G) \leq \min_{\mathcal{P}} \Big(\mathsf{space}(\mathcal{P}) \cdot \log |V(G)|\Big) \quad ?$$

Take-Home Message

$\mathrm{Tree\text{-}CS}$ *and* CS *are different measures but "not too far" from one another*

- $\mathrm{Tree\text{-}CS}\big(\mathrm{Peb}_G[\oplus_2] \vdash \Box\big) \simeq \mathsf{Rev}(G)$
- Separations between $\mathrm{Tree\text{-}CS}$ and CS by graphs G exhibiting separation between $\mathsf{Rev}(G)$ and $\mathsf{Black}(G)$
- $\mathrm{Tree\text{-}CS}(F \vdash \Box) \lesssim \min\limits_{\pi : F \vdash \Box} \mathsf{Rev}(G_\pi)$
- $\mathrm{Tree\text{-}CS}(F \vdash \Box) \lesssim \mathrm{CS}^*(F \vdash \Box)$ for general F

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(*) Some open questions hidden here. We've solved these for Tseitin formulas.

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Thank you for your attention!