Reversible Pebble Games and the Relation Between Tree-Like and General Resolution Space

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Universität Ulm

STACS 2020, Montpellier
Session B3 – March 11, 2020

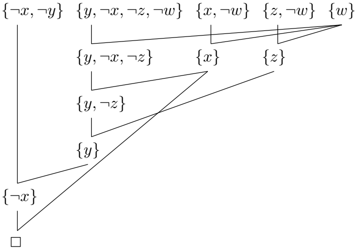
Reversible Pebble Games and the Relation Between Tree-Like and General Resolution Space

Extended Version
Partly on the Blackboard

Part of the *Komplex* Series at Universität Ulm
March 5, 2020

Just to Check We Are on the Same Page...

Proof π



- The proof system resolution has only one derivation rule:

$$\frac{B \vee x \quad C \vee \overline{x}}{B \vee C}$$

- Complexity measures:
  - Length of π = # of clauses in π
  - Clause Space of $\pi = \max \#$ of clauses in memory simultaneously during π
  - $\mu(F \vdash \Box) := \min_{\pi : F \vdash \Box} \mu(\pi)$.
  - Prefix "Tree-" before a complexity measure indicates tree-like resolution.

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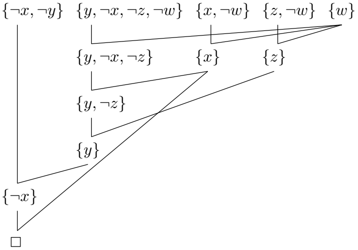

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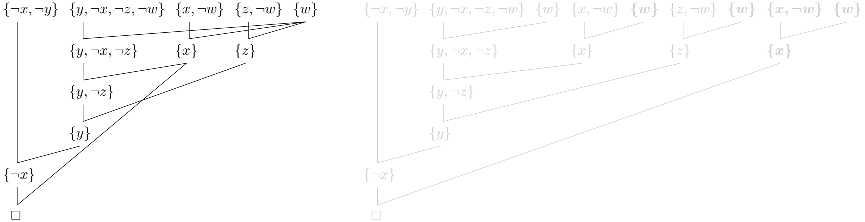
Tree-like vs. General Resolution Refutations

If a clause is needed more than once in a refutation, it has to be rederived each time.



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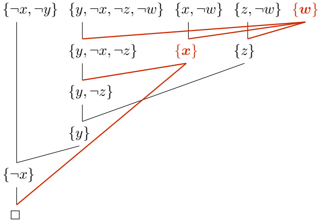
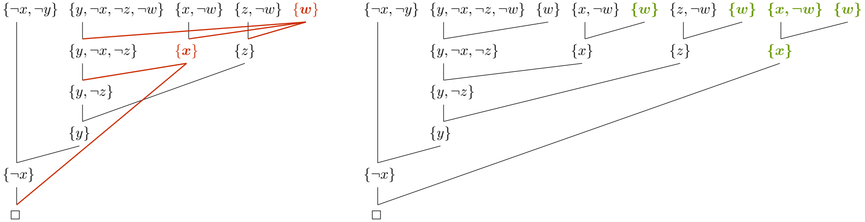


Tree-like vs. General Resolution Refutations

If a clause is needed more than once in a refutation, it has to be rederived each time. ($\rightarrow$ refutation DAG G_π becomes a tree)



A More Formal Definition & Example on the Blackboard

A resolution refutation of an unsatisfiable CNF formula F is an ordered sequence of memory configurations (sets of clauses)

$$\pi = (\mathbb{M}_0, \ldots, \mathbb{M}_t),$$

s. th. $\mathbb{M}_0 = \varnothing$, $\Box \in \mathbb{M}_t$ and for each $i \in [t]$, the configuration $\mathbb{M}_i$ is obtained from $\mathbb{M}_{i-1}$ by applying exactly one of the following rules:

- **Axiom Download:** $\mathbb{M}_i = \mathbb{M}_{i-1} \cup \{C\}$ for some axiom $C \in F$.
- **Erasure:** $\mathbb{M}_i = \mathbb{M}_{i-1} \setminus \{C\}$ for some $C \in \mathbb{M}_{i-1}$.
- **Inference:**

  $$\mathbb{M}_i = \mathbb{M}_{i-1} \cup \{D\}$$

  for some resolvent D inferred from $C_1, C_2 \in \mathbb{M}_i$ by the resolution rule.

A More Formal Definition & Example on the Blackboard

A tree-like resolution refutation of an unsatisfiable CNF formula F is an ordered sequence of memory configurations (sets of clauses)

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- **Tree-like Inference:** [Esteban, Torán]

  $$\mathbb{M}_i = \big(\mathbb{M}_{i-1} \cup \{D\}\big) \setminus \{C_1, C_2\}$$

  for some resolvent D inferred from $C_1, C_2 \in \mathbb{M}_i$ by the resolution rule, i. e., we delete both parent clauses immediately.

A More Formal Definition & Example on the Blackboard

...

After We've Set the Stage: Motivation of This Talk

Thanks to [Ben-Sasson, Impagliazzo, Wigderson '04: Near optimal separation...] we know an almost optimal separation between general and tree-like resolution w. r. t. length:

$\exists$ a family $(F_n)_{n\in\mathbb{N}}$ of unsatisfiable formulas in $\mathrm{O}(n)$ variables with

- resolution refutations of length L (linear in n),
- **but** any tree-like resolution refutation requires length $\exp\big(\Omega(\frac{L}{\log L})\big)$.

Matching upper bound of $\exp\big(\mathrm{O}\big(\frac{L \log\log L}{\log L}\big)\big)$ for tree-like resolution length of any formula that can be refuted in length L by general resolution.

¿What can we say about space?

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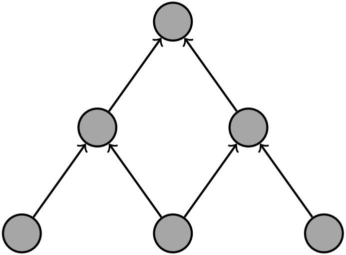
Part I

Separations for Pebbling Formulas

Pebble Games (games played on graphs)

The Black Pebble Game

Goal: Get a single black pebble on the sink of the graph.



$\max$ # of pebbles used at any point:

- **Pebble Placement:** On empty vertex if all direct predecessors have a pebble (in particular: can always pebble sources)
- **Pebble Removal:** At any time

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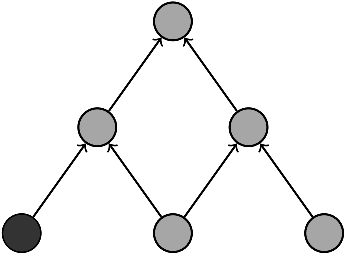

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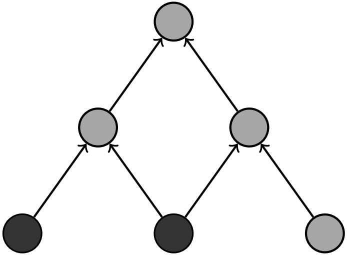

max # of pebbles
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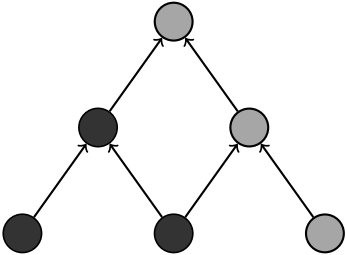

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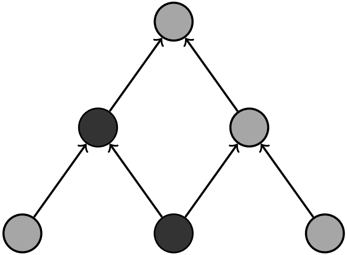

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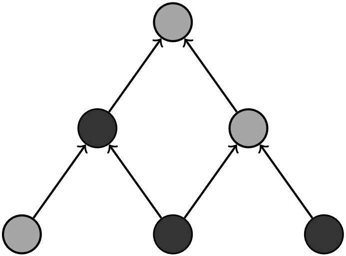

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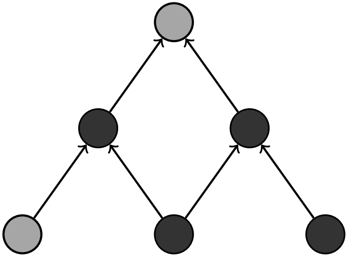

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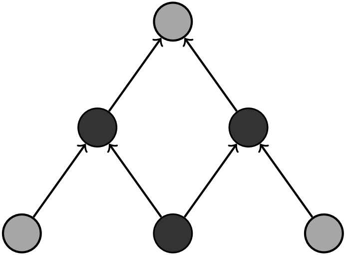

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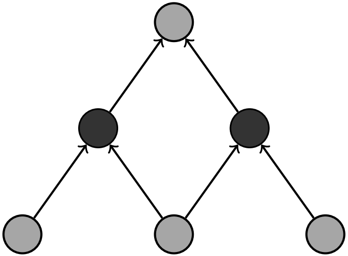

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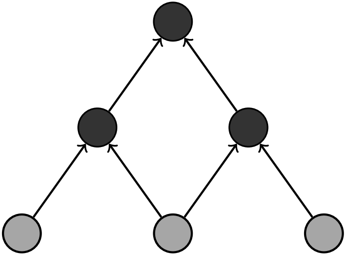

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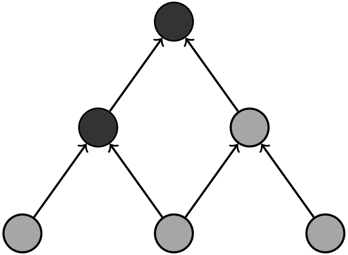

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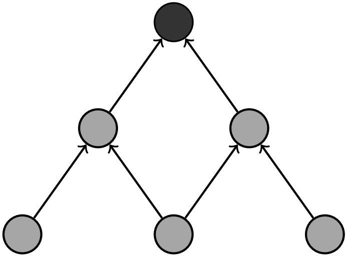

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The Reversible Pebble Game

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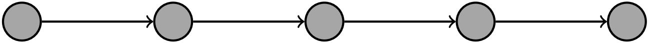
Same measure: $\max$ # of pebbles used at any point:



Different rules:

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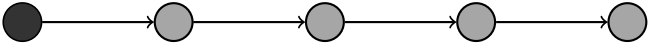

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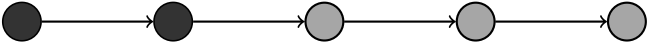

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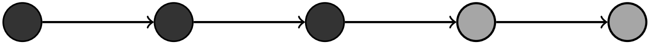

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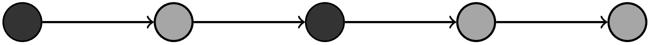

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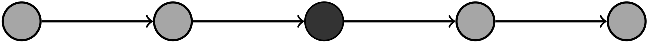

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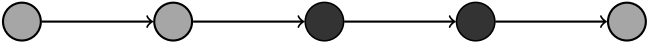

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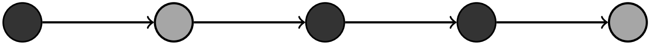

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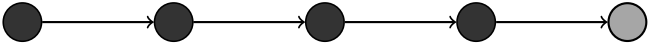

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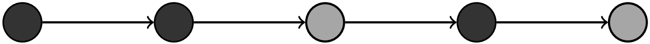

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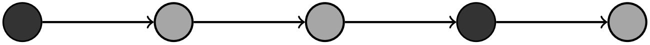

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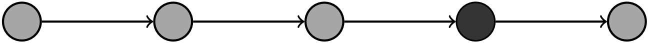

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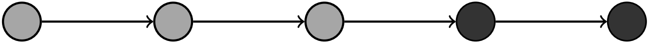

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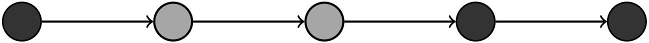

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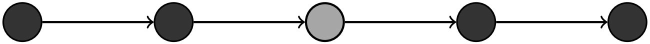

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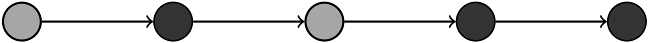
Same measure: $\max$ # of pebbles used at any point: IIII



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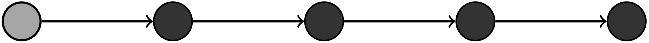

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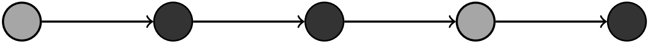

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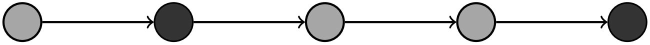

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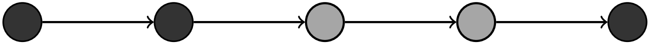

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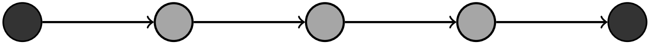

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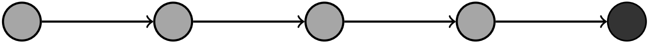

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Complexity Measures for the Pebble Games

$$\text{Black}(G) := \min_{\text{black pebblings } \mathcal{P}} \Big(\max \ \# \text{ of pebbles used at any point in } \mathcal{P} \Big)$$

$$\text{Rev}(G) := \min_{\text{rev. pebblings } \mathcal{P}} \Big(\underbrace{\max \ \# \text{ of pebbles used at any point in } \mathcal{P}}_{=:\text{space}(\mathcal{P})} \Big)$$

$$\text{time}(\mathcal{P}) := \ \# \text{ of moves in } \mathcal{P}$$

Why even care about these pebbling prices?

$\rightsquigarrow$ Plethora connections to resolution! [Nordström '15: New Wine ...]

$$\text{CS}(F \vdash \Box) = \min_{\pi : F \vdash \Box} \text{Black}(G_\pi)$$

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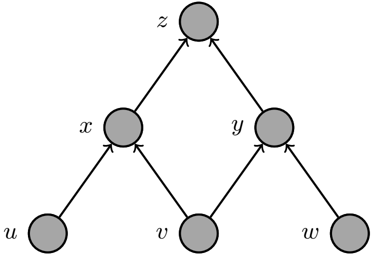
Reversible Pebbling meets Tree-CS in the Special Case of Pebbling Formulas

Pebbling Formula

Clauses of Peb_G:

u
v
w
$(u \land v) \to x = \overline{u} \lor \overline{v} \lor x$
$(v \land w) \to y = \overline{v} \lor \overline{w} \lor y$
$(x \land y) \to z = \overline{x} \lor \overline{y} \lor z$
$\overline{z}$



Encode the rules of the black pebble game in a formula (i. e., formula is defined over an underlying DAG G):

- source vertices are true
- truth propagates upwards
- but the sink vertex is false

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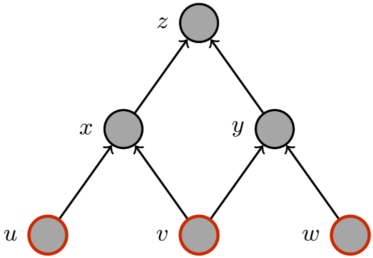

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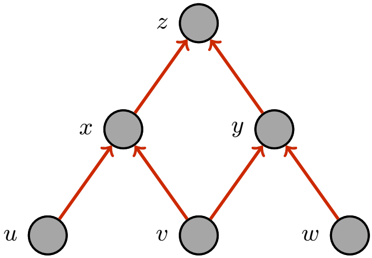

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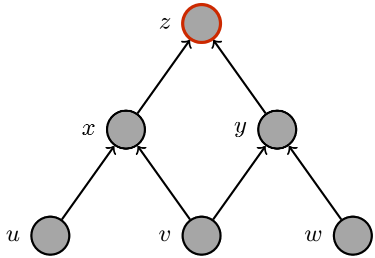

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XORification $\oplus_2$

Make formulas slightly harder to refute

Problem: Pebbling formulas are super easy to refute (in linear length and linear space simultaneously by simple unit propagation).

Solution: Substitute each variable x with e. g. $x_1 \oplus x_2$ and expand result into CNF.

$$\overline{x} \vee y$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$$

$$\Downarrow$$

$$\begin{aligned} &(x_1 \vee \overline{x_2} \vee y_1 \vee y_2) \\ \wedge&(x_1 \vee \overline{x_2} \vee \overline{y_1} \vee \overline{y_2}) \\ \wedge&(\overline{x_1} \vee x_2 \vee y_1 \vee y_2) \\ \wedge&(\overline{x_1} \vee x_2 \vee \overline{y_1} \vee \overline{y_2}) \end{aligned}$$

Reversible Pebbling meets Tree-CS

Theorem

For all DAGs G with a unique sink:

$$\mathsf{Rev}(G) + 2 \leq \mathrm{Tree\text{-}CS}\big(\mathrm{Peb}_G[\oplus_2] \vdash \Box\big) \leq 2 \cdot \mathsf{Rev}(G) + 2.$$

Obtaining Space-Separations with Pebble games (1/5)

Idea:

- $\mathrm{CS}(\mathrm{Peb}_G[\oplus_2] \vdash \Box) = \mathrm{O}\big(\mathsf{Black}(G)\big)$
- $\mathrm{Tree\text{-}CS}\big(\mathrm{Peb}_G[\oplus_2] \vdash \Box\big) = \Omega\big(\mathsf{Rev}(G)\big)$

$\Longrightarrow$ Construct a graph family with a gap between its black and reversible pebbling price

Example: Path graphs P_n of length n

- $\mathsf{Black}(P_n) = \mathrm{O}(1)\ \forall n \in \mathbb{N}$
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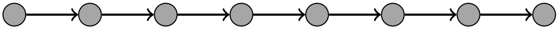

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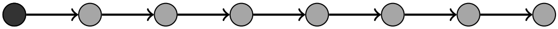

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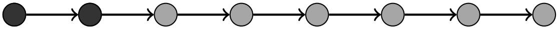

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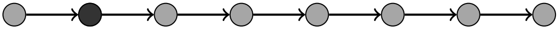

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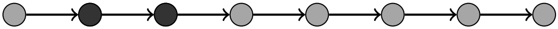

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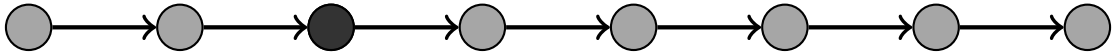

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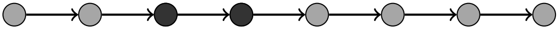

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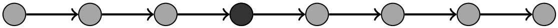

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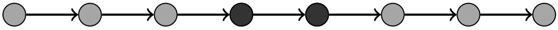

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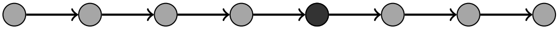

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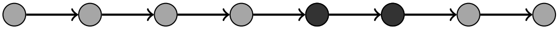

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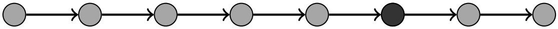

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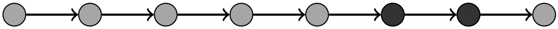

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Obtaining Space-Separations with Pebble games (1/5)

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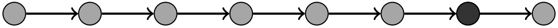

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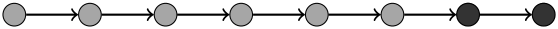

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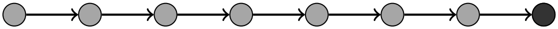

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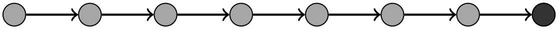

- $\mathsf{Black}(P_n) = \mathrm{O}(1)\ \forall n \in \mathbb{N}$ $\quad\exists$ Results for non-const. space?
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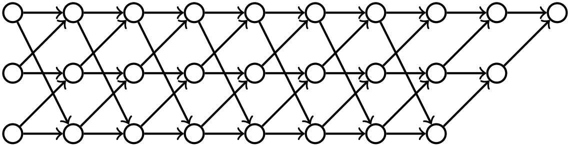
Obtaining Space-Separations with Pebble games (2/5)

Yes! Take the *road graphs*...



- Black$(R_{w,\ell}) = w + 2 \quad \forall w \geq 2$
- Rev$(R_{w,\ell}) = \Omega\big(w \log(\ell/w)\big)$ [Chan, Lauria, Nordström, Vinyals '15]

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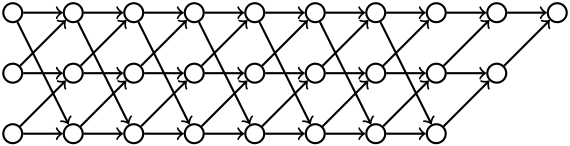

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... **but:** again only $\log$-factor-separation!

Obtaining Space-Separations with Pebble games (3/4)

Can we improve further?

Corollary

For any DAG G with a unique sink vertex it holds

$$\text{Tree-CS}(\text{Peb}_G[\oplus_2] \vdash \Box) = \text{O}\left(\min_{\mathcal{P}} \left(\mathsf{space}(\mathcal{P}) \cdot \log \mathsf{time}(\mathcal{P})\right)\right),$$

where the minimum is taken over all black pebblings $\mathcal{P}$ of G.

Only possible to improve the separation for non one-shot black pebbling strategies (since otherwise we have $\mathsf{time}(\mathcal{P}) = n$).

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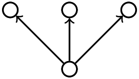
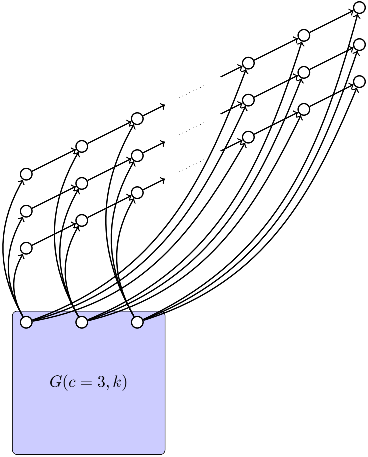
Obtaining Space-Separations with Pebble games (4/5)

Non-constant, non one-shot black pebbling number:





Obtaining Space-Separations with Pebble games (5/5)

Conclusion: The best known separation

For any "slowly enough" growing space function $s(n)$ there is a family of pebbling formulas $\left(\mathrm{Peb}_{G_n}[\oplus_2]\right)_{n=1}^{\infty}$ with $\Theta(n)$ variables such that

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¿Can we actually do any better?

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Part II

Upper Bounds for Tree-CS for General Formulas

An upper bound for Tree-CS

How large can the gap between CS and Tree-CS grow?

Theorem

For any unsatisfiable formula F with n variables it holds (2nd ineq. is tight)

$$\text{Tree-CS}(F \vdash \Box) \leq \min_{\pi : F \vdash \Box} \mathsf{Rev}(G_\pi) + 2, \quad \text{and}$$

$$\min_{\pi : F \vdash \Box} \mathsf{Rev}(G_\pi) \leq \text{Tree-CS}(F \vdash \Box)\Big(\lceil \log n \rceil + 1\Big).$$

Note, that the minimum in the theorem is taken over all possible refutations of F, not only over the tree-like ones.

Recall: $\text{CS}(F \vdash \Box) = \min_{\pi : F \vdash \Box} \mathsf{Black}(G_\pi)$ [ET01] $\rightsquigarrow$ Similarity to Thm!

We will now prove the inequality of the Theorem...

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A combinatorial characterization of Tree-CS (by a game played on formulas)

The Prover-Delayer Game

[Pudlák, Impagliazzo '00: A lower bound for DLL algorithms for k-SAT]

Given: An unsatisfiable CNF formula F

Two players take rounds... until Game Over...

Score of Delayer = # of $*$'s

Prover	**Delayer**
• Wants to falisify $C \in F$ (then Game Over)	
• Queries a variable x of F	
	• Answers – $x = 0$, – $x = 1$ or – $x = *$ ("you choose")
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A Combinatorial Characterisation for Tree-CS

Definition (Game value of the Prover-Delayer game)

Let F be an unsatisfiable CNF formula.

PD(F) $:= \max$ pts. of Delayer on F against optimal strategy of Prover.

Theorem ([Esteban, Torán '03: A combinatorial char. of treelike res. space])

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The equivalence of Rev and R-Mc

Reversible pebbling is hard to analyse

Raz–McKenzie Game to the help

Given: A single sink DAG G

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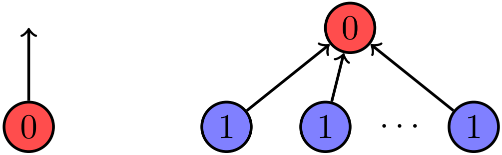

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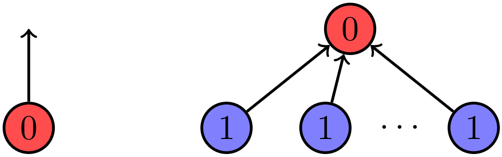

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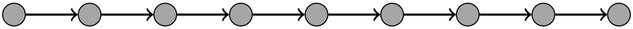

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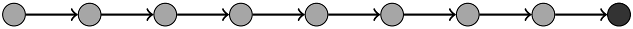

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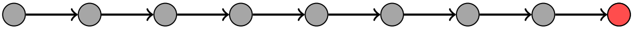

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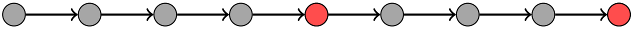

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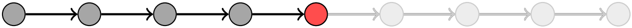

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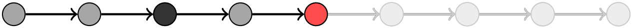

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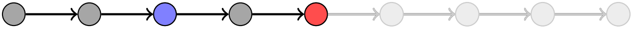

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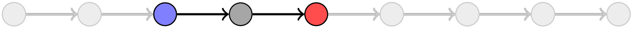
Example: Recall $\mathsf{Rev}(P_n) = \mathsf{R\text{-}Mc}(P_n) = \Theta(\log n)\ \forall n \in \mathbb{N}$



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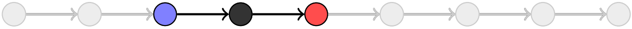

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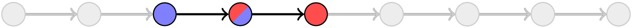

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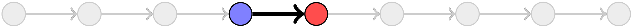

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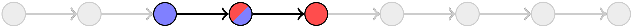

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The Actual Proof

An upper bound for Tree-CS

Proof sketch of $\mathrm{Tree\text{-}CS}(F \vdash \Box) \leq \min_{\pi : F \vdash \Box} \mathsf{Rev}(G_\pi) + 2$

Given: a res. refutation π of F with a ref.-graph G_π and $\mathsf{Rev}(G_\pi) =: k$.

AIM: Give a strategy for Prover in the PD-game under which he has to pay at most k points.
Idea: Simulate the strategy of Pebbler in the Raz–McKenzie game
$\rightarrow$ a falsifying part. assignment α of init. clause will be produced

Stages of the game: Pebbler chooses $C \longrightarrow$ Prover queries vars. in C not yet assigned by α (& extends with Delayer's answers) until either

1. the clause C is sat./fals. by α
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Only left to show: At the end of the game a clause of F is fals. by α.

When Raz–McKenzie finishes:

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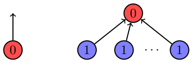

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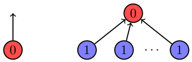

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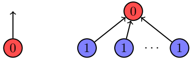

An upper bound for $\mathrm{Tree\text{-}CS}$ in terms of CS^*

[Razborov '18: On space and depth in resolution] introduced amortised clause space:

$$\mathrm{CS}^*(F \vdash \Box) := \min_{\pi : F \vdash \Box} \big(\mathrm{CS}(\pi) \cdot \log \mathrm{L}(\pi)\big)$$

Corollary

$\mathrm{Tree\text{-}CS}(F \vdash \Box) \leq \mathrm{CS}^*(F \vdash \Box) + 2.$

Proof.

- [Královič '04: Time and Space Complexity of Reversible Pebbling] $\mathrm{Rev}(G_\pi) + 2 \leq \min_{\mathcal{P}} \big(\mathrm{space}(\mathcal{P}) \cdot \log \mathrm{time}(\mathcal{P})\big) + 2$, where the minimum is taken over all black pebblings $\mathcal{P}$ of G_π.
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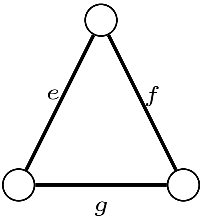
Part III

The Tseitin Formula Case

Tseitin Formulas

"Sum of degrees of vertices in a graph is even"



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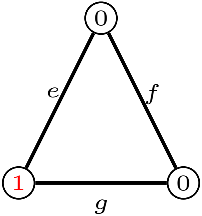

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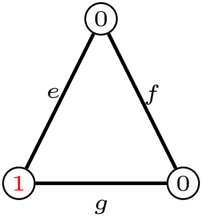

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$$e + f \equiv 0 \pmod 2$$
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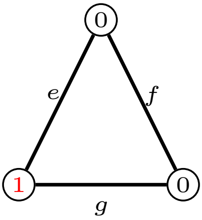

$$e + g \equiv 1 \pmod 2$$
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$$\mathrm{Ts}(G, \chi) := (e \vee f) \wedge (\overline{e} \vee \overline{f}) \wedge (e \vee \overline{f}) \wedge (\overline{e} \vee f) \wedge (f \vee \overline{f}) \wedge (\overline{f} \vee g)$$

Better Bounds for Tseitin Formulas

A quick recap: We have just seen

$$\text{Tree-CS}(F \vdash \Box) \lesssim \min_{\pi: F \vdash \Box} \Big(\text{CS}(\pi) \cdot \log \text{L}(\pi) \Big).$$

Theorem: Matching Upper and Lower Bounds for Tseitin Formulas

- For any connected graph G with n vertices and odd marking χ:

$$\text{Tree-CS}\big(\text{Ts}(G,\chi) \vdash \Box\big) \lesssim \text{CS}\big(\text{Ts}(G,\chi) \vdash \Box\big) \cdot \log n.$$

- $\exists$ a family of Tseitin formulas $\big(\text{Ts}(G_n,\chi_n)\big)_{n=1}^{\infty}$ s. th. $\forall n \in \mathbb{N}$:

$$\text{Tree-CS}\big(\text{Ts}(G_n,\chi_n) \vdash \Box\big) = \Omega\Big(\text{CS}\big(\text{Ts}(G_n,\chi_n) \vdash \Box\big) \cdot \log n\Big).$$

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Open Questions and Conjectures

Is it possible to improve

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where $n = \#$ vertices / formula size?

Conjecture / Strong gut feeling

Yes (and we know: this is the only room for improvement).

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Related Question in the World of Pebbling

Can [Královič]'s bound

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be improved to

$$\mathsf{Rev}(G) \leq \min_{\mathcal{P}} \Big(\mathsf{space}(\mathcal{P}) \cdot \log |V(G)| \Big) \quad ?$$

Take-Home Message

$\mathrm{Tree\text{-}CS}$ *and* CS *are different measures but "not too far" from one another*

- $\mathrm{Tree\text{-}CS}\big(\mathrm{Peb}_G[\oplus_2] \vdash \Box\big) \simeq \mathsf{Rev}(G)$
- Separations between $\mathrm{Tree\text{-}CS}$ and CS by graphs G exhibiting separation between $\mathsf{Rev}(G)$ and $\mathsf{Black}(G)$
- $\mathrm{Tree\text{-}CS}(F \vdash \Box) \lesssim \min_{\pi : F \vdash \Box} \mathsf{Rev}(G_\pi)$
- $\mathrm{Tree\text{-}CS}(F \vdash \Box) \lesssim \mathrm{CS}^*(F \vdash \Box)$ for general F

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Thank you for your attention!