Reversible Pebble Games and the Relation Between Tree-Like and General Resolution Space

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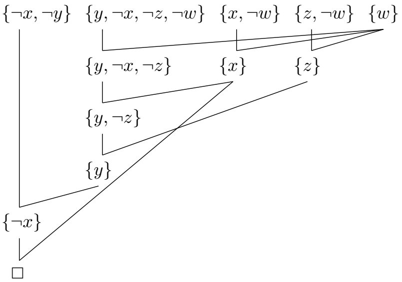
22nd October, 2019

Joint work with **Jacobo Torán**, Universität Ulm

Just to Check We Are on the Same Page...



- Resolution: most studied proof system for refuting CNF formulas
- only one derivation rule:

$$\frac{B \lor x \quad C \lor \overline{x}}{B \lor C}$$

- Length of π = # of clauses in π
- Clause Space of $\pi = \max$ # of clauses in memory simultaneously during π

- Tree-Res, if refutation DAG is a tree ($\rightarrow$ maybe need to rederive clauses)

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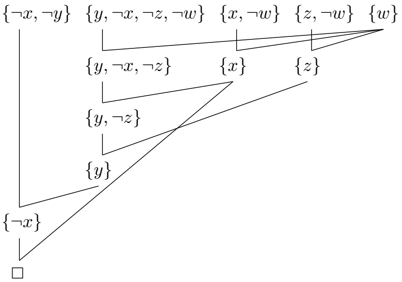

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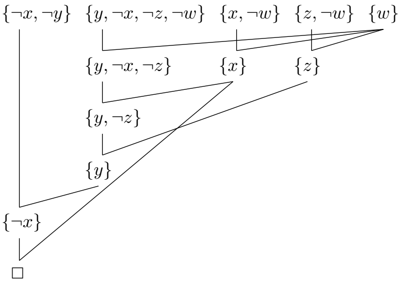

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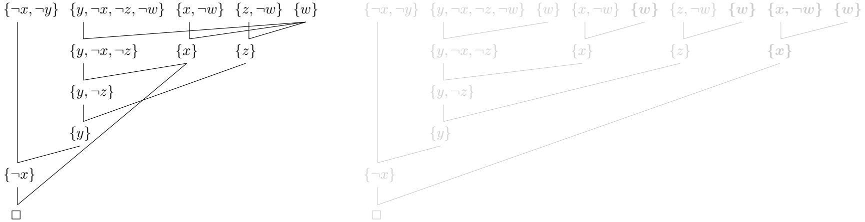
General vs. Tree-like Resolution Refutations

If a clause is needed more than once in a refutation, it has to be rederived each time.



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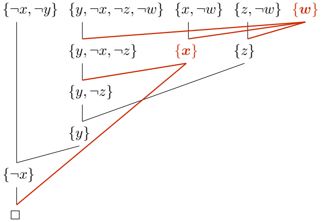

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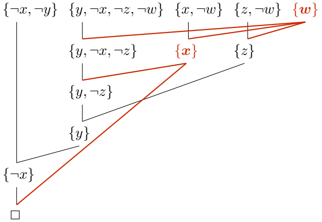

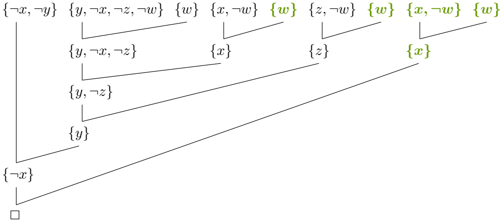

Complexity Measures for Resolution

For a complexity measure μ and a formula F

$$\mu(F \vdash \Box) := \min_{\pi : F \vdash \Box} \mu(\pi).$$

Prefix "Tree-" before a complexity measure indicates tree-like resolution.

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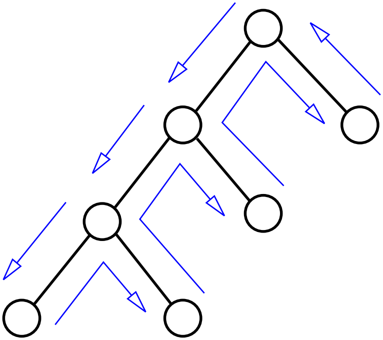
Why Care About these Measures for Resolution?



- After more than 50 years, DPLL is still the basis of most modern SAT Solvers (Chaff, zChaff, GRASP, MiniSAT, . . .).
- Tree-like resolultion and DPLL are p-equivalent proof systems.
- Experimental results (and even theoretical arguments): tree-space measures for resolution correlate well with the hardness of solving formulas with SAT solvers in practice.

[Järvisalo, Matsliah, Nordström, Živný '12: Relating Proof Complexity Measures and Practical Hardness of SAT]

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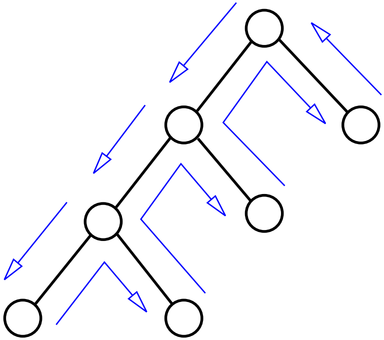

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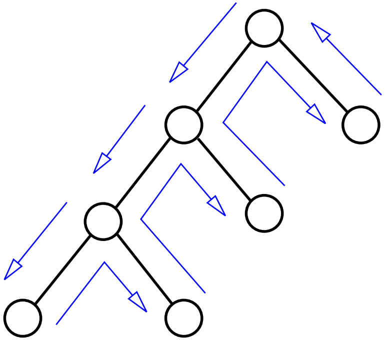

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After We've Set the Stage: Motivation of This Talk

Thanks to [Ben-Sasson, Impagliazzo, Wigderson '04: Near optimal separation...] we know an almost optimal separation between general and tree-like resolution w. r. t. length:

$\exists$ a family $(F_n)_{n \in \mathbb{N}}$ of unsatsfiable formulas in $\mathrm{O}(n)$ variables with

- resolution refutations of length L (linear in n),
- **but** any tree-like resolution refutation requires length $\exp\left(\Omega(\frac{L}{\log L})\right)$.

Matching upper bound of $\exp\left(\mathrm{O}\left(\frac{L \log \log L}{\log L}\right)\right)$ for tree-like resolution length of any formula that can be refuted in length L by general resolution.

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Our Results / Outline of This Talk

In this talk we will:

I. a) characterize $\mathrm{Tree\text{-}CS}$ for special formulas defined over a DAG G in terms of a pebble game played on G.
   b) deduce the best known separation between $\mathrm{Tree\text{-}CS}$ and CS with this new characterization.

II. show that this is almost optimal by proving an upper bound for $\mathrm{Tree\text{-}CS}$ in terms of CS (for general formulas).

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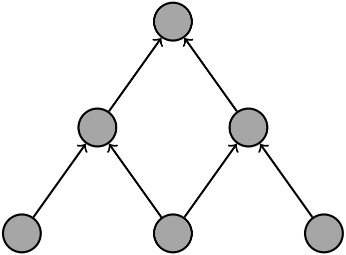
Part I

Separations for Pebbling Formulas

Pebble Games
(games played on graphs)

The Black Pebble Game

Goal: Get a single black pebble on the sink of the graph.



max # of pebbles
used at any point:

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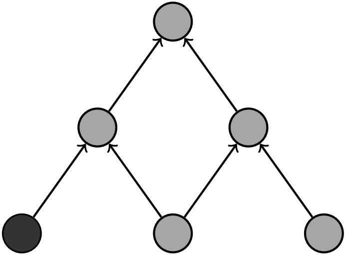

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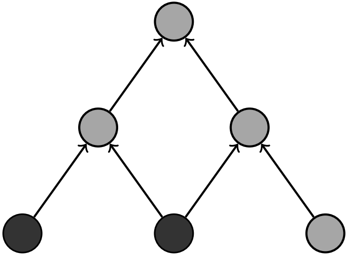

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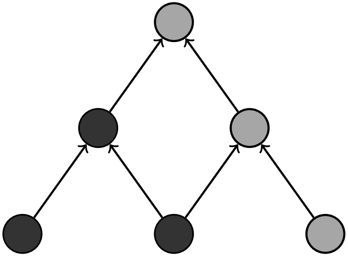

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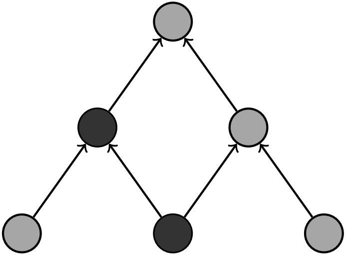

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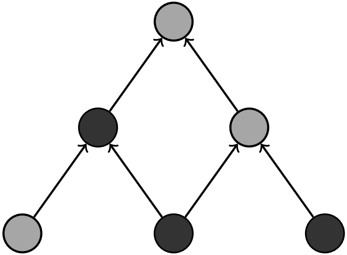

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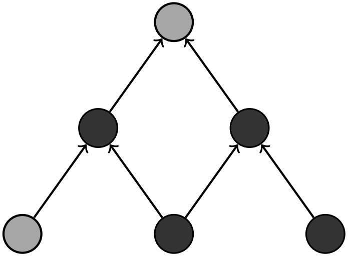

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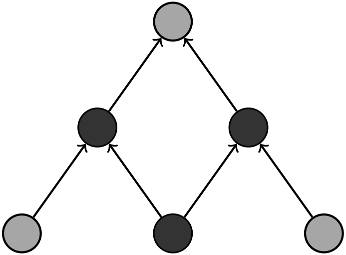

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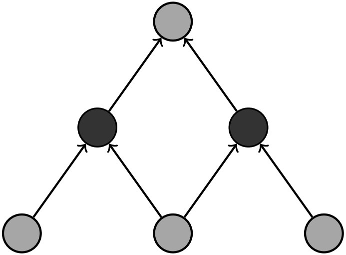

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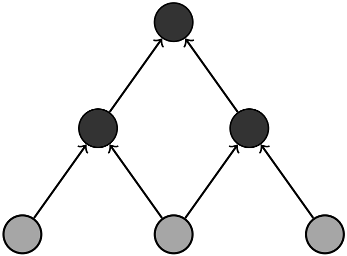

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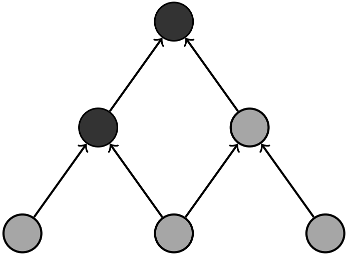

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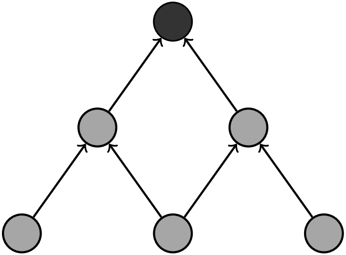

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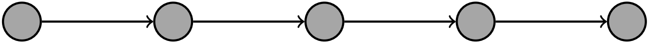
Same measure: $\max$ # of pebbles used at any point:



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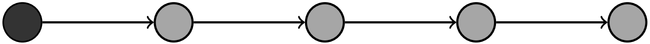

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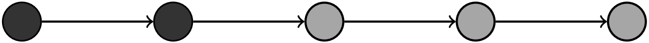

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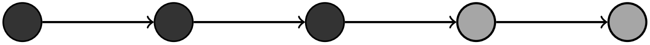

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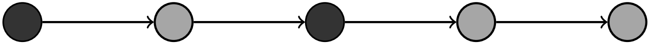

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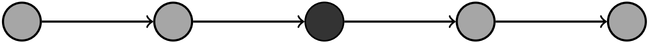

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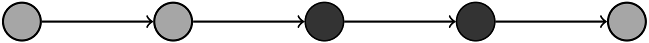

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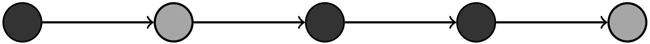

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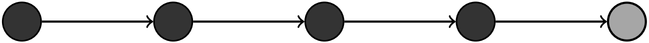

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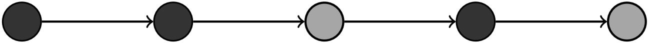

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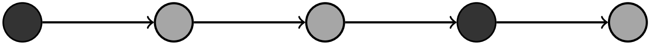

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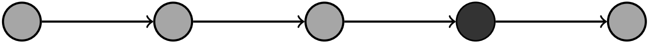

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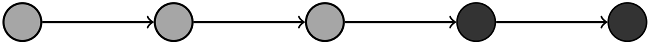

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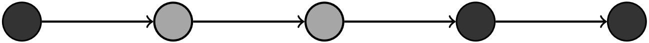

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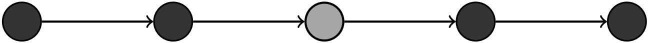

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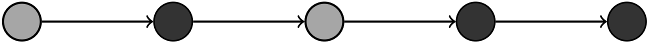

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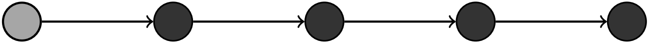

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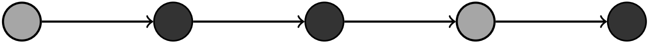

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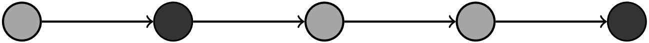

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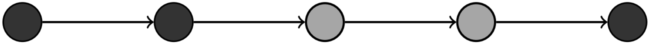

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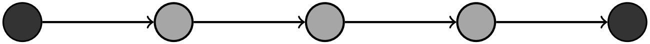

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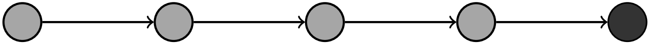

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Complexity Measures for the Pebble Games

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Why even care about these pebbling prices?

Plethora of connections to resolution i. a.:

$\mathrm{CS}(\pi) = \text{Black}(G_\pi)$ $\forall \pi : F \vdash \Box$ [Esteban, Torán '01: Space bounds for resolution].

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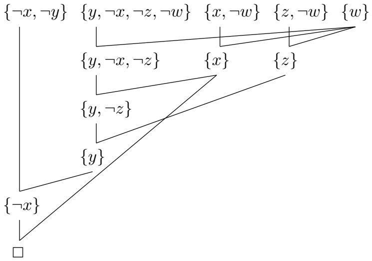

Pebbling Formulas (formulas over DAGs)

Pebbling Formula

Clauses of Peb_G:

u

v

w

$(u \wedge v) \rightarrow x = \overline{u} \vee \overline{v} \vee x$

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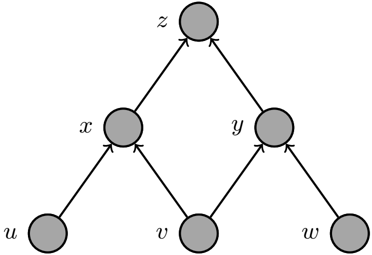

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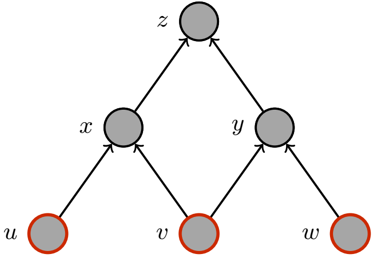

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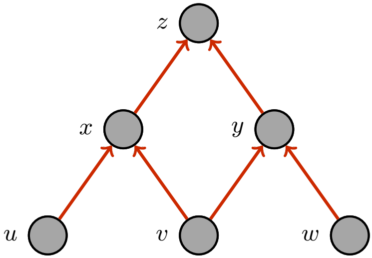

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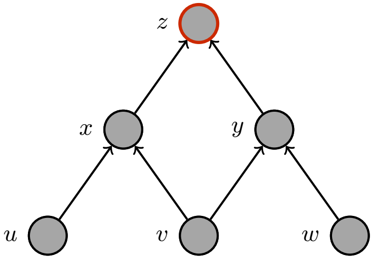

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- For a technical reason we need the XORification of our pebbling formulas.
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Reversible Pebbling meets Tree-CS in the Special Case of Pebbling Formulas

Reversible Pebbling meets Tree-CS

Theorem

For all DAGs G with a unique sink:

$$\mathsf{Rev}(G) + 2 \leq \text{Tree-CS}\big(\text{Peb}_G[\oplus_2] \vdash \Box\big) \leq 2 \cdot \mathsf{Rev}(G) + 2.$$

Obtaining Space-Separations with Pebble games (1/3)

Idea:

- $\mathrm{CS}(\mathrm{Peb}_G[\oplus_2] \vdash \Box) = \mathrm{O}\big(\mathsf{Black}(G)\big)$
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$\Longrightarrow$ Construct a graph family with a gap between its black and reversible pebbling price

Example: Path graphs P_n of length n

- $\mathsf{Black}(P_n) = \mathrm{O}(1)\ \forall n \in \mathbb{N}$
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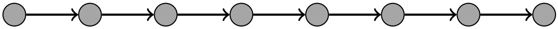

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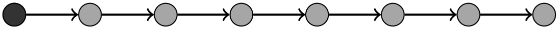

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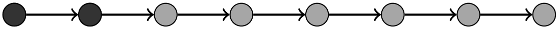

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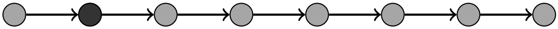

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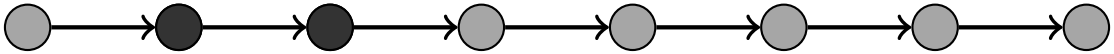

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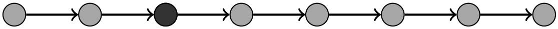

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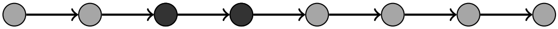

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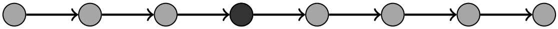

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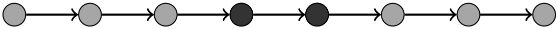

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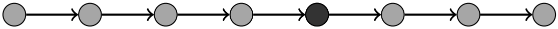

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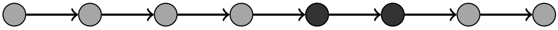

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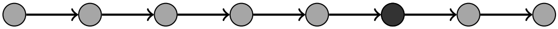

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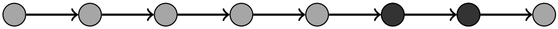

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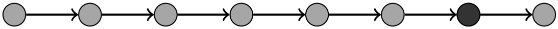

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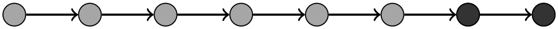

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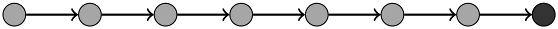

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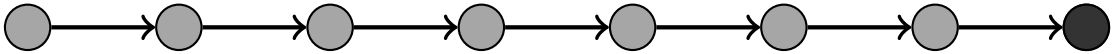

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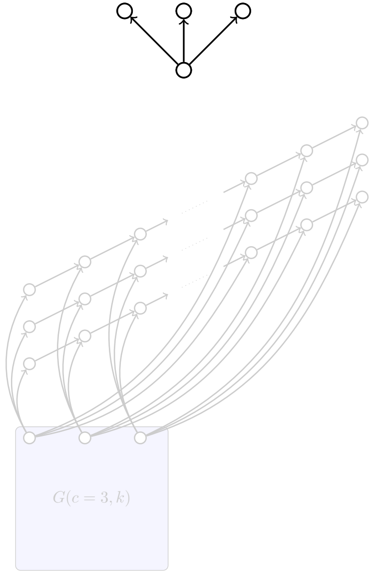

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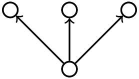

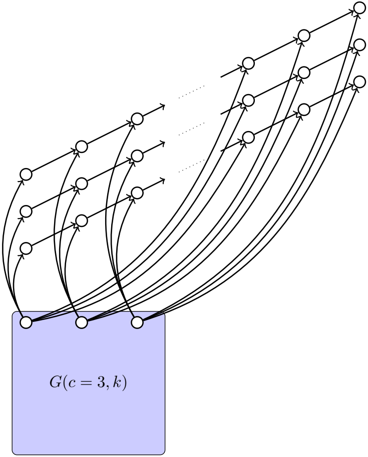

Obtaining Space-Separations with Pebble games (3/3)

Conclusion: The best known separation

For any "slowly enough" growing space function $s(n)$ there is a family of pebbling formulas $\big(\mathrm{Peb}_{G_n}[\oplus_2]\big)_{n=1}^{\infty}$ with $\Theta(n)$ variables such that

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Part II

Upper Bounds for Tree-CS for General Formulas

An upper bound for Tree-CS

How large can the gap between CS and Tree-CS grow?

Theorem

For any unsatisfiable formula F it holds

$$\text{Tree-CS}(F \vdash \Box) \leq \min_{\pi : F \vdash \Box} \mathsf{Rev}(G_\pi) + 2.$$

Note, that the minimum in the theorem is taken over all possible refutations of F, not only over the tree-like ones.

We will now prove this theorem… after introducing yet another two games.

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A combinatorial characterization of Tree-CS (by a game played on formulas)

The Prover-Delayer Game

Given: An unsatisfiable CNF formula F

Two players take rounds... until Game Over...
Score of Delayer = # of $*$'s

Prover	**Delayer**
• Wants to falisify $C \in F$ (then Game Over)	
• Queries a variable x of F	
	• Answers – $x = 0$, – $x = 1$ or – $x = *$ ("you choose")
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Definition (Game value of the Prover-Delayer game)

Let F be an unsatisfiable CNF formula.
$\text{PD}(F) := \max$ pts. of Delayer on F against optimal strategy of Prover.

Theorem ([Esteban, Torán '03: A combinatorial char. of treelike res. space])

Let F be an unsatisfiable CNF formula. Then

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The equivalence of Rev and R-Mc

Rev(G) is hard to compute

Raz–McKenzie Game to the help

Given: A single sink DAG G

Two players take rounds... until Game Over...

Pebbler	Colourer
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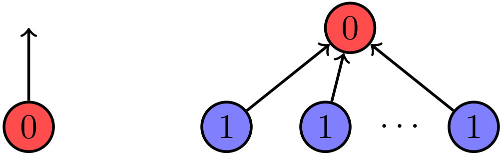

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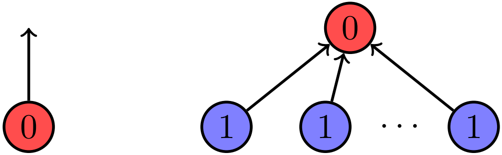

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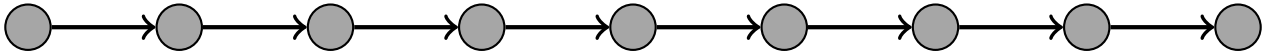

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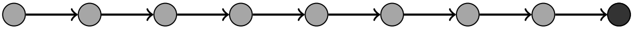

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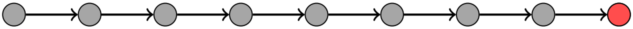

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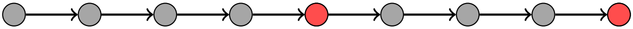

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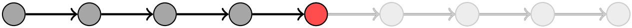

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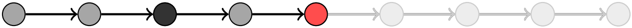

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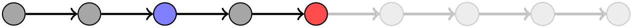

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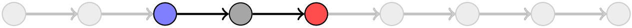

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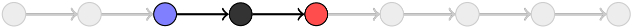

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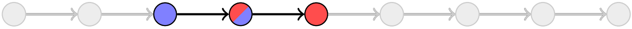

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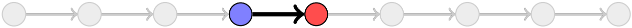

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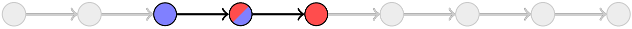

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The Actual Proof

An upper bound for Tree-CS

Proof sketch of $\mathrm{Tree\text{-}CS}(F \vdash \Box) \leq \min_{\pi: F \vdash \Box} \mathsf{Rev}(G_\pi) + 2$

Given: a res. refutation π of F with a ref.-graph G_π and $\mathsf{Rev}(G_\pi) =: k$.

AIM: Give a strategy for Prover in the PD-game under which he has to pay at most k points.

Idea: Simulate the strategy of Pebbler in the Raz–McKenzie game
$\rightarrow$ a falsifying part. assignment α of init. clause will be produced

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After at most k stages the Raz–McKenzie game finished
$\Rightarrow$ Delayer can score at most k points.

Only left to show: At the end of the game a clause of F is fals. by α.

When Raz–McKenzie finishes:

1. either a source vertex in G_π is assigned colour 0 by Colourer,
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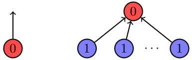

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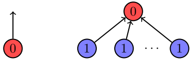

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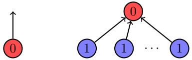

An upper bound for $\mathrm{Tree\text{-}CS}$ in terms of CS^*

[Razborov '18: On space and depth in resolution] introduced amortised clause space:

$$\mathrm{CS}^*(F \vdash \Box) := \min_{\pi : F \vdash \Box} \left(\mathrm{CS}(\pi) \cdot \log \mathrm{L}(\pi)\right)$$

Corollary

$\mathrm{Tree\text{-}CS}(F \vdash \Box) \leq \mathrm{CS}^*(F \vdash \Box) + 2.$

Proof.

- [Královič '04: Time and Space Complexity of Reversible Pebbling] $\mathrm{Rev}(G_\pi) + 2 \leq \min_{\mathcal{P}} \left(\mathrm{space}(\mathcal{P}) \cdot \log \mathrm{time}(\mathcal{P})\right) + 2$, where the minimum is taken over all black pebblings $\mathcal{P}$ of G_π.
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$\mathrm{Tree\text{-}CS}$ and CS are fundamentally different measures

- $\mathrm{Tree\text{-}CS}\big(\mathrm{Peb}_G[\oplus_2] \vdash \Box\big) \simeq \mathsf{Rev}(G)$
- Separations between $\mathrm{Tree\text{-}CS}$ and CS by graphs G exhibiting separation between $\mathsf{Rev}(G)$ and $\mathsf{Black}(G)$
- $\mathrm{Tree\text{-}CS}(F \vdash \Box) \lesssim \mathrm{CS}^*(F \vdash \Box)$ for general F

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(*) Some open questions hidden here. We've solved these for Tseitin formulas.

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- $\mathrm{Tree\text{-}CS}\big(\mathrm{Peb}_G[\oplus_2] \vdash \Box\big) \simeq \mathsf{Rev}(G)$
- Separations between $\mathrm{Tree\text{-}CS}$ and CS by graphs G exhibiting separation between $\mathsf{Rev}(G)$ and $\mathsf{Black}(G)$ (*)
- $\mathrm{Tree\text{-}CS}(F \vdash \Box) \lesssim \mathrm{CS}^*(F \vdash \Box)$ for general F (*)

(*) Some open questions hidden here. We've solved these for Tseitin formulas.

Thank you for your attention!