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Abstract

We consider sets Turing reducible to p-selective sets under various resource bounds and restricted number of queries to the oracle. We show that there is a hierarchy among the sets polynomial-time Turing reducible to p-selective sets with respect to the degree of a polynomial bounding the number of adaptive queries used by a reduction. We give a characterization of EXP/*poly* in terms of exponential-time Turing reducibility to p-selective sets. Finally we show that EXP can not be reduced to the p-selective sets under 2^{lin} time reductions with at most n^k queries for any *fixed* $k \in \mathrm{N}$.

1 Introduction

Selman [13] introduced p-selective sets as a polynomial time analogue to the semirecursive sets as studied by Jockusch [8]. Roughly speaking, a set is p-selective if there is a polynomial-time procedure which decides for a pair of strings which of them is "more likely" to be in the set. Selman used p-selective sets to show that polynomial-time Turing reducibility and many-one reducibility differ on NP, unless E = NE.

Since then much attention has been paid to sets reducible to p-selective sets under various polynomial-time reducibilities. This extends a long line of research of sets reducible to sets of low density, such as tally and sparse sets (for a survey, see [16]). Toda [15] showed various collapses under the assumption that all sets in certain complexity classes are truth-table reducible to some p-selective set. In particular, he showed that if each set in UP is truth-table reducible to some p-selective set then P = UP and if each set in Δ_2^p is truth-table reducible to some p-selective set then P = NP. Recently it has been shown that if each set in NP is truth-table reducible to some p-selective set then P = NP [1, 4, 12]. A rather detailed examination of the relationships between sets reducible or equivalent to the p-selective sets under various reducibilities has been given by Hemaspaandra et al.[6]. For a survey on results concerning p-selective sets we refer to [5]

Here we concentrate on sets Turing reducible to p-selective sets. We start by showing that there is a proper hierarchy between the P_{tt}(P-sel) and P_T(P-sel) with respect to the degree of a polynomial bounding the number of adaptive queries to some p-selective oracle. The proof is by diagonalization based on a simple fact concerning the number of sets selected by a single selector-function.

Selman [13] showed that each tally set is polynomial-time Turing reducible to some p-selective set. Ko [9] showed that the p-selective sets are contained in the nonuniform advice class P/*poly* which in turn is precisely the class of sets polynomial-time Turing reducible to some tally set. Hence *P/poly* can be characterized by the sets polynomial-time Turing reducible to some p-selective set. In Section 4 we adress the question wether a similar characterization of the classes EXP/*poly* and E/*lin* can be given. It should be mentioned that a characterization of EXP/*poly* via tally sets is impossible since *every* set is already exponential-time many-one reducible to some tally set, namely its tally version where each instance is encoded in unary. However, we can show that EXP/*poly* is precisely the class of sets which are exponential-time Turing reducible to some p-selective set using at most polynomially many adaptive queries. Furthermore we show that a set which is Turing reducible in time $O(2^{lin})$ to some p-selective set using at most linearly many queries is contained in E/*lin*. Here the converse fails. Concerning the non-uniform complexity of p-selective sets via linear-length advices there is a recent result by Hemaspaandra and Torenvliet showing P-sel $\subseteq$ NP/*lin* $\cap$ coNP/*lin* [7].

The proofs are based on an observation which informally can be stated as follows: Suppose that V is a finite set and we know the number of strings in $A \cap V$ for some p-selective set A. Then we can decide whether x is in A for each string in V, by simply counting the strings in V that are "more likely" than x in A. We apply this observation to sets Turing reducible to p-selective sets considering various resource bounds. We thereby obtain a close relation between the number of oracle queries to some p-selective set and the length of the advice needed to decide a set reducible to some p-selective set nonuniformly.

In Section 5 we consider the relationship of (uniform) exponential-time complexity classes and sets Turing reducible to some p-selective set. It is an open question whether EXP is included in P/*poly*. Regarding the characterization of P/*poly* in terms of sets polynomial-time Turing reducible to some p-selective set it is natural to ask whether one can settle the relationship between subclasses of P/*poly* and EXP, where this subclasses are obtained from restricting the access to some p-selective oracle. In fact, Toda [15] showed that EXP is not included in the class of sets polynomial-time truth-table reducible to some p-selective set. Here we extend this result to sets Turing reducible to some p-selective set where the reduction may use at most $q(n)$ *adaptive* queries for every *fixed* polynomial $q(n)$.

2 Preliminaries

We write N to denote the set of nonnegative integers. A *string* is a finite sequence of characters over the two letter alphabet $\Sigma = \{0,1\}$. We write $\Sigma^\star$ for the set of all strings including the empty string and $|x|$ for the length of a string $x \in \Sigma^\star$. We use $||A||$ to denote the cardinality of a finite set A. Let $A^{\leq n}$ denote the set of strings in A of length at most n. $A^{=n}$ is the set of strings in A of length n. A *tally* set is a set $A \subseteq \{0^n : n \in \mathrm{N}\}$. Let TALLY denote the class of all tally sets.

We use deterministic-, nondeterministic- and oracle Turing machines and other notions of complexity theory, as can be found in [2]. We are especially interested in the following deterministic time complexity classes $\mathrm{P} = \bigcup_{k \in \mathrm{N}} \mathrm{DTIME}(n^k)$, $\mathrm{E} = \bigcup_{c \in \mathrm{N}} \mathrm{DTIME}(2^{cn})$ and $\mathrm{EXP} = \bigcup_{k \in \mathrm{N}} \mathrm{DTIME}(2^{n^k})$.

A set A is *Turing reducible* to a set B if there exists an oracle Turing machine such that $A = L(M, B)$. A *query tree* of an oracle Turing machine M on input x is a binary tree in which the nodes are labeled with all possible queries M can ask on input x, i.e. the root is labeled with the first query, and for each internal node corresponding to a query q, the left (resp. right) successor corresponds to the next query M asks with a positive (resp. negative) answer on q. For a time bound $t(n)$ and a class of sets $\mathcal{C}$, let $\mathrm{DTIME}(t(n))_T(\mathcal{C})$ denote the class of all sets that are Turing reducible to some set in $\mathcal{C}$ via a $t(n)$ time bounded oracle Turing machine. For a function $q : \mathrm{N} \to \mathrm{N}$ let $\mathrm{DTIME}(t(n))_{q(n)\text{-}T}(\mathcal{C})$ denote the class of all sets in $\mathrm{DTIME}(t(n))_T(\mathcal{C})$ where the oracle Turing machine asks at most $q(n)$ queries on every path of the query tree. We extend this notation to complexity classes and function classes, for example as $\mathrm{P}_{q(n)\text{-}T}(\mathcal{C})$, $\mathrm{P}_{lin\text{-}T}(\mathcal{C})$, $\mathrm{P}_{poly\text{-}T}(\mathcal{C})$ etc.

Throughout a computation an oracle Turing machine may ask queries which depend on the answers to previously asked queries. These kind of queries are called *adaptive* queries. In contrast, a set A is *truth-table reducible* to a set B if there exists an oracle Turing machine M such that $A = L(M, B)$ and all queries asked by M are nonadaptive, i.e. do not depend on the answers to previously asked queries. For a time bound $t(n)$ and a class of sets $\mathcal{C}$, let $\mathrm{DTIME}(t(n))_{tt}(\mathcal{C})$ denote the class of all sets that are truth-table reducible to some set in $\mathcal{C}$ via a $t(n)$ time bounded oracle Turing machine asking only nonadaptive queries.

All time bounds and functions bounding the number of queries are assumed to be monotonic increasing and time constructible.

3 A hierarchy between polynomial-time truth-table and Turing reducibility to p-selective sets

We show that there is a proper hierarchy between the class $\mathrm{P}_{tt}(\text{P-sel})$ and $\mathrm{P}_T(\text{P-sel})$ with respect to the degree of a polynomial bounding the number of queries to some p-selective oracle. First we briefly review the definition and a standard construction of p-selective sets from [13].

Definition 3.1 ([13]). A set $A \subseteq \Sigma^\star$ is *p-selective* if there is a polynomial-time computable function $f : \Sigma^\star \times \Sigma^\star \to \Sigma^\star$ such that for all strings $x, y \in \Sigma^\star$,

1. $f(x,y) = x$ or $f(x,y) = y$, and
2. if $x \in A$ or $y \in A$, then $f(x,y) \in A$

A function f fulfilling Conditions (i) and (ii) is called a *p-selector* for A.

Let P-sel denote the class of all p-selective sets. It immediately follows from the definition that every set in P is p-selective. On the other hand, Selman [13] showed that for every tally set there exists a polynomial-time Turing equivalent p-selective set. Hence there are arbitrarily difficult p-selective sets. The proof of this fact makes use of a of a subclass of the p-selective sets, namely the class of *standard leftcuts* with respect to an infinite binary sequence (cf. [13, 9]). Recall that the dictionary ordering of binary strings over the alphabet $\{0,1\}$ can be defined as follows: $0 \prec 1$, and for $x = x_1 \dots x_m$, $y = y_1 \dots y_n$, $x \prec y$ iff (i) $m = n$ and $(\exists i \leq m)(\forall j < i)[x_j = y_j \; \& \; x_i \prec y_i]$, or (ii) $m < n$ and $x \preceq y_1 \dots y_m$, or (iii) $m > n$ and $x_1 \dots x_n \preceq y$. The standard leftcut L_α with respect to an infinite binary sequence α is the set of strings x which are less than or equal to the initial segment of α of length $|x|$.

Proposition 3.2 ([13]). *Every standard leftcut is p-selective.*

Proof. Every standard leftcut is selected by the p-selector $f(x,y) = x$ if $x \preceq y$ else y. ☐

We go on to consider sets reducible to some p-selective set. It is known that $2^k - 1$ nonadaptive queries to some p-selective set can be simulated by k adaptive queries of a polynomial-time bounded oracle Turing machine. A proof of this fact can be found in [6]. Though it is stated there only for a constant number of queries it can be easily generalized to the case where the nonadaptive queries are bounded only by the running time of the reduction.

Proposition 3.3 ([6]). $\mathrm{P}_{tt}(\text{P-sel}) = \mathrm{P}_{O(\log n)\text{-}T}(\text{P-sel})$.

We next show a hierarchy theorem among the sets polynomial-time Turing reducible to some p-selective set with respect to the number of adaptive queries used by the reduction. In the case of constant number of queries Hemaspaandra et al. [6] showed a tight hierarchy theorem: $\mathrm{P}_{k\text{-}T}(\text{P-sel}) \subset \mathrm{P}_{k+1\text{-}T}(\text{P-sel})$, $k \geq 1$. They use a construction of p-selective sets introduced by Naik et al. [11]. If the number of queries depends on the length of the input we get a less tight hierarchy. The constructed set will be reducible to some p-selective leftcut. We isolate the combinatorial part of the diagonalization in the following lemma (cf. [6]).

Lemma 3.4. *Let f be a* P*-selector and V a finite set of strings. Then*

$$||\{W \subseteq V : f \textit{ selects } W\}|| \leq ||W||.$$

Proof. Suppose that there are more than $||W||$ subsets of W which are selected by f. Hence among these sets there exists distinct sets $W_1, W_2 \subseteq W$ with the same cardinality. That is, for some $x_1, x_2 \in W$, $x_1 \in W_1 - W_2$ and $x_2 \in W_2 - W_1$. It follows that $f(x_1, x_2)$ cannot select both of W_1 and W_2, ☐

Proposition 3.5. $\mathrm{P}_{n^{\frac{k}{2}}\text{-}T}(\text{P-sel}) \subset \mathrm{P}_{n^{\frac{k+1}{2}}\text{-}T}(\text{P-sel})$, $k \geq 1$.

Proof. Let $M_0, M_1, \ldots$ be an enumeration of all polynomial-time oracle machines asking at most $n^{\frac{k}{2}}$ queries on an input of length n. Let $f_0, f_1, \ldots$ be an enumeration of all polynomial-time transducers such that $f(x,y) \in \{x,y\}$.

Let $\mu(s) = 2^{s+4}$, and let F_n denote the lexicographically first $n^{\frac{k}{2}} + (k+1) \cdot log(n)$ strings of length n.

We will construct a set A in stages. The set A will consist only of strings of $F_{\mu(s)}$, $s \geq 0$. In stage $s = \langle i, j \rangle$, we will satisfy the following requirement:

(R_s) for any set B which is selected by f_j, A is not accepted by M_i with oracle B.

The construction proceeds as follows. Let A' denote the set of strings put into A prior to stage s. Let $n = \mu(s)$ and Q_n be the set of all queries in the query-tree of M_i on an all inputs $x \in F_n$. Then $||Q_n|| \leq n^{\frac{k}{2}} + (k+1) \cdot log(n) \cdot 2^{n^{\frac{k}{2}}}$. By Lemma 3.4, there exist at most $||Q_n||$ subsets of Q_n which are selected by f_j. Hence M_i can agree on at most $(n^{\frac{k}{2}} + (k+1) \cdot log(n)) \cdot 2^{n^{\frac{k}{2}}} < n^{k+1} \cdot 2^{n^{\frac{k}{2}}} = 2^{n^{\frac{k}{2}} + (k+1) \cdot log(n)}$ subsets of F_n with some oracle selected by f_j. But there are $2^{n^{\frac{k}{2}} + (k+1) \cdot log(n)}$ distinct subsets of F_n. Hence there exists some (smallest) set $D_n \subseteq F_n$ which cannot be accepted by M_i with any oracle selected by f_j. Setting $A = A' \cup D_n$ we thus established (R_s).

It remains to show that A is Turing reducible to some p-selective set with at most $n^{\frac{k+1}{2}}$ adaptive queries. Let d_n, for $n = \mu(s)$, the string of length $n^{\frac{k}{2}} + (k+1) \cdot log(n)$ denoting the (finite) characteristic function of D_n in the construction of A in stage s. Define a p-selective set B to be the leftcut with respect to the infinite sequence $d_{\mu(0)} d_{\mu(1)} d_{\mu(2)} \ldots$. The membership of some x of length $n = \mu(s)$ in A is fixed by d_n. It thus suffices to reconstruct d_n from the oracle B. By prefix search, this requires at most $\sum_{i=4}^{log(n)} (2^i)^{\frac{k}{2}} + (k+1) \cdot i \leq log(n) \cdot n^{\frac{k}{2}} + (k+1) \cdot log^2(n) \leq \sqrt{n} \cdot n^{\frac{k}{2}} = n^{\frac{k+1}{2}}$ adaptive queries. □

The announced hierarchy between P_{tt}(P-sel) and P_T(P-sel) now follows as a corollary.

Corollary 3.6. *1.* $\mathrm{P}_{tt}(\text{P-sel}) \subset \mathrm{P}_{n\text{-}T}(\text{P-sel})$.

2. $\mathrm{P}_{n^k\text{-}T}(\text{P-sel}) \subset \mathrm{P}_{n^{k+1}\text{-}T}(\text{P-sel})$, $k \geq 1$.

3. $\mathrm{P}_{n^k\text{-}T}(\text{P-sel}) \subset \mathrm{P}_T(\text{P-sel})$, $k \geq 1$.

4 Exponential-time advice classes

We consider the nonuniform complexity of sets Turing reducible to p-selective sets in terms of advice classes. As the main result we obtain a characterization of EXP/*poly* in terms of reducibility to p-selective sets.

Definition 4.1. An *advice function* is a function $h : \mathrm{N} \to \Sigma^*$. For a function $q : \mathrm{N} \to \mathrm{N}$, let $ADV(q(n))$ denote the set of all advice functions h with $|h(n)| \le q(n)$ for all $n \in \mathrm{N}$. For functions $t, q : \mathrm{N} \to \mathrm{N}$, the *advice class* $\mathrm{DTIME}(t(n))/ADV(q(n))$ is the class of sets B for which there exists a set $A \in \mathrm{DTIME}(t(n))$ and an advice function $h(n) \in ADV(q(n))$ such that $B = \{x : \langle x, h(|x|)\rangle \in A\}$.

Using Definition 4.1 we can redefine $\mathrm{P}/poly$ as $\bigcup_{k\in\mathrm{N}} \mathrm{DTIME}(n^k)/ADV(n^k)$. Additionally we consider the advice classes E/lin and $\mathrm{EXP}/poly$ which can be defined similarly.

The following lemma is the key observation which leads to all subsequent results of this paper.

Lemma 4.2. *Let A be a p-selective set with selector-function f. Let V a finite set of strings. Then for each $x \in V$, $x \in A$ if and only if $||\{x' : x' \in V \text{ and } f(x, x') = x'\}|| \le ||A \cap V||$.*

Proof. Fix a string $x \in V$. First assume that x is in A. By the definition of a p-selector, $f(x, x') = x'$ implies $x' \in A$. Therefore, the number of strings x' in V for which $f(x, x') = x'$ is at most $||A \cap V||$. If x is not in A, then for all x' in A, $f(x, x') = x'$. Additionally, $f(x, x) = x$. Hence for more than $||A \cap V||$ strings x' in V it holds that $f(x, x') = x'$. □

Consider a p-selective set A and a p-selector f for A. Let the advice be the binary representation of the number of strings in A of length n. Then the length of the advice is at most $n + 1$. By Lemma 4.2, for each string x of length n, the membership of x A can be decided with the help of the advice by counting the strings x' of length n for which $f(x, x') = x'$. Since f is polynomial-time computable, this can be done in time $O(2^{2n})$. That is, $\text{P-sel} \subseteq \mathrm{DTIME}(2^{2n})/ADV(n + 1)$. This argument can be generalized to sets Turing reducible to p-selective sets, whereby we obtain a relationship between the number of oracle queries and the length of the advice.

Lemma 4.3.

$$\mathrm{DTIME}(t(n))_{q(n)\text{-}T}(\text{P-sel}) \subseteq \bigcup_{k\ge 0} \mathrm{DTIME}(t(n)^k \cdot 2^{2q(n)+2n})/ADV(q(n) + n + 1)$$

Proof. Let A be a set Turing reducible to a p-selective set B via a $O(t(n))$ time bounded oracle Turing machine which asks at most $q(n)$ queries on every path of the query tree on some input of length n. Let f be the p-selector for B and assume that f is computable in time $O(n^k)$ for some constant $k \in \mathrm{N}$.

Let $Q_n = \bigcup_{|x|=n} Q(x)$ where $Q(x)$ denotes the set of all queries in the query-tree of M on an input x. Define the advice function $h : \mathrm{N} \to \Sigma^*$ to be the binary representation of $||B \cap Q_n||$. Thus the length of $h(n)$ is less or equal to $q(n) + n + 1$.

For a string x of length n, we decide $x \in A$ by the following algorithm. First generate a list of all queries $q \in Q_n$ by traversing the query trees of M for all inputs of length n. In order to avoid counting a single query more than once in the following step, eliminate multiple occurrences of queries in this list. Now simulate M on x. Whenever M asks a query q, count the strings q' in the list such that $f(q, q') = q'$. If this number is less or equal to $h(n)$, continue with the answer "YES", otherwise continue with "NO". Accept if and only if M accepts x.

By Lemma 4.2, we always continue with the correct answer. Therefore, we accept x iff $x \in A$. To compute the list of all queries in Q_n, we have to generate successively $Q(x)$ for all x of length n. This can be done in time $O(2^n \cdot 2^{q(n)} \cdot t(n))$. Eliminating multiple occurrences of queries in this list requires additionally time $O((2^n \cdot 2^{q(n)})^2 \cdot t(n))$. To determine the answer for a query q, we have to compute the p-selector f on at most $2^n \cdot 2^{q(n)}$ strings of length less or equal to $t(n)$. Since f is computable in time $O(n^k)$, this requires time $O((t(n))^k \cdot 2^n \cdot 2^{q(n)})$. We conclude that the whole algorithm runs in time $O(t(n)^k \cdot 2^{2q(n)+2n})$. □

Applying Lemma 4.3 and the standard leftcut construction we obtain the characterization of EXP/*poly*.

Theorem 4.4. $\mathrm{EXP}_{poly\text{-}T}(\text{P-sel}) = \mathrm{EXP}/poly$

Proof. By Lemma 4.3, $\mathrm{EXP}_{poly\text{-}T}(\text{P-sel}) \subseteq \mathrm{EXP}/poly$. To see the inverse inclusion let A be accepted by an exponential-time Turing machine M with the advice function h. Define an infinite binary sequence $\alpha = \overline{h(0)}01\overline{h(1)}01\ldots$, where $\overline{x}$ denotes the string x with each bit doubled. In order to decide $x \in A$ with the p-selective leftcut L_α, first compute $\overline{h(0)}01\overline{h(1)}01\ldots01\overline{h(|x|)}$ from L_α by prefix search. Then simulate M with input x and the advice $h(|x|)$. Since the length of $h(|x|)$ is polynomially bounded, both the number and the length of the queries are bounded by some polynomial. It follows that $h(|x|)$ can be obtained from the oracle L_α in polynomial-time. Hence the whole algorithm runs in exponential-time. □

Remark 4.5. Note that in the above proof the queries used by the exponential-time reduction are polynomially length bounded. Thus it follows from Theorem 4.4 that every set in $\mathrm{EXP}_{poly\text{-}T}(\text{P-sel})$ can be exponential-time Turing reduced to some p-selective set where *both* the number and the length of the queries are polynomially bounded.

A characterization similar to Theorem 4.4 fails for E/*lin*. Adapting a proof in [3], we show that $\mathrm{E}_{lin\text{-}T}(\text{P-sel})$ is properly included in E/*lin*. In the proof we use a Kolmogorov-random sequence. For the definition of Kolmogorov complexity and related facts we refer to [10].

Theorem 4.6. $\mathrm{E}_{lin\text{-}T}(\text{P-sel}) \subset \mathrm{E}/lin$

Proof. The inclusion $\mathrm{E}_{lin\text{-}T}(\text{P-sel}) \subseteq \mathrm{E}/lin$ follows from Lemma 4.3. Moreover, replacing $Q_n = \bigcup_{|x|=n} Q(x)$ by $Q_{\leq n} = \bigcup_{|x|\leq n} Q(x)$ in the proof of Lemma 4.3 we see that for a set in $\mathrm{E}_{lin\text{-}T}(\text{P-sel})$ all strings *up* to length n can be decided in exponential-time with an advice of linear length. This implies that the Kolmogorov complexity of all initial segments of the characteristic sequence up to strings of length n is at most linear in n.

Now consider a binary Kolmogorov random infinite sequence ρ. That is, a sequence such that the Kolmogorov complexity of its prefixes is at least linear in the length of the prefixes infinitely often. We define a set A which contains at most the first n strings of length n. Let x be the lexicographically i-th string, $i \leq n$, of length n. Then $x \in A$ if and only if the $(\frac{n(n-1)}{2} + i)$-th bit of ρ is 1. That is, we divide ρ into consecutive subsequences of length $1, 2, \ldots, n, \ldots$ and the n-th subsequence of length n denotes the membership of the first n strings of length n in A. It follows that A is in E/lin. We use a prefix of length n^2 of ρ to define A up to strings of length n. Hence the Kolmogorov complexity of the prefixes of the characteristic sequence of A up to strings of length n is at least quadratic in n infinitely often. Thus A is not in $\mathrm{E}_{lin\text{-}T}(\text{P-sel})$. $\square$

Remark 4.7. Throughout this paper we consider only ***bounded*** query reductions to p-selective sets. The reason is that if we do ***not*** restrict the number of queries, then ***every*** set is reducible in linear exponential time to p-selective sets. In order to see that, fix any set A. Then A can be many-one reduced in linear exponential-time to its tally version $tally(A) = \{0^i : s_i \in A\}$, where s_i is the i-th string in the lexicographical ordering on $\Sigma^\star$. Furthermore, every tally set can be Turing reduced to some p-selective set in polynomial time [13]. Hence A is in $\mathrm{E}_m(\mathrm{P}_T(\text{P-sel}))$. Since $\mathrm{E}_m(\mathrm{P}_T(\text{P-sel})) \subseteq \mathrm{E}_T(\text{P-sel})$, we conclude that A is in $\mathrm{E}_T(\text{P-sel})$.

5 Turing reducibility to p-selective sets and uniform exponential-time complexity

We locate sets Turing reducible to p-selective sets in (uniform) exponential-time classes using Lemma 4.3 and the following proposition.

Proposition 5.1. *Let $f : \mathrm{N} \to \mathrm{N}$ be a function with $n \leq f(n) < 2^n$. Then* $\mathrm{DTIME}(2^{4 \cdot f(n)}) \not\subseteq \mathrm{DTIME}(2^{f(n)})/ADV(f(n))$.

Proof. Let $M_0, M_1, \ldots$ be an enumeration of all Turing machines M_i running in time $i2^{f(n)}$. We will construct a set A in stages. Each stage determines the membership of all strings of length n. In stage n, we will satisfy the following requirement:

(R_n) for any advice function $h \in ADV(f(n))$, A is not accepted by M_n with advice h.

This clearly implies $A \notin \mathrm{DTIME}(2^{f(n)})/ADV(f(n))$. Let A' denote the set of strings put into A prior to stage n. There are at most $2^{f(n)} < 2^{2^n}$ sets of strings of length n which can be accepted by M_n with some advice of length $f(n)$. Since there are 2^{2^n} distinct subsets of $\Sigma^{=n}$ there is a (smallest) set $D_n \subseteq \Sigma^{=n}$ which is not accepted by M_n with some advice of length $f(n)$. Setting $A = A' \cup D_n$ we thus established (R_n).

In order to decide $x \in A$ (uniformly) for some string x of length n we only have to determine the set D_n in the above construction. But this can be done in time $O(2^{f(n)} \cdot 2^n \cdot n \cdot 2^{f(n)})$, hence A is in $\mathrm{DTIME}(2^{4f(n)})$. $\square$

Theorem 5.2. *Fix $c, k \in$ N. Then*

1. $\mathrm{E} \not\subseteq \mathrm{P}_{cn\text{-}T}(\text{P-sel})$
2. $\mathrm{EXP} \not\subseteq \mathrm{E}_{n^k\text{-}T}(\text{P-sel})$

Proof. (1) By Lemma 4.3, for every $c \in$ N, $\mathrm{P}_{cn\text{-}T}(\text{P-sel}) \subseteq \mathrm{DTIME}(2^{c'n})/ADV(c'n)$ for some constant $c' \in$ N. But $E \not\subseteq DTIME(2^{c'n})/ADV(c'n)$ by Proposition 5.1. The proof of (2) is similar. □

Remark 5.3. Theorem 5.2 (1) holds not only for polynomial-time reducibility, but also for super polynomial-time bounds. More precisely, for all $t(n)$ such that, for all $k \in$ N, $t(n)^k \in O(2^n)$, $\mathrm{E} \not\subseteq \mathrm{DTIME}(t(n))_{cn\text{-}T}(\text{P-sel})$.

6 Conclusion

We showed that there is a hierarchy among the sets Turing reducible to p-selective sets with respect to the degree of the polynomial bounding the number of adaptive queries used by a reduction. Furthermore, we gave a characterization of EXP/*poly* in terms of Turing reducibility to p-selective sets.

Furthermore, we extended Toda's result $\mathrm{EXP} \not\subseteq \mathrm{P}_{tt}(\text{P-sel})$ to $\mathrm{EXP} \not\subseteq \mathrm{E}_{n^k\text{-}T}(\text{P-sel})$ for every fixed $k \in$ N. Wilson [14] constructed an oracle relative to which $\mathrm{EXP}^{\mathrm{NP}}$ (and hence EXP) is included in $\mathrm{P}/poly = \mathrm{P}_T(\text{P-sel})$. Thus our separation seems to be the best possible without using nonrelativizing techniques. However, since $\mathrm{EXP} \subseteq \mathrm{P}/poly$ if and only if $\mathrm{EXP}/poly \subseteq \mathrm{P}/poly$, our characterization might shed some light on the $\mathrm{EXP} \subseteq \mathrm{P}/poly$ question.

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