On Toda's Theorem in structural communication complexity

Henning Wunderlich

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Abstract

We prove Toda's Theorem in the context of structural communication complexity, i.e. $\mathbf{PH}^{cc} \subseteq \mathrm{BP} \cdot \oplus\mathbf{P}^{cc} \subseteq \mathbf{P}^{cc}(\#\mathbf{P}^{cc}) = \mathbf{P}^{cc}(\mathbf{PP}^{cc})$. The class $\mathbf{PSPACE}^{cc}$ was defined via alternating protocols with $\mathcal{O}(\log n)$ many alternations. We consider the class $\mathrm{BP} \cdot \oplus\mathbf{P}^{cc}$ of Toda's Theorem, and show that every language in this class can be decided with alternating protocols using $\mathcal{O}(\log n/\log\log n)$ many alternations. The proof is based on a new alternating protocol for the inner product function IP with $\mathcal{O}(\log n/\log\log n)$ many alternations.

1 Introduction

The main contribution of this paper is to establish Toda's Theorem in the setting of communication complexity, i.e. we prove $\mathbf{PH}^{cc} \subseteq \mathrm{BP} \cdot \oplus\mathbf{P}^{cc} \subseteq \mathbf{P}^{cc}(\#\mathbf{P}^{cc}) = \mathbf{P}^{cc}(\mathbf{PP}^{cc})$. This might be useful in the search for a solution of the famous $\mathbf{PH}^{cc}$ vs. $\mathbf{PSPACE}^{cc}$ problem, because no communication complexity measures/lower bound methods are known for alternating classes, while for the classes $\mathbf{BPP}^{cc}$ and $\oplus\mathbf{P}^{cc}$ lower bound methods are available. Thus, it might be easier to come up with a measure for $\mathrm{BP} \cdot \oplus\mathbf{P}^{cc}$, a class not based on the concept of alternation, than to develop a measure for alternation. Of course, it might be the case that $\mathrm{BP} \cdot \oplus\mathbf{P}^{cc} = \mathbf{PSPACE}^{cc}$, but we show that every language in $\mathrm{BP} \cdot \oplus\mathbf{P}^{cc}$ can be decided with alternating protocols using only $\mathcal{O}(\log n/\log\log n)$ many alternations, i.e. substantially less than allowed for $\mathbf{PSPACE}^{cc}$. The proof is based on a new alternating protocol for the inner product function IP with $\mathcal{O}(\log n/\log\log n)$ many alternations.

1.1 Structural complexity

For introductions to the broad field of structural complexity see [3, 2, 6, 14, 10]. Nice surveys on a variety of topics in this field can be found in [18, 19], especially on counting complexity in [15, 7] by Schöning and Fortnow, respectively. The parity class $\oplus\mathbf{P}$ was defined by Papadimitriou and Zachos in [12], where it was shown that $\oplus\mathbf{P}(\oplus\mathbf{P}) = \oplus\mathbf{P}$. One can define operators on complexity classes, e.g. the BP-operator, which was defined by Schöning [16]. Using the BP-operator

*Universität Ulm, Fakultät für Ingenieurwissenschaften und Informatik, Institut für Theoretische Informatik, Oberer Eselsberg, D-89069 Ulm, e-mail: Henning.Wunderlich@uni-ulm.de

and the Valiant-Vazirani-Lemma [21] Toda [20] was able to prove his celebrated theorem

$$\mathbf{PH} \subseteq \mathrm{BP} \cdot \oplus\mathbf{P} \subseteq \mathbf{P}(\#\mathbf{P}) = \mathbf{P}(\mathbf{PP}) \ ,$$

which tells us that counting (mod 2 with random source) is at least as powerful as the whole polynomial-time hierarchy **PH**. See also [17] for a simplified proof.

1.2 Communication complexity

For a thorough introduction to communication complexity we refer the reader to the book of Kushilevitz and Nisan [11].

1.2.1 Basic definitions and notation.

We only work with the *binary alphabet* $\mathbb{B} := \{0,1\}$. The length of a string $x \in \mathbb{B}^*$ is denoted by $|x|$. A prefix-free encoding of x is $\overline{x} := 0^{|x|}1x$. In order to encode pairs of strings $x, y \in \mathbb{B}^*$ we use the pairing function $\langle x, y\rangle := \overline{x}y$. The *set of pairs of strings of equal length* is denoted by $\mathbb{B}^{**} := \{(x,y) \mid x, y \in \mathbb{B}^*, |x| = |y|\}$. A *language* L is a subset of $\mathbb{B}^{**}$, its *characteristic function* χ^L is defined as $\chi^L := (\chi^L_n)$, where $\chi^L_n \colon \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{N}$, $\chi^L_n(x,y) := 1$, if $(x,y) \in L$, and 0 otherwise. We write $(x,y) \in L$ if $\chi^L_{|x|}(x,y) = 1$. The *set of all languages* is denoted by $\mathcal{L}$. A *(communication) complexity class* is a subset $\mathcal{C} \subseteq \mathcal{L}$. We define $\mathbf{poly} := \{f \colon \mathbb{R}^+ \to \mathbb{R}^+ \mid \exists\text{polynomial } p \colon f \le p\}$, the *set of functions with polynomial growth.* With log we denote the logarithm to the basis 2.

1.2.2 Yao's model.

We consider the basic model of communication complexity, introduced by Yao [22]. In this model, there are two players (parties) Alice and Bob, who want to cooperatively compute a function $f \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are finite sets. Both have complete information about f and unlimited computational power but receive only parts of the inputs. Alice is given $x \in \mathcal{X}$, Bob is given y, and they exchange messages in order to compute $f(x,y)$. Each message solely depends on the player's input and the messages communicated so far. The communication is carried out according to a fixed *protocol* Π (over domain $\mathcal{X} \times \mathcal{Y}$ with range $\mathcal{Z}$).

1.2.3 Protocols.

There are four kinds of protocols, namely *deterministic*, *randomized*, *nondeterministic* and *alternating* ones. We only describe deterministic protocols in detail: A *deterministic protocol* is a labeled binary tree, where an inner node specifies the player who sends a bit of communication next. If v is an inner node, then it is labeled either by a function $a_v \colon \mathcal{X} \to \{0,1\}$ or by a function $b_v \colon \mathcal{Y} \to \{0,1\}$. Each leaf l is labeled with an output value $z_l \in \mathcal{Z}$. The value of the protocol Π on input (x,y) is the label of the leaf reached by starting from the root, and walking on the tree. At each internal node v labeled by a_v Alice sends $a_v(x)$ and they walk left if $a_v(x) = 0$ and right if $a_v(x) = 1$. Analogously, if v is labeled with b_v. The cost of the protocol Π on input (x,y) is the length of the path taken on this input. In a *randomized protocol* Alice and Bob have access to a public or private source of randomness (*random string*). The functions a_v, b_v are arbitrary functions of the inputs and the random strings. In a *nondeterministic protocol*, we have $\mathcal{Z} = \{0, 1\}$, and each player gets a *guess string* in addition to the input. Here, a_v and b_v are arbitrary functions of the inputs and the guess strings. For nondeterministic protocols there exist different *accepting modes*. For example, a nondeterministic protocol accepts a language L in the *nondeterministic accepting mode*, if for all $(x, y) \in L$ there exist guess strings g_A and g_B such that Alice on input x and guess string g_A and Bob on input y and guess string g_B reach a leaf labeled with 1, and if for all $(x, y) \notin L$ there do not exist any guess strings such that the players reach a 1-leaf. Another example is the *parity accepting mode*: Here, an input is accepted iff the number of guess strings such that the players reach a 1-leaf is odd. For a definition of *alternating protocols*, see [1, p.339]. For formal definitions concerning protocols, cost and complexity measures, and accepting modes, see [11, Def. 1.1, p.4; Chap. 3, p.28; Chap. 2, p.18] and [5]. A protocol over domain $\mathcal{X} \times \mathcal{Y}$ is an n*-bit* protocol, if $\mathcal{X} = \mathcal{Y} = \mathbb{B}^n$. A protocol family $(\Pi_n)_{n\in\mathbb{N}}$ of n-bit protocols Π_n *decides* a language L if each Π_n computes χ_n^L.

1.2.4 Communication complexity classes.

Each protocol type and acceptance mode leads to a complexity measure, e.g. $\mathrm{D}(f)$ for the deterministic communication complexity of f, or $\oplus\mathrm{D}(f)$, which is the minimum cost of a nondeterministic protocol deciding f in parity accepting mode. If a problem can be solved with communication polylogarithmically in the input size, then we consider this as *efficient*. The communication complexity classes are defined as sets of languages that can be decided efficiently according to a fixed measure. For example, $\mathbf{P}^{cc}$ is the class of languages such that $L \in \mathbf{P}^{cc}$ iff there exists a bound $b \in \mathbf{poly}$ with $\mathrm{D}(\chi_n^L) \leq p(\log n)$, and $\oplus\mathbf{P}^{cc}$ is the class of languages such that $L \in \oplus\mathbf{P}^{cc}$ iff there exists a bound $b \in \mathbf{poly}$ with $\oplus\mathrm{D}(\chi_n^L) \leq p(\log n)$.

1.2.5 Oracle protocols.

A deterministic, randomized, nondeterministic or alternating protocol Π over $\mathcal{X}$, $\mathcal{Y}$ is an *oracle protocol* with oracle family $O = (O_m)_{m\in\mathbb{N}}$, if Π contains *oracle nodes* in its protocol tree. Associated with an oracle node v are two functions $a_v\colon \mathcal{X} \to \mathbb{B}^{m_v}$ and $b_v\colon \mathcal{Y} \to \mathbb{B}^{m_v}$. If Alice and Bob reach an oracle node v during a computation on input $(x, y) \in X \times \mathcal{Y}$, they compute by themselves $x' := a_v(x)$ and $y' := b_v(y)$, respectively, and call O_{m_v} on (x', y'). The oracle node v has exactly $|\mathrm{range}(O)|$ many successors. Alice and Bob continue the computation on one of them according to the returned value $O(x', y')$. The communication costs for each oracle call are $\lceil \log |\mathrm{range}(O)| \rceil$. If a language L is used as an oracle family, we write L instead of χ^L. *Relativized communication complexity classes* are defined via efficient oracle protocol families. For example, $\mathbf{P}^{cc}(L')$ contains all languages L which can be decided by a protocol family $(\Pi_n)_{n\in\mathbb{N}}$ of deterministic n-bit oracle protocols with L' as the oracle.

1.3 Structural communication complexity

Research in the field of structural communication complexity started with the article of Babai, Frankl and Simon [1], where some analogies between Turing machine classes like **P**, **NP**, **PP**, **PSPACE**, the polynomial hierachy $\mathbf{PH} = \bigcup_k \Sigma_k^p$, etc. and the corresponding communication complexity classes $\mathbf{P}^{cc}$, $\mathbf{NP}^{cc}$, $\mathbf{PP}^{cc}$, $\mathbf{PSPACE}^{cc}$, $\mathbf{PH}^{cc} = \bigcup_k \Sigma_k^{cc}$, etc. were shown. For more ground work, especially on closure properties, the boolean communication hierarchy, or counting communication complexity classes like $\mathrm{MOD}_m\mathbf{P}$, see Halstenberg and Reischuk [8] or Damm et al. [5]. In [9] Klauck established separation results between $\mathbf{MA}^{cc}$ and $\mathbf{NP}^{cc}$, $\mathbf{MA}^{cc}$ and $\mathbf{APP}^{cc}$, and $\mathbf{APP}^{cc}$ and $\mathbf{PP}^{cc}$, respectively. In recent research, Buhrman et al. [4] showed $\Sigma_2^{cc}, \Pi_2^{cc} \not\subseteq \mathbf{PP}^{cc}$. This was improved to $\Sigma_2^{cc}, \Pi_2^{cc} \not\subseteq \mathbf{UPP}^{cc}$ by Razborov and Sherstov [13].

1.3.1 Reductions.

We introduce different kinds of reductions between languages. The many-one reductions are also called *rectangular reductions*. The disjunctive reductions are not needed in the sequel but defined only for the sake of completeness.

Definition 1.1. *(Reductions) Let L and L' be languages.*

1. *L is* many-one reducible *to L', if there exists a bound $b \in$ **poly** and a family of function pairs $\{(f_n, g_n)\}_{n\in\mathbb{N}}$, $f_n, g_n \colon \mathbb{B}^n \to \mathbb{B}^{\lceil 2^{b(\log n)}\rceil}$, such that for all $(x,y) \in (\mathbb{B}^n)^2$ it holds: $(x,y) \in L$ iff $(f_n(x), g_n(y)) \in L'$.*

2. *L is* Turing reducible *to L', if $L \in \mathbf{P}^{cc}(L')$.*

3. *L is* majority reducible *to L', if there exist bounds $b, t \in$ **poly** and a family of function pairs $\{(f_n, g_n)\}_{n\in\mathbb{N}}$, $f_n, g_n \colon \mathbb{B}^n \to (\mathbb{B}^{\lceil 2^{b(\log n)}\rceil})^{\lceil t(\log n)\rceil}$, such that for all $(x,y) \in (\mathbb{B}^n)^2$ it holds: $(x,y) \in L$ iff $((f_n(x))_i, (g_n(y))_i) \in L'$ for the majority of the indices $i \in [\lceil t(\log n)\rceil]$.*

4. *L is* conjunctively reducible *to L', if there exist bounds $b, t \in$ **poly** and a family of function pairs $\{(f_n, g_n)\}_{n\in\mathbb{N}}$, $f_n, g_n \colon \mathbb{B}^n \to (\mathbb{B}^{\lceil 2^{b(\log n)}\rceil})^{\lceil t(\log n)\rceil}$, such that for all $(x,y) \in (\mathbb{B}^n)^2$ it holds: $(x,y) \in L$ iff $((f_n(x))_i, (g_n(y))_i) \in L'$ for all indices $i \in [\lceil t(\log n)\rceil]$.*

5. *L is* disjunctively reducible *to L', if there exist bounds $b, t \in$ **poly** and a family of function pairs $\{(f_n, g_n)\}_{n\in\mathbb{N}}$, $f_n, g_n \colon \mathbb{B}^n \to (\mathbb{B}^{\lceil 2^{b(\log n)}\rceil})^{\lceil t(\log n)\rceil}$, such that for all $(x,y) \in (\mathbb{B}^n)^2$ it holds: $(x,y) \in L$ iff $((f_n(x))_i, (g_n(y))_i) \in L'$ for at least one of the indices $i \in [\lceil t(\log n)\rceil]$.*

1.4 Organization of this paper

In **Section 1** we prove Toda's Theorem in the setting of communication complexity. In **Section 2** we present an alternating protocol for the inner product function IP with $\mathcal{O}(\log n/\log\log n)$ many alternations, which gives us an upper bound on the number of alternations for languages in the class $\mathrm{BP}\cdot\oplus\mathbf{P}^{cc}$ of Toda's Theorem.

2 Toda's Theorem

In order to prove Toda's Theorem we need to define different kinds of operators on communication complexity classes. Readers familiar with communication complexity might wonder why the operators are defined in a *public coin style*, i.e. both players get the same witness/random string. Of course, one can define the operators such that each player gets his/her own witness/random string (*private coin style*). The reason is that these definitions are equivalent, if the operators are simulated by a protocol. Alice can guess Bob's witness and send it to him, or she can send him her random string, because the length of witnesses/random strings is bounded polylogarithmically in the length of the input.

Definition 2.1. *(Complexity class operators) For a language* $L \subseteq \mathbb{B}^{**}$ *and bounds* $p, q \in \mathbf{poly}$ *we define*

$$\begin{array}{rcl} \forall^p(L) & := & \{(x,y) \in \mathbb{B}^{**} \mid \forall w \in \mathbb{B}^{\lceil p(\log|x|)\rceil} \colon (\langle x,w\rangle, \langle y,w\rangle) \in L\} \ , \\ \exists^p(L) & := & \{(x,y) \in \mathbb{B}^{**} \mid \exists w \in \mathbb{B}^{\lceil p(\log|x|)\rceil} \colon (\langle x,w\rangle, \langle y,w\rangle) \in L\} \ , \\ \mathrm{Mod}_k^p(L) & := & \{(x,y) \in \mathbb{B}^{**} \mid |\{w \in \mathbb{B}^{\lceil p(\log|x|)\rceil} \mid (\langle x,w\rangle, \langle y,w\rangle) \in L\}| \bmod k \neq 0\}, \\ \oplus^p(L) & := & \mathrm{Mod}_2^p(L) \ . \end{array}$$

For a communication complexity class $\mathcal{C} \subseteq \mathcal{L}$ *we define*

$$\begin{array}{rcl} \mathrm{co} \cdot \mathcal{C} & := & \{\overline{L} \mid L \in \mathcal{C}\} \ , \\ \forall \cdot \mathcal{C} & := & \{\forall^p(L) \mid L \in \mathcal{C}, p \in \mathbf{poly}\} \ , \\ \exists \cdot \mathcal{C} & := & \{\exists^p(L) \mid L \in \mathcal{C}, p \in \mathbf{poly}\} \ , \\ \mathrm{Mod}_k \cdot \mathcal{C} & := & \{\mathrm{Mod}_k^p(L) \mid L \in \mathcal{C}, p \in \mathbf{poly}\} \ , \\ \oplus \cdot \mathcal{C} & := & \mathrm{Mod}_2 \cdot \mathcal{C} \ . \end{array}$$

A language L *is in* $\mathrm{BP} \cdot \mathcal{C}$ *if there exists a language* $L' \in \mathcal{C}$ *and a bound* $q \in \mathbf{poly}$ *such that for all* $(x,y) \in (\mathbb{B}^n)^2$ *it holds:*

$(x,y) \in L$ *implies* $|\{r \in \mathbb{B}^{\lceil q(\log n)\rceil} \mid (\langle x,r\rangle, \langle y,r\rangle) \in L'\}| / 2^{\lceil q(\log n)\rceil} \geq \frac{2}{3}$.

$(x,y) \notin L$ *implies* $|\{r \in \mathbb{B}^{\lceil q(\log n)\rceil} \mid (\langle x,r\rangle, \langle y,r\rangle) \in L'\}| / 2^{\lceil q(\log n)\rceil} \leq \frac{1}{3}$.

We give a definition of the polynomial hierarchy suitable for our purposes based on the class operators defined above. Note that this definition is equivalent to the one given in [1].

Definition 2.2. *(Polynomial hierarchy)* $\mathbf{PH}^{cc} := \bigcup_{k \geq 0} \Sigma_k^{cc}$, *where* $\Sigma_0^{cc} := \mathbf{P}^{cc}$ *and* $\Sigma_{k+1}^{cc} := \exists \cdot \mathrm{co} \cdot \Sigma_k^{cc}$.

We observe the following properties of the communication complexity class operators. The proofs are easy, so we omit most of them for space reasons.

Observation 2.3. *(Probability amplification) Let* $\mathcal{C} \subseteq \mathcal{L}$ *be a communication complexity class closed under majority reductions, and let* $b \in \mathbf{poly}$. *If a language* L *is in* $\mathrm{BP} \cdot \mathcal{C}$, *then there exists a language* $L' \in \mathcal{C}$ *and a bound* $q \in \mathbf{poly}$ *such that for all* $(x,y) \in (\mathbb{B}^n)^2$ *it holds:*

$(x,y) \in L \Rightarrow |\{r \in \mathbb{B}^{\lceil q(\log n)\rceil} \mid (\langle x,r\rangle, \langle y,r\rangle) \in L'\}| \ / \ 2^{\lceil q(\log n)\rceil} \geq 1 - 2^{-b(\log n)}$.

$(x,y) \notin L \Rightarrow |\{r \in \mathbb{B}^{\lceil q(\log n)\rceil} \mid (\langle x,r\rangle, \langle y,r\rangle) \in L'\}| \ / \ 2^{\lceil q(\log n)\rceil} \leq 2^{-b(\log n)}$.

Observation 2.4. *(Inclusion) Let* $\mathcal{C} \subseteq \mathcal{L}$ *be a communication complexity class closed under many-one reductions. Then* $\mathcal{C} \subseteq \mathrm{Op} \cdot \mathcal{C}$ *for every operator* $\mathrm{Op} \in \{\forall, \exists, \mathrm{Mod}_k, \oplus, \mathrm{BP}\}$.

Observation 2.5. *(Monotonicity) Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{L}$ be communication complexity classes such that $\mathcal{C} \subseteq \mathcal{D}$. Then $\mathrm{Op} \cdot \mathcal{C} \subseteq \mathrm{Op} \cdot \mathcal{D}$ for every operator $\mathrm{Op} \in \{\mathrm{co}, \forall, \exists, \mathrm{Mod}_k, \oplus, \mathrm{BP}\}$.*

Observation 2.6. *(Idempotency) Let $\mathcal{C} \subseteq \mathcal{L}$ be a communication complexity class closed under many-one reductions. Then $\mathrm{Op} \cdot \mathrm{Op} \cdot \mathcal{C} = \mathrm{Op} \cdot \mathcal{C}$ for every operator $\mathrm{Op} \in \{\forall, \exists, \oplus\}$.*

The idempotency of the BP-operator follows from its probability amplification property (Observation 2.3).

Observation 2.7. *(Idempotency of BP) Let $\mathcal{C} \subseteq \mathcal{L}$ be a communication complexity class closed under majority reductions. Then $\mathrm{BP} \cdot \mathrm{BP} \cdot \mathcal{C} = \mathrm{BP} \cdot \mathcal{C}$.*

Observation 2.8. *(co$\cdot$ vs. $\cdots$) Let $\mathcal{C} \subseteq \mathcal{L}$ be a communication complexity class. Then $\mathrm{co} \cdot \exists \cdot \mathcal{C} = \forall \cdot \mathrm{co} \cdot \mathcal{C}$, $\mathrm{co} \cdot \forall \cdot \mathcal{C} = \exists \cdot \mathrm{co} \cdot \mathcal{C}$, and $\mathrm{co} \cdot \mathrm{BP} \cdot \mathcal{C} = \mathrm{BP} \cdot \mathrm{co} \cdot \mathcal{C}$.*

Definition 2.9. *(Intersection and Union) Let $\mathcal{C}$ and $\mathcal{C}'$ be communication complexity classes. $\mathcal{C}$ is* closed under $\mathcal{C}'$-intersection *iff for all $L \in \mathcal{C}$ and $L' \in \mathcal{C}'$ we have $L \cap L' \in \mathcal{C}$. $\mathcal{C}$ is* closed under $\mathcal{C}'$-union *iff for all $L \in \mathcal{C}$ and $L' \in \mathcal{C}'$ we have $L \cup L' \in \mathcal{C}$.*

Observation 2.10. *(co$\cdot$ vs. $\oplus$) Let $\mathcal{C} \subseteq \mathcal{L}$ be a communication complexity class that contains $\mathbf{P}^{cc}$, is closed under $\mathbf{P}^{cc}$-intersection, $\mathbf{P}^{cc}$-union, and many-one reductions. Then $\mathrm{co} \cdot \oplus \cdot \mathcal{C} = \oplus \cdot \mathcal{C}$.*

Proof. Let $L \in \oplus \cdot \mathcal{C}$. There exist a bound $p \in \mathbf{poly}$ and a language $L_1 \in \mathcal{C}$ such that $L = \oplus^p(L_1)$. Define

$$
\begin{aligned}
L_2 &:= \{(\langle x, b_1 w_1\rangle, \langle y, b_2 w_2\rangle) \mid b_1, b_2 \in \mathbb{B}, (\langle x, w_1\rangle, \langle y, w_2\rangle) \in L_1\} \ , \\
L_3 &:= \{(\langle x, 1w_1\rangle, \langle y, 1w_2\rangle) \mid |x| = |y| = n, |w_1| = |w_2| = \lceil p(\log n)\rceil\} \ , \\
L_4 &:= \{(\langle x, 0w_1\rangle, \langle y, 0w_2\rangle) \mid |x| = |y| = n, w_1 = w_2 = 0^{\lceil p(\log n)\rceil}\} \ .
\end{aligned}
$$

Then L_2 is in $\mathcal{C}$, because $\mathcal{C}$ is closed under many-one reductions, and $L_3, L_4 \in \mathbf{P}^{cc}$. The language $L_5 := (L_2 \cap L_3) \cup L_4$ is in $\mathcal{C}$, because $\mathcal{C}$ is closed under $\mathbf{P}^{cc}$-intersection and $\mathbf{P}^{cc}$-union. Define $L' := \oplus^{p+1}(L_5)$. Clearly, $\overline{L} = L' \in \oplus \cdot \mathcal{C}$. □

Observation 2.11. *If $\mathcal{C} \subseteq \mathcal{L}$ is a communication complexity class closed under conjunctive reductions, then $\oplus \cdot \mathcal{C}$ is closed under conjunctive reductions.*

Using Observations 2.10 and 2.11 one can prove the result of Papadimitriou and Zachos [12] in the setting of communication complexity as in time complexity (see also [10, Prop. 4.8, p.125]).

Fact 2.12. *(Papadimitriou & Zachos) Let $\mathcal{C} \subseteq \mathcal{L}$ be a communication complexity class, which contains $\mathbf{P}^{cc}$, is closed under $\mathbf{P}^{cc}$-intersection, $\mathbf{P}^{cc}$-union, and conjunctive reductions. Then $\oplus\mathbf{P}^{cc}(\oplus \cdot \mathcal{C}) = \oplus \cdot \mathcal{C}$.*

The following observation shows that the names used for the operators are compatible with the names of classical communication complexity classes, if the operators are applied to $\mathbf{P}^{cc}$.

Observation 2.13. *(Compatibility)*

$$
\begin{aligned}
\mathbf{NP}^{cc} &= \exists \cdot \mathbf{P}^{cc} \ , & \oplus\mathbf{P}^{cc} &= \oplus \cdot \mathbf{P}^{cc} \ , \\
\textbf{co–NP}^{cc} &= \forall \cdot \mathbf{P}^{cc} \ , & \mathbf{BPP}^{cc} &= \mathrm{BP} \cdot \mathbf{P}^{cc} \ .
\end{aligned}
$$

Swapping lemmata are well known in the field of structural complexity theory. Below, we give a proof of a lemma of this type for the sake of completeness. The main ingredient is the probability amplification property of the BP-operator (Observation 2.3).

Lemma 2.14. *(Swapping) Let $\mathcal{C} \subseteq \mathcal{L}$ be a communication complexity class closed under majority reductions. Then $\oplus \cdot \mathrm{BP} \cdot \mathcal{C} \subseteq \mathrm{BP} \cdot \oplus \cdot \mathcal{C}$.*

Proof. Let L be a language in $\oplus \cdot \mathrm{BP} \cdot \mathcal{C}$. Then there exists a language L' in $\mathrm{BP} \cdot \mathcal{C}$ and a bound $p' \in \mathbf{poly}$ such that $L = \oplus^{p'}(L')$. As $L' \in \mathrm{BP} \cdot \mathcal{C}$ and $\mathcal{C}$ is closed under majority reductions we use probability amplification to obtain a language L'' in $\mathcal{C}$ and a bound $p'' \in \mathbf{poly}$ such that

$$(\langle x, w\rangle, \langle y, w\rangle) \in L' \quad \Rightarrow \quad \mathrm{Pr}_r[(\langle\langle x, w\rangle, r\rangle, \langle\langle y, w\rangle, r\rangle) \in L''] \geq 1 - 2^{-l'_n - 2} \ , \text{ and}$$
$$(\langle x, w\rangle, \langle y, w\rangle) \notin L' \quad \Rightarrow \quad \mathrm{Pr}_r[(\langle\langle x, w\rangle, r\rangle, \langle\langle y, w\rangle, r\rangle) \in L''] \leq 2^{-l'_n - 2} \ .$$

for every input $(x, y) \in (\mathbb{B}^n)^2$ and witness w. Here, $l'_n := \lceil p'(\log n)\rceil$, and the random string r is uniformly drawn from $\mathbb{B}^{l''_n}$, where $l''_n := \lceil p''(\log n)\rceil$. We define $W_{(x,y)} := \{w \in \mathbb{B}^{l'_n} \mid (\langle x, w\rangle, \langle y, w\rangle) \in L'\}$ and $\mathrm{Good}_n := \bigcap_{w \in \mathbb{B}^{l'_n}} \mathrm{Good}_{n,w}$, where $\mathrm{Good}_{n,w} := \{r \in \mathbb{B}^{l''_n} \mid \forall (x, y) \in (\mathbb{B}^n)^2 \colon (\langle\langle x, w\rangle, r\rangle, \langle\langle y, w\rangle, r\rangle) \in L'' \Leftrightarrow w \in W_{(x,y)}\}$. For a fixed w_0 we get

$$\mathrm{Pr}_r[r \notin \mathrm{Good}_n] \quad \leq \quad 2^{l'_n} \cdot \mathrm{Pr}_r[r \notin \mathrm{Good}_{n,w_0}] \leq 2^{l'_n} \cdot 2^{l'_n - 2} \leq \frac{1}{4} \ .$$

Thus, $\mathrm{Pr}_r[r \in \mathrm{Good}_n] \geq \frac{3}{4}$. The language

$$L''' := \{(\langle\langle x, r\rangle, w\rangle, \langle\langle y, r'\rangle, w'\rangle) \mid (\langle\langle x, w\rangle, r\rangle, \langle\langle y, w'\rangle, r'\rangle) \in L''\}$$

is in $\mathcal{C}$ (closure under many-one reductions).
In case $(x, y) \in L$ we have

$$\begin{aligned}
& \mathrm{Pr}_r[(\langle x, r\rangle, \langle y, r\rangle) \in \oplus^{p'}(L''')] \\
= \ & \mathrm{Pr}_r[|\{w \mid (\langle\langle x, w\rangle, r\rangle, \langle\langle y, w\rangle, r\rangle) \in L''\}| \text{ odd}] \\
\geq \ & \mathrm{Pr}_r[\forall w \colon w \in W_{(x,y)} \Leftrightarrow (\langle\langle x, w\rangle, r\rangle, \langle\langle y, w\rangle, r\rangle) \in L''] \qquad (1) \\
\geq \ & \mathrm{Pr}_r[\forall (x, y) \colon \forall w \colon w \in W_{(x,y)} \Leftrightarrow (\langle\langle x, w\rangle, r\rangle, \langle\langle y, w\rangle, r\rangle) \in L''] \\
= \ & \mathrm{Pr}_r[r \in \mathrm{Good}_n] \geq \frac{3}{4} \ ,
\end{aligned}$$

where (1) follows from $(x, y) \in L \Leftrightarrow |W_{(x,y)}| \text{ odd}$.
The case $(x, y) \notin L$ is treated similarly. We conclude $L \in \mathrm{BP} \cdot \oplus \cdot \mathcal{C}$. □

The Valiant-Vazirani-Lemma is well known in structural complexity theory, and there exist many proof ideas for this important result. The solution we propose is an adaptation of an algebraic proof due to Fortnow in [7, p.88, Lemma 3.12].

Lemma 2.15. *(Valiant-Vazirani) Let $\mathcal{C} \subseteq \mathcal{L}$ be a communication complexity class containing $\mathbf{P}^{cc}$ and closed under $\mathbf{P}^{cc}$-intersection, $\mathbf{P}^{cc}$-union and conjunctive reductions. Then $\exists \cdot \mathcal{C} \subseteq \mathrm{BP} \cdot \oplus \cdot \mathcal{C}$.*

Proof. Let L be a language in $\exists \cdot \mathcal{C}$. There exists a language $L' \in \mathcal{C}$ and a bound $p \in \mathbf{poly}$ such that $L = \exists^p(L')$. Define $l_n := \lceil p(\log n)\rceil$. We fix an input

$(x, y) \in L$, $|x| = |y| = n$. Let $S := \{w \in \mathbb{B}^{l_n} \mid (\langle x, w\rangle, \langle y, w\rangle) \in L'\}$ be the set of witnesses of (x, y) and $d := |S|$ its size. We pick a natural number m such that $2l_n d < m \leq 4l_n d$ and encode the witnesses as polynomials over $F := \mathrm{GF}(2^m)$, the finite field with 2^m elements. We then consider pairs $(a, b) \in F^2$ and show that for a sizable fraction of them there will be exactly one polynomial p representing a witness such that $p(a) = b$. The statement follows by choosing m, a and b at random. For a string $s = s_1 \cdots s_l$ we define the polynomial $p_s(X) := \sum_{i=1}^{l} s_i X^i$. We fix a witness w in S. An element a of F is called w-good, if for all witnesses $w' \neq w$ in S we have $p_w(a) \neq p_{w'}(a)$. Since p_w and $p_{w'}$ can agree on at most l_n elements, there are at least $|F| - l_n d$ many w-good elements in F. Consider the set A_w containing all pairs $(a, p_w(a))$ for w-good elements a. The sets A_w and $A_{w'}$ are disjoint for different strings w and w'. Define $A := \bigcup_{w\in S} A_w$. Then $|A| \geq d(|F| - l_n d)$. We define the language L'' in $\mathcal{C}$ by

$$\begin{aligned} L'' \quad := \quad & \{(\langle\langle x, r\rangle, w\rangle, \langle\langle y, r\rangle, w\rangle) \mid n := |x| = |y|, r = \langle m^*, a, b\rangle, m^* \in [2l_n], \\ & a, b \in \mathrm{GF}(2^{m^*}), |w| = l_n, p_w(a) = b, (\langle x, w\rangle, \langle y, w\rangle) \in L'\} \ , \end{aligned}$$

where $r = \langle m^*, a, b\rangle$ means that we use r as an encoding of a natural number m^* and field elements a and b. Furthermore, define $L''' := \oplus^p(L'') \in \oplus \cdot \mathcal{C}$.
If $(x, y) \notin L$ then for all w and r the pair $(\langle\langle x, r\rangle, w\rangle, \langle\langle y, r\rangle, w\rangle)$ is not in L'', and thus $(x, y) \notin L'''$.
If $(x, y) \in L$ then with probability $1/2l_n$ we have $m = m^*$ as $m \leq \log 4l_n d \leq 2l_n$. In case $m = m^*$ there is exactly one witness w for $(\langle x, r\rangle, \langle y, r\rangle)$ showing $(x, y) \in L'''$. The size of A is at least $l_n d^2$, the size of F^2 is at most $16 l_n^2 d^2$. If we choose (a, b) at random in F^2 we have a $1/16 l_n$ chance of being in A. Thus, for fixed input (x, y) the probability of choosing r at random such that $m = m^*$ and $(a, b) \in A$ is at least $1/32 l_n^2$.
The class $\oplus\cdot\mathcal{C}$ is closed under majority reductions by Fact 2.12. Thus, probability amplification is possible, and we get $L \in \mathrm{BP} \cdot \oplus \cdot \mathcal{C}$. $\square$

Theorem 2.16. *(Toda)* $\mathbf{PH}^{cc} \subseteq \mathrm{BP} \cdot \oplus \cdot \mathbf{P}^{cc}$.

Proof. We prove $\Sigma_k^{cc} \subseteq \mathrm{BP} \cdot \oplus \cdot \mathbf{P}^{cc}$ by induction on k:
Case $k = 0$: The class $\mathbf{P}^{cc}$ is closed under many-one reductions. The class $\oplus \cdot \mathbf{P}^{cc}$ is also closed under many-one reductions by Fact 2.12, because $\mathbf{P}^{cc}$ is closed under $\mathbf{P}^{cc}$-intersection, $\mathbf{P}^{cc}$-union, and conjunctive reductions. Thus, $\Sigma_0^{cc} = \mathbf{P}^{cc} \subseteq \oplus \cdot \mathbf{P}^{cc} \subseteq \mathrm{BP} \cdot \oplus \cdot \mathbf{P}^{cc}$ by the inclusion property of the $\oplus$- and BP-operator (Observation 2.4).
Case $k \to k + 1$: It holds

$$\begin{aligned} \Sigma_{k+1}^{cc} &= \exists \cdot \mathrm{co} \cdot \Sigma_k^{cc} && (2) \\ &\subseteq \exists \cdot \mathrm{co} \cdot \mathrm{BP} \cdot \oplus \cdot \mathbf{P}^{cc} && (3) \\ &= \exists \cdot \mathrm{BP} \cdot \mathrm{co} \cdot \oplus \cdot \mathbf{P}^{cc} && (4) \\ &= \exists \cdot \mathrm{BP} \cdot \oplus \cdot \mathbf{P}^{cc} && (5) \\ &\subseteq \mathrm{BP} \cdot \oplus \cdot \mathrm{BP} \cdot \oplus \cdot \mathbf{P}^{cc} && (6) \\ &\subseteq \mathrm{BP} \cdot \mathrm{BP} \cdot \oplus \cdot \oplus \cdot \mathbf{P}^{cc} && (7) \\ &= \mathrm{BP} \cdot \mathrm{BP} \cdot \oplus \cdot \mathbf{P}^{cc} && (8) \\ &= \mathrm{BP} \cdot \oplus \cdot \mathbf{P}^{cc} && (9) \end{aligned}$$

(2) By Definition 2.2.

(3) By the induction hypothesis for Σ_k^{cc} and monotonicity (Observation 2.5) of the operators co$\cdot$ and $\exists\cdot$.

(4) By Observation 2.8.

(5) By closure under complement of $\oplus \cdot \mathbf{P}^{cc}$ (Observation 2.10).

(6) By the Valiant-Vazirani-Lemma (Lemma 2.15). Its application is possible, because $\mathrm{BP} \cdot \oplus \cdot \mathbf{P}^{cc}$ is closed under conjunctive reductions and $\mathbf{P}^{cc}$-intersection.

(7) By the Swapping-Lemma (Lemma 2.14) and monotonicity of the BP-operator (Observation 2.5). The Swapping-Lemma can be applied, because $\oplus \cdot \mathbf{P}^{cc}$ is closed under majority reductions.

(8) By idempotency of the $\oplus$-operator (Observation 2.6).

(9) By idempotency of the BP-operator (Observation 2.7). This holds because $\oplus \cdot \mathbf{P}^{cc}$ is closed under majority reductions.

□

Fact 2.17. $\mathbf{P}^{cc}(\mathbf{PP}^{cc}) = \mathbf{P}^{cc}(\#\mathbf{P}^{cc})$.

Proof. Alice and Bob can compute every $\#\mathbf{P}^{cc}$-function f by binary search with polylog communication asking oracle queries to $\mathrm{Graph}_{\leq}(f) := \{(\langle x, v\rangle, \langle y, v\rangle) \mid (v)_2 \leq f(x,y)\} \in \mathbf{PP}^{cc}$. □

Theorem 2.18. *(Toda)* $\mathbf{PH}^{cc} \subseteq \mathrm{BP} \cdot \oplus\mathbf{P}^{cc} \subseteq \mathbf{P}^{cc}(\#\mathbf{P}^{cc}) = \mathbf{P}^{cc}(\mathbf{PP}^{cc})$.

Proof. Let $\mathrm{acc}_\Pi(x,y)$ denote the number of accepting paths of a nondeterministic protocol Π on input (x,y). The class $\#\mathbf{P}^{cc}$ contains all constant functions and is closed under addition and multiplication. If $(\Pi_n)_{n\in\mathbb{N}}$ is an efficient nondeterministic protocol family with $\mathrm{acc}_\Pi := (\mathrm{acc}_{\Pi_n})_{n\in\mathbb{N}}$ in $\#\mathbf{P}^{cc}$, and if we choose $p \in \mathbf{poly}$, then there exists an efficient nondeterministic protocol family $(\Pi'_n)_{n\in\mathbb{N}}$ such that $\mathrm{acc}_{\Pi'_n}(x,y) = (1 + \mathrm{acc}_{\Pi_n}(x,y)^{\lceil p(\log n)\rceil})^{\lceil p(\log n)\rceil}$. This proves $\mathrm{BP} \cdot \oplus\mathbf{P}^{cc} \subseteq \mathbf{P}^{cc}(\#\mathbf{P}^{cc})$ as in the time complexity setting. □

Let IP denote the *inner product* function (see [11, Ex. 1.25, p.12]), and let MAJ denote the *majority function* (see e.g. [9]). The corollary below considers the consequences of the unlikely case that the inner product or majority function can be computed with a constant number of alternations.

Corollary 2.19. *It holds:*

1. $\mathrm{IP} \in \mathbf{PH}^{cc}$ *iff* $\mathbf{PH}^{cc} = \mathrm{BP} \cdot \oplus\mathbf{P}^{cc}$.
2. *If* $\mathbf{PH}^{cc} = \mathbf{PSPACE}^{cc}$ *then* $\mathbf{PH}^{cc} = \mathrm{BP} \cdot \oplus\mathbf{P}^{cc}$.
3. $\mathrm{MAJ} \in \mathbf{PH}^{cc}$ *iff* $\mathbf{PH}^{cc} = \mathrm{BP} \cdot \oplus\mathbf{P}^{cc} = \mathbf{BP}^{cc}(\mathbf{PP}^{cc})$.
4. $\mathrm{IP} \in \mathbf{PH}^{cc}$ *iff* $\mathrm{MAJ} \in \mathbf{PH}^{cc}$.

Proof. 1. $\Rightarrow$: $\mathrm{IP} \in \mathbf{PH}^{cc}$ implies $\oplus\mathbf{P}^{cc} \subseteq \mathbf{PH}^{cc}$ because IP is complete for $\oplus\mathbf{P}^{cc}$ under many-one reductions. Applying the BP-operator yields $\mathrm{BP}\cdot\oplus\mathbf{P}^{cc} \subseteq \mathrm{BP}\cdot\mathbf{PH}^{cc} = \mathbf{PH}^{cc}$. $\Leftarrow$: $\mathrm{IP} \in \oplus\mathbf{P}^{cc} \subseteq \mathrm{BP}\cdot\oplus\mathbf{P}^{cc} = \mathbf{PH}^{cc}$.
2. Follows from $\oplus\mathbf{P}^{cc} \subseteq \mathbf{PSPACE}^{cc}$ and (1.).
3. $\Rightarrow$: $\mathrm{MAJ} \in \mathbf{PH}^{cc}$ implies $\mathbf{PP}^{cc} \subseteq \mathbf{PH}^{cc}$ because MAJ is complete for $\mathbf{PP}^{cc}$ under many-one reductions. We obtain $\mathbf{BPP}^{cc}(\mathbf{PP}^{cc}) \subseteq \mathbf{BPP}^{cc}(\mathbf{PH}^{cc}) = \mathbf{PH}^{cc} \subseteq \mathrm{BP}\cdot\oplus\mathbf{P}^{cc} \subseteq \mathbf{P}^{cc}(\mathbf{PP}^{cc}) \subseteq \mathbf{BPP}^{cc}(\mathbf{PP}^{cc})$. $\Leftarrow$: $\mathrm{MAJ} \in \mathbf{PP}^{cc} \subseteq \mathrm{BPP}(\mathbf{PP}^{cc}) = \mathbf{PH}^{cc}$.
4. Follows from (1.) and (3.). □

3 An alternating protocol for IP

The class $\mathbf{PSPACE}^{cc}$ was defined as the class of languages which can be recognized with protocols using $(\log n)^{\mathcal{O}(1)}$ communication and $\mathcal{O}(\log n)$ alternations. In this section we show that languages in the class $\mathrm{BP}\cdot\oplus\mathbf{P}^{cc}$ can be recognized by alternating protocols using only $\mathcal{O}(\log n/\log\log n)$ many alternations. It is enough to give such a protocol for the inner product function, because IP is complete for $\oplus\mathbf{P}^{cc}$, and Schöning's generalization $\mathrm{BP}\cdot\mathcal{C} \subseteq \exists\cdot\forall\cdot\mathcal{C} \cap \forall\cdot\exists\cdot\mathcal{C}$ of the well known result of Lautemann, which is easily transferred into the communication complexity context. For a proof, see [10, Prop. 2.24, p.76]. Fix an odd natural number k. Alice has input $x = x_0 \dots x_{n-1}$ and Bob has input $y = y_0 \dots y_{n-1}$. They execute the following alternating protocol $I_k(s, t, b)$ on their inputs:

If $(k \geq t - s + 1)$ **then** Alice and Bob determine if $\mathrm{IP}_{t-s+1}(x_s \dots x_t, y_s \dots y_t) = b$ using the trivial protocol (Alice sends her input; both compute the value by themselves). They return the value of the trivial protocol.

else Alice guesses the following strings and sends them to Bob:

1. $\exists S \subseteq \{0, \dots, k-1\}, |S|$ odd: (branch disjunctively)
2. $\exists \tilde{b} \in \{0, 1\}$: (branch disjunctively)
3. $\forall i \in S$: (branch conjunctively)
4. $\forall j \in \overline{S}$: (branch conjunctively)
5. $\forall h \in \{i, j\}$: (branch conjunctively)

return $I_k(s_1, t_1, b_1)$, where $d := t - s + 1$, $B := \lceil \frac{d}{k} \rceil$, $s_1 := h \cdot B$, $t_1 := \min\{n-1, (h+1)\cdot B - 1\}$, and if $(h = i)$ then $b_1 := b$ else $b_1 := \tilde{b}$.

Correctness. Divide each input x and y in an odd number k of blocks of approximately equal sizes, i.e. $x = x^{(1)} \cdots x^{(k)}$, $y = y^{(1)} \cdots y^{(k)}$. It holds $\mathrm{IP}(x, y) = \sum_{i\in[k]} \mathrm{IP}(x^{(i)}, y^{(i)}) \bmod 2$. If $\mathrm{IP}(x, y)$ evaluates to 1, there exists an odd number of blocks $S' \subseteq [k]$ where IP evaluates to 1 and the values of IP cancel on $\overline{S'}$. There are three cases:

1. $\mathrm{IP}(x^{(j)}, y^{(j)}) = 0$ for all $j \in \overline{S'}$. We set $S := S'$ and $\tilde{b} := 0$.
2. $\mathrm{IP}(x^{(j)}, y^{(j)}) = 1$ for all $j \in \overline{S'}$. We set $S := S'$ and $\tilde{b} := 1$.
3. There exist $j_0, j_1 \in \overline{S}$ such that $\mathrm{IP}(x^{(j_0)}, y^{(j_0)}) = 0$ and $\mathrm{IP}(x^{(j_1)}, y^{(j_1)}) = 1$. The number of $j \in \overline{S'}$ with $\mathrm{IP}(x^{(j)}, y^{(j)}) = 1$ has to be even. We set $S := S' \cup \{j \in \overline{S'} \mid \mathrm{IP}(x^{(j)}, y^{(j)}) = 1\}$ and $\tilde{b} := 0$. Note that $|S|$ is odd.

In all three cases we have obtained a set $S \subseteq [k]$ of odd cardinality and a $\tilde{b}$ such that $\mathrm{IP}(x^{(i)}, y^{(i)}) = 1$ for all $i \in S$ and $\mathrm{IP}(x^{(j)}, y^{(j)}) = \tilde{b}$ for all $j \in \overline{S}$. The case when $\mathrm{IP}(x, y)$ evaluates to 0 is analogous. Thus, the protocol $I_k(s, t, b)$ accepts iff $\mathrm{IP}_{t-s+1}(x_s \ldots x_t, y_s \ldots y_t) = b$. The protocol $I_k(0, n-1, 1)$ computes $\mathrm{IP}_n(x, y)$.

Communication costs. There are two alternations in each round and the number of rounds is bounded by $t = \log_2 n / \log_2 k$. If we choose an odd k of size $(\log n)^c$ then the communication costs in each round are $\mathcal{O}(k)$ bits and the number of alternations is $\mathcal{O}(\log n / \log \log n)$. If $\mathbf{AComm}_{\mathcal{A}}(\mathcal{F})$ denotes the class of languages which can be recognized by alternating protocols using communication bounded by a function in $\mathcal{F}$ and a number of alternations bounded by a function in $\mathcal{A}$, we have obtained

Theorem 3.1. $\mathrm{BP} \cdot \oplus\mathbf{P}^{cc} \subseteq \mathbf{AComm}_{\mathcal{O}(\log n / \log \log n)}((\log n)^{\mathcal{O}(1)})$.

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