Invariants of 4-dimensional singularities of integrable Hamiltonian systems

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2-atom and molecule

Definition

Let $f$ be a Morse function on a closed symplectic manifold $M^2$ and $c$ a singular value of $f$.

By **2-atom** $(P^2, f)$ we call the foliation of a neighbourhood $P^2$ of a singular fiber $f$ defined by inequality $c - \varepsilon \leq f \leq c + \varepsilon$ for sufficiently small $\varepsilon > 0$ foliated by the level lines of $f$.

We consider 2-atoms up to the fiber equivalence.

Figure: Torus and its molecule
2-atom and molecule

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The molecule of $(M^2, f)$ is the graph whose vertices are 2-atoms and edges are one-parametric families of circles.

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**f-graph**

**Definition**

For each 2-atom one can construct f-graph as follows.

Construction of f-graph:

**Figure:** Atom $C_1$ and its f-graph

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1. Mark by black color the atom boundary circles corresponding to the $c + \varepsilon$ value of the function $f$. Fix the orientation on the atom (the manifold $M^2$ is orientable) that induces the orientations of black circles.

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### Remark

Each f-graph vertex is incident to just one incoming, one outgoing, and one nonoriented edge.

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A Hamiltonian system $\dot{x} = \{x, H\}$ on symplectic manifold $M^4$ is called **integrable** iff there exists an additional independent integral $f$ in involution with Hamiltonian $H$. 

Remark

Any 2-atom is the base of Siefert bundle of a 3-atom.
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Suppose that both $H$ and $f$ are Bott functions. The foliation on the **isoenergetic surface** $Q^3_h = \{H = h\}$ generated by $f$ is called **Liouville foliation**. Suppose that this foliation has a singular fiber $L = f^{-1}(c)$. Using $f$ and $Q^3_h$ instead of $f$ and $M^2$ we define **3-atom** word-by-word as 2-atom.
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**Circular molecule**

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Suppose that \((0, 0)\) is a zero rank singular value of the momentum map. Consider the preimage of the curve in bifurcation diagram goes around \((0, 0)\). One can construct **circular molecule of the singularity** corresponding to the preimage in the same way as in the 2-dimensional case.

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Conjecture (A. T. Fomenko)

*The circular molecule completely defines 4-dimensional singularity up to Liouville equivalence.*

Figure: Marked circular molecule
Almost direct product

Definition

Suppose that a finite group $G$ symplectically acts on both atoms $(V_1, f_1)$ and $(V_2, f_2)$ and both $f_i$ are $G$-invariant.

Consider the action of $G$ on $V_1 \times V_2$ defined by 

$$\varphi(g)(x_1, x_2) = (\varphi_1(g)(x_1), \varphi_2(g)(x_2)),$$

where $\varphi_i$ is the action of $G$ on the atom $V_i$. The almost direct product of atoms $V_1$ and $V_2$ is the factor $(V_1 \times V_2)/G$ defined by such action of group $G$. 

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Definition

The triple $(V_1, V_2, G)$ is called irreducible if there is no elements in the group $G$, which act non-trivially just at one of $V_i$. 

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Invariants of 4-dimensional singularities of integrable Hamiltonian systems
The connection between integrable Hamiltonian systems and almost direct products

**Theorem (Nguen Tien Zung)**

*Any 4-dimensional singularity can be represented as the almost direct product $(V_1 \times V_2)/G$; for irreducible triple $(V_1, V_2, G)$ the representation is unique up to Liouville equivalence.*

**Figure: Bifurcation diagram of almost direct product**

**Figure: Marked circular molecule of direct product $C_1 \times C_1$**
Three examples of almost direct products with the same marked circular molecule

**Theorem (A. V. Grabezhnoy)**

Almost direct products \( (C_1 \times C_1')/\mathbb{Z}_2 \), \( (C_1 \times C_4')/\mathbb{Z}_4 \) and direct product \( C_1 \times C_1 \) have the same marked circular molecule.

*Figure:* f-graphs of atoms \( C_1' \) and \( C_4' \) and atom \( C_4' \) visualized in Wolfram Mathematica
An infinite series of almost direct products with the same circular molecule

**Theorem**

*Almost direct products* $(C_1^n \times C_1^n) / \mathbb{Z}_n$ (for every $n > 1$) *and direct product* $C_1 \times C_1$ *have the same circular molecules.*

**Figure:** f-graph of atom $M_2$

**Figure:** f-graph of atom $C_1^n$
Main definitions

Relations between 4-dim singularity and its boundary

A. V. Grabezhnoy

Results

Lemma

Let the generator of a finite cyclic group G acts by composition of rotations on $\frac{2\pi k}{n}$ and $\frac{2\pi p}{q}$ along the basic cycles, where $\frac{k}{n}$ and $\frac{p}{q}$ is irreducible fractions.

Figure: Example: torus and group action
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1. The images of basic cycles under factorization intersect each other in $GCD(n, q)$ points.

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1. **The images of basic cycles under factorization intersect each other in $\text{GCD}(n, q)$ points.**

2. **Orient the cycles corresponding to the rotations. Divide the first and the second cycles by $n$ and $q$ parts respectively and numerated them from 0 (the intersection point) w.r.t. the orientation. Suppose that the point number 1 w.r.t. the first circle is $a$ w.r.t. the second. Than $a$ satisfied the following system:**

\[
\begin{align*}
px & \equiv 1 \pmod{q} \\
 a & \equiv -kx \pmod{n}.
\end{align*}
\]

(1)
Remark

**The number** $a$ **is the numerator of** $r$-**mark**, $\text{GCD}(n, q)$ **is the denominator.**

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2. Lemma gives simple way to calculate $r$-marks for circular molecules of almost direct products. Also it is enough to know for this calculation only action of the group $G$ on boundary circles of the corresponding atoms. The structure of atoms gives restrictions on the choice of the group $G$.

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**Theorem**

All $r$-marks of the marked circular molecule for every almost direct product with group $G$ are finite and have the form $\frac{k}{|G|}$.

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px & \equiv 1 \pmod{q} \\
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Thank you for your attention!