A dynamic approach to heterogeneous elastic wires

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Generalized Euler-Bernoulli energy

Model a heterogeneous elastic wire by a closed planar **curve** γ with **density** ρ. Taking into account the interplay between shape and heterogeneity,

$$\mathcal{E}_\mu(\gamma,\rho) = \frac{1}{2}\int_\gamma \left(\beta(\rho)(\kappa - c_0)^2 + \mu\,(\partial_s\rho)^2\right) \mathrm{d}s \tag{1}$$

with the arc-length parameter s and the curvature κ of γ (see also [1]).

Model parameters:

- real analytic bending stiffness β, $\beta > 0$,
- spontaneous curvature $c_0 \in \mathbb{R}$,
- diffusivity $\mu > 0$ of the density.

- Further, we fix the length $L > 0$,
- the rotation index $\omega \in \mathbb{Z}$,
- the integral of the density as the total mass $\nu L \in \mathbb{R}$.

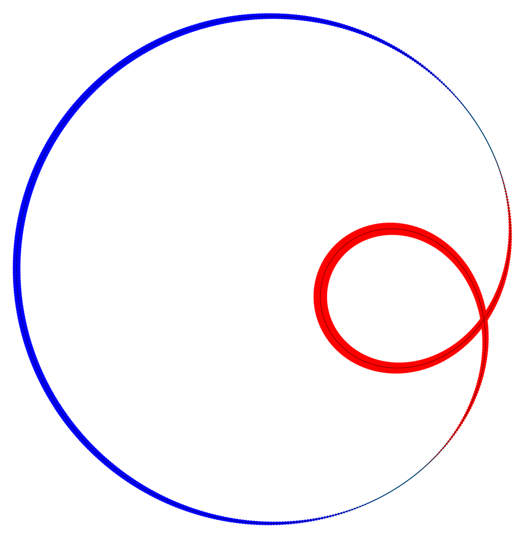

Fig. 1: Heterogeneous curve with $\omega = 2$. Color and thickness indicate the density.

Order reduction

Consider the arclength-parametrization $\gamma\colon [0, L] \to \mathbb{R}^2$ described by an inclination angle function $\theta\colon [0, L] \to \mathbb{R}$ such that

$$\partial_s\gamma = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}.$$

Then $\theta(L) - \theta(0) = 2\pi\omega$ and $\kappa = \partial_s\theta$.

Express (1) in terms of $\theta\colon [0, L] \to \mathbb{R}$ and $\rho\colon [0, L] \to \mathbb{R}$ by

$$\boxed{\mathcal{E}_\mu(\theta,\rho) = \frac{1}{2}\int_0^L \left(\beta(\rho)(\partial_s\theta - c_0)^2 + \mu\,(\partial_s\rho)^2\right) \mathrm{d}s.} \tag{2}$$

Coupled system

Decrease (2) by evolving an admissible initial datum (θ_0, ρ_0) by the associated L^2-gradient flow.

$$\begin{cases} \partial_t\theta = \partial_s\left(\beta(\rho)(\partial_s\theta - c_0)\right) + \lambda_{\theta 1}\sin\theta - \lambda_{\theta 2}\cos\theta & \text{in } (0,T)\times[0,L],\\ \partial_t\rho = \mu\partial_s^2\rho - \frac{1}{2}\beta'(\rho)(\partial_s\theta - c_0)^2 - \lambda_\rho & \text{in } (0,T)\times[0,L],\\ \theta(\cdot,L) - \theta(\cdot,0) = 2\pi\omega, \quad \rho(\cdot,L) = \rho(\cdot,0) & \text{on } [0,T),\\ \partial_s\theta(\cdot,L) = \partial_s\theta(\cdot,0), \quad \partial_s\rho(\cdot,L) = \partial_s\rho(\cdot,0) & \text{on } [0,T),\\ \theta(0,\cdot) = \theta_0, \quad \rho(0,\cdot) = \rho_0 & \text{on } [0,L]. \end{cases} \tag{3}$$

Nonlocal Lagrange multipliers: Define $\lambda_{\theta 1}(\theta,\rho)$, $\lambda_{\theta 2}(\theta,\rho)$ and $\lambda_\rho(\theta,\rho)$ such that

$$\int_0^L \sin\theta\,\mathrm{d}s = \int_0^L \cos\theta\,\mathrm{d}s = \int_0^L \rho\,\mathrm{d}s - \nu L = 0 \quad \text{for all } t \geq 0,$$

which ensures the closedness and the fixed total mass constraint.

Existence and convergence

Theorem 1 ([2]). The initial boundary value problem (3) is locally well-posed. Moreover, the solution (θ, ρ) exists globally and converges for $t \to \infty$ to a stationary solution $(\theta_\infty, \rho_\infty)$.

(Non)preservation analysis

!! Decisive advantage of working with the angle function θ: Both equations are of second order (not fourth).

→ Parabolic maximum principles are available for both equations.

Theorem 2 (Zeroset of κ). Let $c_0 = 0$. Then both the number of zeros of $\kappa = \partial_s\theta$ and the number of inflection points of the associated curve are nonincreasing in time.

Theorem 3 (Preservation of convexity). Let $c_0 = 0$. Then $\partial_s\theta_0 = \kappa_0 \geq 0$ ($\kappa_0 > 0$) on $[0, L]$ implies $\kappa \geq 0$ ($\kappa > 0$) on $[0, \infty) \times [0, L]$.

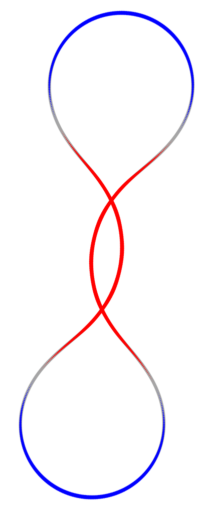

Fig. 2: 4 inflection points.

Similarly: Preservation of positivity of the density under appropriate assumptions on β.

Remarkably: If $c_0 \neq 0$, Theorem 2 and 3 are false in general!

Theorem 4 (Preservation of symmetry). Let $\omega = 1$ and $k \geq 2$. If (θ_0, ρ_0) describes a k-fold rotationally symmetric heterogeneous curve, then so does (θ, ρ) for all $t \in (0, \infty)$. Likewise, the property of being axially symmetric is transferred from (θ_0, ρ_0) to (θ, ρ).

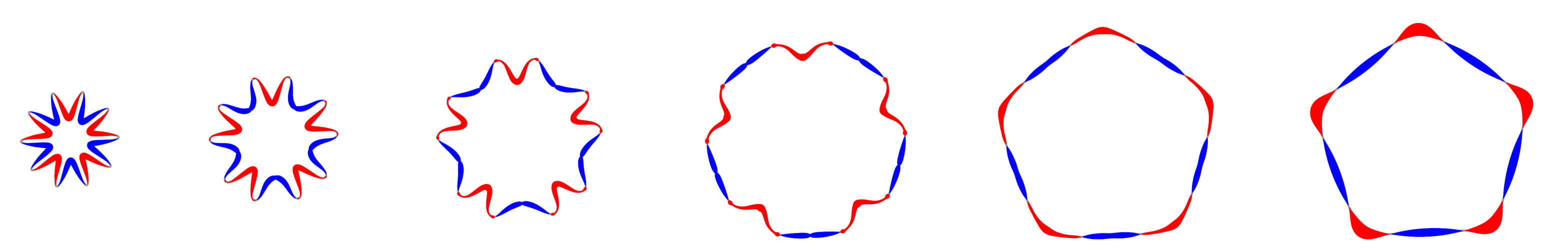

Fig. 3: Preservation of 5-fold rotational symmetry. Time increases from left to right. Parameters: $\beta(x) = e^x$, $c_0 = 0$, $\mu = 10^{-3}$, $\nu = 0$, $\omega = 1$.

Theorem 5. Let $\omega = 1$ and $k \geq 2$. If (θ_0, ρ_0) describes a k-fold rotationally symmetric heterogeneous curve with $\kappa_0 \geq c_0$ ($\kappa_0 \leq c_0$) on $[0, L]$, then $\kappa \geq c_0$ ($\kappa \leq c_0$) on $[0, \infty) \times [0, L]$.

Asymptotic behavior

Theorem 6 (Growth assumptions on β). Let β be such that

$$\beta'(x)(\nu - x) \leq \bar{C}\beta(x)(\nu - x)^2 \quad \text{for all } x \in \mathbb{R},$$

for some $\bar{C} \geq 0$. Let (θ_0, ρ_0) satisfy $\bar{C}L\mathcal{E}_\mu(\theta_0, \rho_0) < \mu$. Then, ρ converges exponentially fast to $\rho_\infty \equiv \nu$ in $L^2(0, L)$ as $t \to \infty$. Moreover, the limit θ_∞ describes an ω-fold covered circle if $\omega \neq 0$ or a multifold covered figure eight elastica if $\omega = 0$.

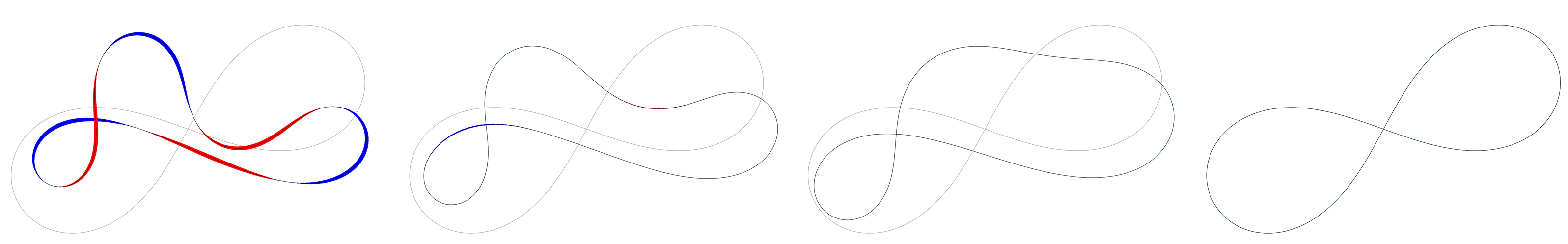

Fig. 4: Convergence to figure eight elastica with constant density. Time increases from left to right. Parameters: $\beta(x) = 0.1 + x^2$, $c_0 = 2$, $\mu = 10^{-1}$, $\nu = 0$, $\omega = 0$.

Theorem 7 (Large μ). Let $\omega \neq 0$ and $\rho_0 \equiv \nu$. If μ is large enough, then the limit $(\theta_\infty, \rho_\infty)$ describes an ω-fold covered circle with constant density.

Note: In general, the constant initial density does not remain constant.

Idea of the proof: For μ large enough, the ω-fold covered circle with constant density is the unique minimizer of (2) and locally the unique constrained critical point.

References:

[1] Brazda, Jankowiak, Schmeiser, Stefanelli. *Bifurcation of elastic curves with modulated stiffness.* Europ. J. of Appl. Math. (2022)

[2] Dall'Acqua, Langer, Rupp, *A dynamic approach to heterogeneous elastic wires.* arXiv:2205.06587 (2022)