A dynamic approach to heterogeneous elastic wires

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**Generalized Euler-Bernoulli energy**

Model a heterogeneous elastic wire by a closed planar curve \( \gamma \) with density \( \rho \). Taking into account the interplay between shape and heterogeneity,

\[
\mathcal{E}_\mu(\gamma, \rho) = \frac{1}{2} \int_0^L \left[ \beta(\rho)(\kappa - \gamma)^2 + \mu (\partial_\nu \rho)^2 \right] ds
\]

with the arc-length parameter \( s \) and the curvature \( \kappa \) of \( \gamma \) (see also [1]).

**Model parameters:**
- real analytic bending stiffness \( \beta, \gamma > 0 \),
- spontaneous curvature \( c_0 \in \mathbb{R} \),
- diffusivity \( \mu > 0 \) of the density.
- Further, we fix the length \( L > 0 \),
- the rotation index \( \omega \in \mathbb{Z} \),
- the integral of the density as the total mass \( \nu L \in \mathbb{R} \).

**Order reduction**

Consider the arclength-parametrization \( \gamma : [0, L] \to \mathbb{R}^2 \) described by an inclination angle function \( \theta : [0, L] \to \mathbb{R} \) such that

\[
\partial_s \gamma = \left( \cos \theta, \sin \theta \right)
\]

Then \( \theta(L) - \theta(0) = 2\pi \omega \) and \( \kappa = \partial_s \theta \).

Express (1) in terms of \( \theta : [0, L] \to \mathbb{R} \) and \( \rho : [0, L] \to \mathbb{R} \) by

\[
\mathcal{E}_\mu(\theta, \rho) = \frac{1}{2} \int_0^L \left( \beta(\rho)(\partial_\nu \theta - c_0)^2 + \mu (\partial_\nu \rho)^2 \right) ds
\]

**Coupled system**

Decrease (2) by evolving an admissible initial datum \( (\theta_0, \rho_0) \) by the associated \( L^2 \)-gradient flow:

\[
\begin{align*}
\partial_t \theta &= \partial_s (\beta(\rho)(\partial_\nu \theta - c_0)) + \lambda_{\theta_1} \sin \theta - \lambda_{\theta_2} \cos \theta & \text{in } (0, T) \times [0, L], \\
\partial_t \rho &= \mu \partial_\nu^2 \rho - \frac{1}{2} \beta'(\rho)(\partial_\nu \theta - c_0)^2 - \lambda_{\rho_1} \sin \theta && \text{in } (0, T) \times [0, L], \\
\theta(s, 0) &= \theta_0(s), \quad \rho(s, 0) &= \rho_0(s) & \text{on } [0, T),
\end{align*}
\]

**Nonlocal Lagrange multipliers:** Define \( \lambda_{\theta_1}(\theta, \rho), \lambda_{\theta_2}(\theta, \rho) \) and \( \lambda_{\rho_1}(\theta, \rho) \) such that

\[
\int_0^L \sin \theta ds = \int_0^L \cos \theta ds = \int_0^L \rho ds - \nu L = 0 \quad \text{for all } t \geq 0,
\]

which ensures the closedness and the fixed total mass constraint.

**Existence and convergence**

**Theorem 1** ([2]). The initial boundary value problem (3) is locally well-posed. Moreover, the solution \( (\theta, \rho) \) exists globally and converges for \( t \to \infty \) to a stationary solution \( (\theta_\infty, \rho_\infty) \).

**Order reduction**

Decouple (2) by solving the system of first order equations (3).

**(Non)preservation analysis**

!! Decisional advantage of working with the angle function \( \theta \).
- Both equations are of second order (not fourth).
- Parabolic maximum principles are available for both equations.

**Theorem 2** (Zeroset of \( \kappa \)). Let \( c_0 = 0 \). Then both the number of zeros of \( \kappa = \partial_s \theta \) and the number of inflection points of the associated curve are nonincreasing in time.

**Theorem 3** (Preservation of convexity). Let \( c_0 = 0 \). Then \( \partial_s \theta = \kappa_0 \geq 0 \) \( (\kappa_0 \geq 0) \) on \( [0, L] \) implies \( \kappa \geq 0 \) \( (\kappa \geq 0) \) on \( [0, \infty) \times [0, L] \).

Similarly: Preservation of positivity of the density under appropriate assumptions on \( \beta \).

**Remarkably:** If \( c_0 \not= 0 \), Theorem 2 and 3 are false in general!

**Theorem 4** (Preservation of symmetry). Let \( \omega = 1 \) and \( \kappa \geq 2 \). If \( (\theta_0, \rho_0) \) describes a \( k \)-fold rotationally symmetric heterogeneous curve, then so does \( (\theta, \rho) \) for all \( t \in (0, \infty) \). Likewise, the property of being axially symmetric is transferred from \( (\theta_0, \rho_0) \) to \( (\theta, \rho) \).

**Asymptotic behavior**

**Theorem 6** (Growth assumptions on \( \beta \)). Let \( \beta \) be such that

\[
\beta'(x)(v - x) \leq C \beta(x)(v - x)^2
\]

for all \( x \in \mathbb{R} \), for some \( C \geq 0 \). Let \( (\theta_0, \rho_0) \) satisfy \( C L \mathcal{E}_\mu(\theta_0, \rho_0) < \mu \). Then, \( \rho \) converges exponentially fast to \( \rho_\infty \equiv v \) in \( L^2(0, L) \) as \( t \to \infty \). Moreover, the limit \( \theta_\infty \) describes an \( \omega \)-fold covered curve if \( \omega \neq 0 \) or a multifold covered figure eight elastica if \( \omega = 0 \):

**Theorem 7** (Large \( \mu \)). Let \( \mu \not= 0 \) and \( \rho_0 \equiv v \). If \( \mu \) is large enough, then the limit \( (\theta_\infty, \rho_\infty) \) describes an \( \omega \)-fold covered curve with constant density.

**Note:** In general, the constant initial density does not remain constant.

**Idea of the proof:** For \( \mu \) large enough, the \( \omega \)-fold covered curve with constant density is the unique minimizer of (2) and locally the unique constrained critical point.