A dynamic approach to heterogeneous elastic wires

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Model

Model a heterogeneous elastic wire by a closed planar **curve** γ with **density** ρ. Extending [1], the energy of the wire is

$$\mathcal{E}_\mu(\gamma,\rho) = \frac{1}{2}\int_\gamma \left(\beta(\rho)(\kappa - c_0)^2 + \mu\,(\partial_s\rho)^2\right) \mathrm{d}s \qquad (1)$$

with the arc-length parameter s and the curvature κ of γ.

Model parameters:

- density-modulated bending stiffness $\beta \in C^\infty(\mathbb{R})$, $\beta > 0$,
- spontaneous curvature $c_0 \in \mathbb{R}$,
- diffusivity $\mu > 0$ of the density.

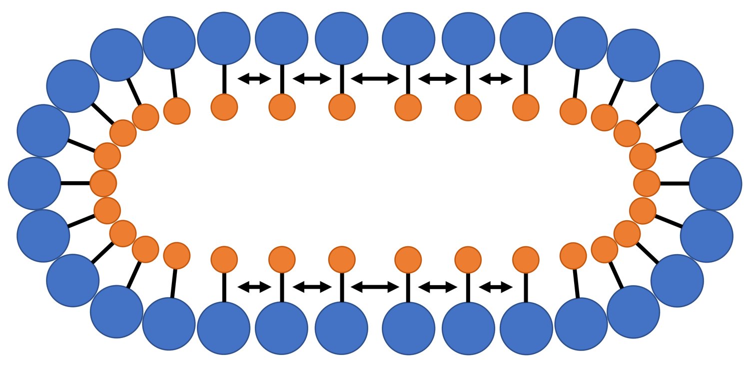

Figure 1: Visualisation of a heterogeneous wire with $c_0 \neq 0$.

This energy is invariant under orientation-preserving reparametrizations.

Order reduction

Describe the arclength-parametrization $\gamma\colon [0,L] \to \mathbb{R}^2$ by an inclination angle function $\theta\colon [0,L] \to \mathbb{R}$ such that

$$\partial_s\gamma = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}.$$

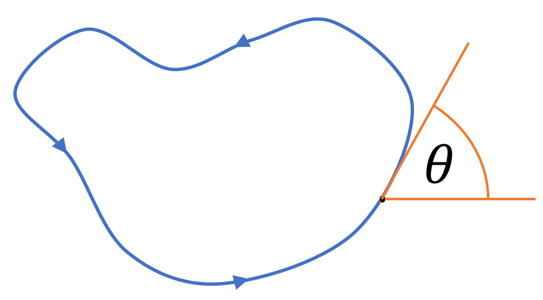

Figure 2: Angle function θ.

The angle function θ represents a C^1-closed curve if and only if

$$\int_0^L \cos\theta\,\mathrm{d}s = \int_0^L \sin\theta\,\mathrm{d}s = 0 \quad \text{and} \quad \theta(L) - \theta(0) = 2\pi\omega, \qquad (2)$$

for some $\omega \in \mathbb{Z}$, the rotation index of the curve.

Express (1) in terms of $\theta\colon [0,L] \to \mathbb{R}$ and $\rho\colon [0,L] \to \mathbb{R}$ by

$$E_\mu(\theta,\rho) = \frac{1}{2}\int_0^L \left(\beta(\rho)(\partial_s\theta - c_0)^2 + \mu\,(\partial_s\rho)^2\right) \mathrm{d}s, \qquad (3)$$

using that $\kappa = \partial_s\theta$.

Goal

Use a dynamic approach to minimize (3) by evolving (θ,ρ) by the associated L^2-gradient flow. Thereby, require that

- the curve γ remains C^2-closed,
- the density ρ remains C^1-periodic and $\int_0^L \rho\,\mathrm{d}s = \int_0^L \rho_0\,\mathrm{d}s. \qquad (4)$

Boundary value problem

$$\begin{cases} \partial_t\theta = \partial_s\left(\beta(\rho)(\partial_s\theta - c_0)\right) + \lambda_{\theta 1}\sin\theta - \lambda_{\theta 2}\cos\theta & \text{in } (0,T)\times[0,L],\\ \partial_t\rho = \mu\partial_s^2\rho - \frac{1}{2}\beta'(\rho)(\partial_s\theta - c_0)^2 - \lambda_\rho & \text{in } (0,T)\times[0,L],\\ \theta(\cdot,L) - \theta(\cdot,0) = 2\pi\omega, \quad \rho(\cdot,L) = \rho(\cdot,0) & \text{on } [0,T),\\ \partial_s\theta(\cdot,L) = \partial_s\theta(\cdot,0), \quad \partial_s\rho(\cdot,L) = \partial_s\rho(\cdot,0) & \text{on } [0,T),\\ \theta(0,\cdot) = \theta_0, \quad \rho(0,\cdot) = \rho_0 & \text{on } [0,L] \end{cases} \qquad (5)$$

☹ The L^2-gradient flow is a quasilinear coupled parabolic system that is nonlocal due to the Lagrange multipliers $\lambda_{\theta 1}$, $\lambda_{\theta 2}$ and λ_ρ.

☺ Working with the angle function θ reduces the order of the equation for γ from fourth to second order.

Nonlocal Lagrange multipliers

Closedness-constraint: Lagrange multipliers $\lambda_{\theta 1}$ and $\lambda_{\theta 2}$. Define

$$\begin{pmatrix}\lambda_{\theta 1}(t)\\ \lambda_{\theta 2}(t)\end{pmatrix} := \Pi^{-1}(\theta)(t)\int_0^L \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \partial_s\left(\beta(\rho)(\partial_s\theta - c_0)\right)\mathrm{d}s,$$

where $\Pi^{-1}(\theta)(t)$ denotes the inverse of the matrix

$$\Pi(\theta)(t) := \begin{pmatrix} \int_0^L \sin^2\theta\,\mathrm{d}s & -\int_0^L \cos\theta\sin\theta\,\mathrm{d}s\\ -\int_0^L \cos\theta\sin\theta\,\mathrm{d}s & \int_0^L \cos^2\theta\,\mathrm{d}s \end{pmatrix}.$$

This ensures (2) along the flow.

Fixed total mass-constraint: Lagrange multiplier λ_ρ. With

$$\lambda_\rho(t) := -\frac{1}{2L}\int_0^L \beta'(\rho)(\partial_s\theta - c_0)^2\,\mathrm{d}s,$$

a solution (θ,ρ) satisfies (4).

Local well-posedness

Theorem 1 ([2]). Let $(\theta_0,\rho_0) \in h^{1+\alpha}([0,L])$ (little Hölder space) for some $\alpha \in (0,1)$ such that

$$\theta_0(L) - \theta_0(0) = 2\pi\omega, \quad \partial_s\theta_0(L) = \partial_s\theta_0(0),$$
$$\rho_0(L) = \rho_0(0), \quad \partial_s\rho_0(L) = \partial_s\rho_0(0)$$

and θ_0 satisfies (2). Then there exist $T_0 > 0$ and a unique solution

$$(\theta,\rho) \in C^\infty((0,T_0)\times[0,L])$$

of (5) on $(0,T_0)\times[0,L]$ satisfying the initial condition in the sense that

$$\lim_{t\to 0}(\theta(t),\rho(t)) = (\theta_0,\rho_0) \quad \text{in } C^{1+\alpha}([0,L]).$$

Moreover, the solution depends continuously on the initial datum.

Idea of the proof

Local Existence: Transform (5) to an equivalent problem in a periodic setting to get rid of the boundary conditions. Apply the Inverse Function Theorem between appropriate time-weighted Hölder spaces.

!! Working with time-weights makes the arguments more challenging, but provides well-posedness for weak initial data.

Global existence

Theorem 2 ([2]). The unique smooth solution (θ,ρ) of Theorem 1 exists up to $T = \infty$ and subconverges to a stationary solution of (5).

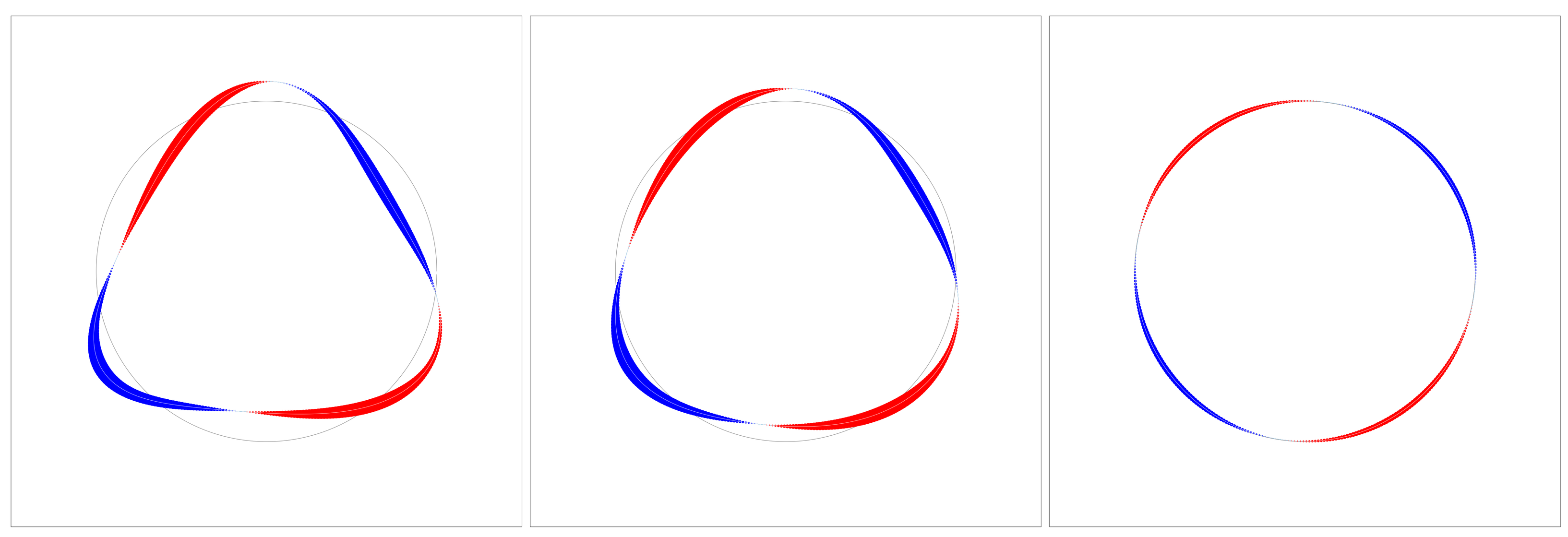

Figure 3: Numerical experiments on the convergence (thickness and color of the curve describe ρ) by Gaspard Jankowiak, University of Constance.

References:

[1] Brazda, Jankowiak, Schmeiser, Stefanelli. *Bifurcation of elastic curves with modulated stiffness.* Europ. J. of Appl. Math. (2022)

[2] Dall'Acqua, Langer, Rupp, *A dynamic approach to heterogeneous elastic wires.* arXiv:2205.06587 (2022)

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