Advanced Topics in the Calculus of Variations
Blatt 5

A28a)

<u>Def</u> (a) f hat in $\mathcal{F}$ eine C^0 - feine Approximation

$\iff \forall \varepsilon: \Omega \to \mathbb{R}_+ \quad \exists g \in \mathcal{F}: \quad |g(x) - f(x)| < \varepsilon(x) \quad \forall x \in \Omega.$

<u>Bew</u> Wähle $\varepsilon_n: \Omega \to \mathbb{R}_+ \quad \varepsilon_n(x) := \frac{1}{n}$.

$\Rightarrow \forall n \in \mathbb{N} \quad \exists g_n \in \mathcal{F} \quad : \quad |g_n(x) - f(x)| < \varepsilon_n(x) \quad \forall x \in \Omega$

$\Rightarrow \forall n \in \mathbb{N} \; \exists (g_n)_{n \in \mathbb{N}} \in \mathcal{F} \; : \; \|g_n - f\|_\infty \leq \frac{1}{n}$

$\Longrightarrow g_n \xrightarrow{n \to \infty} f$ glm. und $(g_n)_{n \in \mathbb{N}} \subset \mathcal{F}$.

(b) Setze $f :\equiv 0$ und $f_n: (0,1) \to \mathbb{R}$, $f_n \equiv 1/n$

$f_n \to f$ glm auf $(0,1)$ aber:

Sei: $\varepsilon(x) := x$. Dann $\varepsilon: (0,1) \to \mathbb{R}_+$ da

$\varepsilon(x) > 0 \; \forall x \in \Omega = (0,1)$.

Aber

$\forall n \in \mathbb{N} \qquad |f_n(x) - f(x)| = \frac{1}{n} \not< x \qquad \forall x \in (0,1)$

(c) Sei $\varepsilon: \Omega \to \mathbb{R}_+ \quad \varepsilon(x) := \operatorname{dist}(x, \Omega^c)$.

Da Ω offen gilt $\varepsilon(x) > 0 \; \forall x \in \Omega$.

<u>Nun</u> $\exists \; g \in \mathcal{F}: \quad |f(x) - g(x)| < \operatorname{dist}(x, \Omega^c)$

<u>Beh 1</u> $\forall (x_n) : \exists y \in \partial\Omega : x_n \to y$ gilt $(f(x_n))_{n=1}^\infty$ Cauchy

Beachte: Da g stetige Fortsetzung auf $\bar{\Omega}$ hat ist

Bew sei $(x_n) \to y$ beliebig

$$|f(x_n) - f(x_m)| \le |f(x_n) - g(x_n)| + |g(x_n) - g(x_m)|$$

$$+ |f(x_m) - g(x_m)|$$

$$\le \operatorname{dist}(x_n, \Omega^c) + \operatorname{dist}(x_m, \Omega^c) + |g(x_n) - g(x_m)|$$

Nun sei $\varepsilon > 0$. Dann $\exists\, n_0 \in \mathbb{N} : \left(n \ge n_0 \right.$

$$\left. \Longrightarrow \operatorname{dist}(x_n, \Omega^c) < \frac{\varepsilon}{3} \right)$$

Auch $\exists\, n_1 \in \mathbb{N} \quad : \left(n, m \ge n_1 \Rightarrow |g(x_n) - g(x_m)| < \frac{\varepsilon}{3} \right)$

Seien nun $n, m \ge \max(n_0, n_1)$

$$\Longrightarrow |f(x_n) - f(x_m)| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$$\Longrightarrow \exists \lim_{n \to \infty} f(x_n).$$

Nun für $y \in \partial\Omega$ wähle $x_n \xrightarrow{n \to \infty} y$, $x_n \subset \Omega$

und setze $f(y) := \lim_{n \to \infty} f(x_n)$.

Z $f(y)$ hängt nicht von der Wahl der Folge $(x_n)_{n \in \mathbb{N}}$ ab.

Beweis Seien $x_n \to y$, $\hat{x}_n \to y$

Z $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(\hat{x}_n)$

Sei $z_n := \begin{cases} x_k & 2k = n \\ \tilde{x}_k & 2k+1 = n \end{cases}$

Dann $z_n \to y$! (y muss der einzige Häufungswert sein)

$\Longrightarrow \exists \lim_{n\to\infty} f(z_n)$.

Da $(x_n), (\tilde{x}_n)$ ~~Folgen~~ Teilfolgen von z_n sind

so gilt $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(\tilde{x}_n) = \underbrace{\lim_{n\to\infty} f(z_n)}_{= f(y)}$.

Außerdem $g|_{\partial\Omega} = f|_{\partial\Omega}$ da $|g(x) - f(x)| \leq \operatorname{dist}(x, \partial\Omega^c)$ $\forall x \in \Omega$. & $g, f \in C(\bar{\Omega})$

(d) Wähle $f \equiv 0$ und $f_n : (0,1) \to \mathbb{R}$:

$$f_n(x) := \begin{cases} 0 & x \in (0, \frac{1}{2n}) \\ 2(x - \frac{1}{2n}) & x \in (\frac{1}{2n}, \frac{1}{n}) \\ \frac{1}{n} - 2(x - \frac{1}{n}) & x \in (\frac{1}{n}, \frac{3}{2n}) \\ 0 & x \in (\frac{3}{2n}, 1) \end{cases}$$

f_n glm stetig! $f_n(0) = f(0)$, $f_n(1) = f(1)$

Sei aber nun $\varepsilon(x) := \frac{x}{2}$, dann $\varepsilon : (0,1) \to \mathbb{R}_+$

aber

$\nexists n \in \mathbb{N} : |f_n(x) - f(x)| < \varepsilon(x)$

da $f_n(\frac{1}{n}) = \frac{1}{n}$, $\varepsilon(\frac{1}{n}) = \frac{1}{2n}$!

$\boxed{\text{Aufgabe 29}}$

(a) Seien $x, y \in \mathbb{R}^M$, $\lambda \in (0,1)$

$\underline{\underline{Z}}$ $\operatorname{dist}(\lambda x + (1-\lambda)y, Z) \leq \lambda \operatorname{dist}(x, Z) + (1-\lambda) \operatorname{dist}(y, Z)$,

<u>Bew</u> Sei $(x_n)_{n=1}^{\infty} \subset Z$: ~~dist (x, Z~~ $\|x_n - x\| \longrightarrow \operatorname{dist}(x, Z)$

$(y_n)_{n=1}^{\infty} \subset Z$: $\|y_n - y\| \longrightarrow \operatorname{dist}(y, Z)$.

Dann gilt $(\lambda x_n + (1-\lambda) y_n)_{n=1}^{\infty} \subset Z$ da Z konvex

$\Longrightarrow \operatorname{dist}(\lambda x + (1-\lambda)y, Z) \leq \|\lambda x + (1-\lambda)y - (\lambda x_n + (1-\lambda)y_n)\|$

$\leq \lambda \|x - x_n\| + (1-\lambda) \|y - y_n\|$

$\xrightarrow{(n\to\infty)} \lambda \operatorname{dist}(x, Z) + (1-\lambda) \operatorname{dist}(y, Z)$

$\Longrightarrow$ Beh.

(b) ~~Sei~~ ~~u_2~~ Angenommen $A, B, C \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$ sind so dass

$\forall \delta \in \left(0, \frac{|A-B|}{2}\right)$ hat $x \mapsto Cx + b$ eine

C^0-feine Approximation in

$\mathcal{F} := \{ f : \Omega \to \mathbb{R}^m$ stückweise affin :

$\operatorname{dist}(Du(x), \{A, B\}) < \delta \}$

Sei $j \in \mathbb{N}$. Wähle $\delta = 1/j$

Dann gilt $\exists\, u_j : \Omega \to \mathbb{R}^m$ stückweise affin mit

$\operatorname{dist}(Du_j(x), \{A, B\}) < \frac{1}{j}$, $\|u_j - u\|_{L^\infty} < \frac{1}{j}$

(Wegen der Definition einer C^0-feinen Approximation für $\varepsilon(x) := \frac{1}{j}$) dann $\|Du_j\|_{L^\infty} \leq \max(\|A\|, \|B\|) + \frac{1}{j}$

Nun gilt $(u_j) \subset W^{1,\infty}(\Omega)$ beschränkt und $u_j \xrightarrow{L^\infty(\Omega)} u$

Nach Auswahl einer Teilfolge gilt auch $u_j \xrightarrow{W^{1,2}(\Omega)} u$.

in $W^{1,2}(\Omega)$, da $(u_j) \subset W^{1,2}(\Omega)$ beschränkt und $u_j \xrightarrow{L^2(\Omega)} u$.

Sei nun $x_0 \in \Omega$ und $r > 0$ so, dass $\overline{B_r(x_0)} \subset \Omega$.

Dann gilt

$$\int_{B_r(x_0)} Du_j(x)\, dx = \int_{\partial B_r(x_0)} \left(u_j \otimes \frac{x - x_0}{r}\right) dS(x)$$

$$\xrightarrow{j \to \infty} \int_{\partial B_r(x_0)} \left((Cx + b) \otimes \frac{x - x_0}{r}\right) dS(x)$$

$$= \int_{B_r(x_0)} D(Cx + b)\, dx$$

$$= C\, |B_r(x_0)|$$

$$\Rightarrow C = \lim_{j \to \infty} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} Du_j(x)\, dx \ \overset{u_j \xrightarrow{W^{1,2}} u}{=\!=} \ \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} Du(x)\, dx$$

<u>Beh</u> $Du(x) \in \underbrace{\operatorname{Conv}(\{A, B\})}_{:= \{\lambda A + (1-\lambda) B \mid \lambda \in [0,1]\}}$ für

Bew

Sei $\mathcal{E}: L^2(\Omega) \longrightarrow \mathbb{R}$ gegeben durch

$$\mathcal{E}(v) := \int \mathrm{dist}(v(x), \mathrm{Conv}(\{A, B\})\, dx$$

Nun AZ9a $\Longrightarrow \mathcal{E}$ konvex (da $\mathrm{dist}(\cdot, \mathrm{Conv}(\{A,B\}))$ konvex!)

Auch: $\mathcal{E}$ unterhalbstetig denn sei $v_n \longrightarrow v$ in

L^2. Nun $\exists\, (v_{n_k})$ TF: $\liminf\limits_{n\to\infty} \mathcal{E}(v_n) = \lim\limits_{k\to\infty} \mathcal{E}(v_{n_k})$

$\Longrightarrow \exists$ TF $(v_{n_{k_\ell}})$: $v_{n_{k_\ell}} \longrightarrow v$ f.ü.

$$\Longrightarrow \mathcal{E}(v) = \int \mathrm{dist}(v(x), \mathrm{Conv}(\{A, B\})), dx$$

$$= \int \liminf_{\ell\to\infty} \mathrm{dist}(v_{n_{k_\ell}}(x), \mathrm{Conv}(\{A,B\})$$

$$\overset{\text{Fatou}}{\leq} \liminf_{\ell\to\infty} \int \mathrm{dist}(v_{n_{k_\ell}}(x), \mathrm{Conv}(\{A,B\}))\, dx$$

$$\leq \liminf_{\ell\to\infty} \mathcal{E}(v_{n_{k_\ell}}) = \liminf_{n\to\infty} \mathcal{E}(v_n)$$

$\Longrightarrow \mathcal{E}(v) \leq \liminf\limits_{n\to\infty} \mathcal{E}(v_n)$

$\Longrightarrow \mathcal{E}$ konvex und unterhalbstetig

$\Longrightarrow \mathcal{E}$ schwach unterhalbstetig.

Nun $Du_n \xrightarrow{L^2(\Omega)} Du$

$$\Longrightarrow \int_\Omega \operatorname{dist}\left(\nabla u(x), \operatorname{Conv}(\{A,B\})\right) dx$$

$$= \mathcal{E}(\nabla u) \leq \liminf_{j \to \infty} \mathcal{E}(\nabla u_j) = \liminf_{j \to \infty} \int_\Omega \underbrace{\operatorname{dist}\left(\nabla u_j(x), \operatorname{Conv}(\{A,B\})\right)}_{\substack{\leq \operatorname{dist}(\nabla u_j, \{A,B\}) \\ \leq \frac{1}{j}}} dx$$

$$\leq \liminf_{j \to \infty} \frac{|\Omega|}{j} = 0.$$

Aufgabe 30

(a) <u>Abgeschlossenheit</u>

Sei $(x_n) \subset K^{rc}$: $x_n \longrightarrow x$ z.z. $x \in K^{rc}$

<u>Beweis</u> Sei $f|_K \leq 0$, f Rang 1-konvex.

z.z. $f(x) \leq 0$

$$f(x) \xrightarrow[\Rightarrow \text{ stetig}]{\text{Rang 1 konvex}} \lim_{n\to\infty} \underbrace{f(x_n)}_{\leq 0 \text{ da } x_n \in K^{rc}} \leq 0$$

$$\Longrightarrow f(x) \leq 0 \Longrightarrow x \in K^{rc}$$

<u>Beschränktheit</u> $\exists R > 0 : K \subset B_R(0)$.

Nun $f : \mathbb{R}^{m \times d} \longrightarrow \mathbb{R}$ mit

$f(X) := |X|^2 - R$ ist

Rang 1-konvex (da konvex) und

$$f|_K \leq 0 \overset{\text{Def}}{\Longrightarrow} f|_{K^{rc}} \leq 0$$

$$\Longrightarrow \forall X \in K^{rc} \quad |X|^2 - R < 0$$

$$\Longrightarrow K^{rc} \subset B_R(0)$$

(b) z.z. $K^{rc} \subset K^{co}$.

Sei ~~$f \in K^r$~~ $X \in K^{rc}$. z.z. $X \in K^{co}$

~~$\forall$ Sei K^{co}~~

Nun $Y \in K^{co} \iff \Big[f: \mathbb{R}^{m \times d} \to \mathbb{R}$ konvex $f|_K \leq 0 \Rightarrow f(Y) \leq 0 \Big]$.

Sei nun $f: \mathbb{R}^{m\times d} \to \mathbb{R}$ konvex mit

$f|_K \leq 0$

$\Longrightarrow f: \mathbb{R}^{m \times d} \to \mathbb{R}$ rang-1-konvex mit

$f|_K \leq 0$

$\Longrightarrow f(X) \leq 0 \implies$ Beh ($X \in K^{co}$)

A30(c)

$K := \{ A \in O_2(\mathbb{R}) \mid \det A = -1 \}$

<u>Beh</u> $0 \in K^{co}$ aber $0 \notin K^{rc}$

$0 \in K^{co}$ denn

$$0 = \frac{1}{2} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\in O_2(\mathbb{R}) \\ \det = -1}} + \frac{1}{2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\substack{\in O_2(\mathbb{R}) \\ \det = -1}}$$

Nun $0 \notin K^{rc}$ da

$\det A + 1 \leq 0$ auf K und

$A \mapsto \det A + 1$ rang-1-konvex

$\Longrightarrow \det A + 1 \leq 0$ auf K^{rc}

nach Definition von K^{rc}

$\Longrightarrow \det A \leq -1 \quad \forall A \in K^{rc}$

$\Longrightarrow 0 \notin K^{rc}$

$\Longrightarrow K^{rc} \neq K^{co}$

$\overset{\text{A30(b)}}{\Longrightarrow} K^{rc} \subsetneq K^{co}$

Nun für

$\tilde{K} := \{A \in O_2(\mathbb{R}) \mid \det A = 1\}$

gilt $O \in \tilde{K}^{co}$ da

$O = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

aber $O \notin \tilde{K}^{rc}$ da

$1 - \det A \leq 0$ auf $\tilde{K}$ und

$A \longmapsto 1 - \det A$ rang-1-konvex (Warum? :D)

$\Longrightarrow 1 - \det A \leq 0$ auf $\tilde{K}^{rc}$

$\Longrightarrow \det A \geq 1$ auf $\tilde{K}^{rc}$

$\Longrightarrow O \notin \tilde{K}^{rc}$

Aufgabe 31

Z Sei $K \subset \mathbb{R}^{m \times d}$ kompakt, $U \supset K$ offen

$\nu \in \mathcal{P}^{rc}(K)$, $\delta > 0$, $b \in \mathbb{R}^m$. Dann hat

$x \mapsto \bar{\nu} x + b$ eine C^0-feine

Approximation in

$\mathcal{F} := \{ u : \Omega \to \mathbb{R}^m$ stückweise affin mit
$Du(x) \in U^{rc}$ f.ü. $\}$

Bew Wir zeigen $(x \mapsto \bar{\nu} x + b) \in \mathcal{F}$.
Dann hat diese Funktion sicherlich eine
C^0-feine Approximation in $\mathcal{F}$ (nämlich
wird sie beliebig gut durch sich selbst
approximiert!)

Bew $x \mapsto \bar{\nu} x + b$ stückweise affin ✓

$\nabla(\bar{\nu} x + b) = \bar{\nu} \in K^{rc}$ laut Aufgabe 22(a

$\overset{K^{rc} \subset U^{rc}}{\Longrightarrow} \nabla(\bar{\nu} x + b) \in U^{rc}$ fü

$\Longrightarrow (x \mapsto \bar{\nu} x + b) \in \mathcal{F}$

$\Longrightarrow$ Beh.

A32 (α) Wir zeigen „$\subset$" und „$\supset$"

1) „$\subset$" $\partial B_\alpha(0)^{rc} \subset \overline{\partial B_\alpha(0)^{c_0}} = \overline{B_\alpha(0)}$

2) „$\supset$" Sicherlich gilt $\partial B_\alpha(0) \subset \partial B_\alpha(0)^{rc}$

Noch z.z. $B_\alpha(0) \subset \partial B_\alpha(0)^{rc}$

Bew Sei $X \in \mathbb{R}^{m \times d}$: $\|X\|^2 < \alpha^2$

$\Rightarrow \|X\|^2 < \alpha^2$

$\Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{d} (x_{ij})^2 < \alpha^2$

Nun

$$\|X + t(e_1 \otimes e_1)\|^2 = \|X\|^2 + 2t\,(X, e_1 \otimes e_1)_{\mathbb{R}^{m \times d}} + t^2 =: \tilde{f}(t)$$

Setze $f(t) := \|X + t(e_1 \otimes e_1)\|^2 - \alpha^2$

Nun $\lim_{t \to \pm\infty} f(t) = \infty$, f quadratisches Polynom und $f(0) < 0$.

$\Longrightarrow$ f hat zwei Reelle Nullstellen

$\Longrightarrow \exists\, \beta_1, \beta_2 \in \mathbb{R} : f(\beta_1) = f(\beta_2) = 0$ $\beta_1 < 0 < \beta_2$

$\Leftarrow$ Setze

$$A_1 := X + \beta_1 (e_1 \otimes e_1)$$
$$A_2 := X + \beta_2 (e_1 \otimes e_1)$$

Dann $A_1, A_2 \in \partial B_\alpha(0)$ und

A_1, A_2 rang-1-verbunden. Beachte $0 < \beta_2$, $\beta_1 < 0$

<u>Nun</u> $\dfrac{\beta_2}{\beta_2 - \beta_1}, \dfrac{-\beta_1}{\beta_2 - \beta_1} \in (0,1)$

und sie addieren sich zu 1

$$\Longrightarrow \partial B_\alpha(0)^{rc} \ni \frac{\beta_2}{\beta_2 - \beta_1} A_1 + \frac{-\beta_1}{\beta_2 - \beta_1} A_2$$

$$= \frac{\beta_2}{\beta_2 - \beta_1} \left(X + \beta_1 (e_1 \otimes e_1)\right) + \frac{-\beta_1}{\beta_2 - \beta_1} \left(X + \beta_2 (e_1 \otimes e_1)\right)$$

$$= X$$

$$\Longrightarrow X \in \partial B_\alpha(0)^{rc}$$

$$\Longrightarrow B_\alpha(0) \subset \partial B_\alpha(0)^{rc}$$

$$\Longrightarrow \overline{B_\alpha(0)} \subset \partial B_\alpha(0)^{rc}$$

1b) ~~Zeige~~ Gromov-Hausdorff-artige Konvergenz

(A) $\lim_{n\to\infty} \sup_{x\in U_n} \operatorname{dist}(x, \partial B_1(0))$

$\leq \limsup_{n\to\infty} \left(\frac{1}{n}\right) = 0.$

(B) Z $U_{n+1}^{rc} \supset U_n \quad \forall n\in\mathbb{N}.$

Bew

$$U_{n+1}^{rc} = \bigcup_{K\subset U_{n+1}} K^{rc} \supset \partial B_{1-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)}(0)^{rc}$$

$$\overset{A\,32(a)}{=\!=} \overline{B_{1-\frac{1}{2}\left(\frac{1}{n+2}+\frac{1}{n+1}\right)}(0)}$$

$$\supset \overline{B_{1-\frac{1}{n+1}}(0)} \supset U_n.$$

$\Longrightarrow$ Beh,

A33

(a) In Approximation von $\{-1, 1\}$

$\hat{=}$ In-Approximation von $\partial B_1(0)$

Setze $U_1 = B_{\frac{1}{2}}(0)$

$U_2 = B_{\frac{2}{3}}(0) \setminus B_{\frac{1}{2}}(0)$

$U_3 = B_{\frac{3}{4}}(0) \setminus B_{\frac{2}{3}}(0)$

$\vdots$

$\xrightarrow[\text{A32(b)}]{\text{wie}}$ $(U_n)_{n=1}^{\infty}$ In-Approximation von $\{-1, 1\} \hat{=} \partial B_1(0)$

Nun Theorem 24 (Konvexe Integration)

$K \subset \mathbb{R}^{m \times d}$ kompakt, $(U_i)_{i=1}^{\infty}$ In-Approximation für K

$v: \Omega \to \mathbb{R}^m$ $C^1_{(\bar{\Omega})}$-Funktion : $Dv(x) \in U_1 \ \forall x \in \Omega$

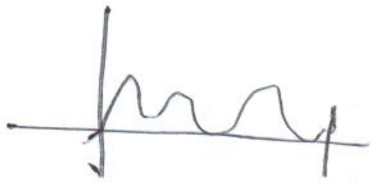
$\Rightarrow \exists$ C^0-feine Approximation ~~mit U:Ω~~

in $\mathcal{F} := \{ u: \Omega \to \mathbb{R}^m$ Lipschitz : $Du(x) \in K$ f.ü.$\}$.

Sei $K = \{-1, 1\}$, $(U_n)_{n=1}^{\infty}$ wie davor

Setze $V(x) \equiv 0$ $\nabla V(x) = 0 \in U_1$

$\overset{\text{Thm } 24}{\Longrightarrow} \exists\, C^0$-feine Approximation in $\mathbb{F}$



6.1. **A toy example.**

We will consider here a toy model, where the considerations above can be demonstrated and which has been suggested by the anonymous referee to [30]. Consider the problem of exhibiting functions $u : [0,1] \to \mathbb{R}$ such that $|u| = 1$. In the context of (23)-(24) this corresponds to $K = \{-1, 1\}$ and the differential constraint being void. The following scheme aims at producing such functions. We assume to start with a given function $u_0 : [0,1] \to \mathbb{R}$ and build a sequence with the following iteration scheme:

$$u_{k+1}(x) = u_k(x) + \tfrac{1}{2}\left[1 - u_k^2(x)\right] s(\lambda_k x),$$

where $s : \mathbb{R} \to \mathbb{R}$ is the 1-periodic extension of $\mathbf{1}_{(0,1/2]} - \mathbf{1}_{(1/2,1]}$ and $\lambda_k > 1$ is a sequence of frequencies still to be fixed. The following assertions are straightforward:

- If $\sup_{[0,1]} |u_k| < 1$, then also $\sup_{[0,1]} |u_{k+1}| < 1$.
- If $\sup_{[0,1]} |u_0| < 1$ and $u_k \to u_\infty$ in $L^1(0,1)$, then $|u_\infty| = 1$ a.e.

Therefore, in order to produce a solution to our toy problem, it suffices to choose the sequence $\{\lambda_k\}$ so to ensure the strong convergence of u_k. To this end observe that

$$\int_0^1 |u_{k+1}|^2\, dx = \int_0^1 \left(|u_k|^2 + \tfrac{1}{4}(1 - u_k^2)^2 + u_k(1 - u_k^2)s(\lambda_k x)\right) dx.$$

Moreover, as $\lambda \to \infty$, we have $s(\lambda x) \rightharpoonup 0$ in $L^2(0,1)$. Therefore, by choosing λ_k sufficiently large (depending on u_k), we can ensure that

$$\int_0^1 |u_{k+1}|^2\, dx \geq \int_0^1 |u_k|^2\, dx + \int_0^1 \tfrac{1}{8}(1 - u_k^2)^2\, dx.$$

The strong convergence follows then easily. Here we see that choosing λ_k to be a rapidly increasing sequence "helps" the strong convergence of the scheme.

However, it is also clear that for any additional regularity of the limit u_∞ one should choose λ_k to increase as slowly as possible. More precisely, the optimal regularity that is reachable via this iteration scheme will depend on the connection between the choice of λ_k with the rate of convergence of the scheme. To see this, observe that – roughly speaking – fractional Sobolev regularity of u_∞ will follow from interpolating between the norms

$$\begin{aligned}
\|u_{k+1} - u_k\|_{L^1} &\sim \int_0^1 (1 - u_k^2)\, dx \\
\|u_{k+1} - u_k\|_{BV} &\sim \lambda_k \int_0^1 (1 - u_k^2)\, dx
\end{aligned}$$

Therefore the following statement is of interest, showing that exponential growth of the frequencies leads to exponential convergence of the scheme:

Lemma 6.1. *Let $\lambda_k = 2^k$. Then*

$$\int_0^1 (1-u_k^2)\,dx \le \left(\tfrac{7}{8}\right)^k \int_0^1 (1-u_0^2)\,dx.$$

Proof. By the choice of the oscillatory function s we see that for any continuous function $f \in C(-1,1)$ we have

$$\int_0^1 f(u_{k+1})\,dx = \int_0^1 \tfrac{1}{2}\left[f\left(u_k - \tfrac{1}{2}(1-u_k^2)\right) + \tfrac{1}{2}f\left(u_k + \tfrac{1}{2}(1-u_k^2)\right)\right] dx$$

Now set $f(u) := (1-u^2)^{1/2}$. By direct calculation we obtain

$$f''(u) = -(1-u^2)^{-3/2}$$

and $\frac{d^4 f}{du^4} \le 0$ on $(-1,1)$. It follows that $f''(u)(1-u^2)^2 = -f(u)$ and

$$\tfrac{1}{2}f(u-v) + \tfrac{1}{2}f(u+v) \le f(u) + \tfrac{1}{2}f''(u)v^2.$$

In particular, setting $v = \frac{1}{2}(1-u^2)$ we obtain

$$\tfrac{1}{2}f\Big(u - \tfrac{1}{2}(1-u^2)\Big) + \tfrac{1}{2}f\Big(u + \tfrac{1}{2}(1-u^2)\Big) \le \frac{7}{8}f(u).$$

We conclude

$$\int_0^1 f(u_{k+1})\,dx \le \frac{7}{8}\int_0^1 f(u_k)\,dx$$

and the lemma follows. □

6.2. $C^{1,\alpha}$ isometric immersions.

The question of a sharp regularity threshold has been the object of investigation for the isometric embedding of surfaces as well (see for instance [40], [81]). As already mentioned, the isometric embeddings of $\mathbb{S}^2$ into $\mathbb{R}^3$ are rigid in the class C^2, whereas the h-principle holds for C^1. Borisov investigated embeddings of class $C^{1,\alpha}$, and proved the rigidity for $\alpha > 2/3$ ([8], [10]) and the h-principle for $\alpha < 1/13$ (although the latter was announced in 1965, see[11], a partial proof only appeared in 2004 [12]). In [25] we returned to this problem, and gave a more modern PDE proof of the h-principle for $\alpha < 1/7$, namely

Theorem 6.2 (Local existence). *Let $n \in \mathbb{N}$ and $g_0 \in \mathrm{sym}_n^+$. There exists $r > 0$ such that the following holds for any smooth bounded open set $\Omega \subset \mathbb{R}^n$ and any Riemannian metric $g \in C^\beta(\overline{\Omega})$ with $\beta > 0$ and $\|g - g_0\|_{C^0} \le r$. There exists a constant $\delta_0 > 0$ such that, if $u \in C^2(\overline{\Omega};\mathbb{R}^{n+1})$ and α satisfy*

$$\|u^\sharp e - g\|_0 \le \delta_0^2 \qquad \text{and} \qquad 0 < \alpha < \min\left\{\frac{1}{1+2n_*}, \frac{\beta}{2}\right\},$$

then there exists a map $v \in C^{1,\alpha}(\overline{\Omega};\mathbb{R}^{n+1})$ with

$$v^\sharp e = g \qquad \text{and} \qquad \|v - u\|_{C^1} \le C\,\|u^\sharp e - g\|_{C^0}^{1/2}.$$