Symmetric Willmore surfaces of revolution satisfying natural boundary conditions

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April 29, 2009

Abstract

We consider the Willmore-type functional

$$\mathcal{W}_\gamma(\Gamma) := \int_\Gamma H^2 \, dA - \gamma \int_\Gamma K \, dA,$$

where H and K denote mean and Gaussian curvature of a surface Γ, and $\gamma \in [0,1]$ is a real parameter. Using direct methods of the calculus of variations, we prove existence of surfaces of revolution generated by symmetric graphs which are solutions of the Euler-Lagrange equation corresponding to $\mathcal{W}_\gamma$ and which satisfy the following boundary conditions: the height at the boundary is prescribed, and the second boundary condition is the natural one when considering critical points where only the position at the boundary is fixed. In the particular case $\gamma = 0$ the boundary conditions are arbitrary positive height α and zero mean curvature.

Keywords. Natural boundary conditions, Willmore surfaces of revolution.
AMS classification. 49Q10; 53C42, 35J65, 34L30.

1 Introduction

For a smooth, immersed surface $\Gamma \subset \mathbb{R}^3$ and real parameters γ, μ, H_0, Nitsche in [N1] and [N2] considered the functional

$$\mathcal{F}(\Gamma) = \int_\Gamma \Phi(H,K) \, dA \quad \text{with } \Phi(H,K) = \mu + (H - H_0)^2 - \gamma K, \tag{1.1}$$

where H is the mean curvature of the immersion, K its Gauss curvature, and dA its area element. In many applications, Γ is an idealised model for the interface occurring in real materials. The energy $\mathcal{F}(\Gamma)$ then reflects the surface tension and, therefore, elastic properties of this interface. Similar versions of this functional as model for elastic energies of thin plates were already studied by Poisson [P] in 1812, or Germain [G] in 1921. For a concise presentation we refer to Love's textbook [L]. In 1973, Helfrich [H] studied a functional quite similar to $\mathcal{F}$ from (1.1) as a model for biological bilayer membranes, see also [O] for a more recent survey on this subject. Therefore, $\mathcal{F}$ is sometimes referred to as *Helfrich functional.* Detailed historical information can also be found in Nitsche [N1] and [N2].

From the mathematical point of view it is natural to assume a certain definiteness condition for the functional $\mathcal{F}$. More precisely, we require existence of a constant $C > -\infty$ such that $\mathcal{F}(\Gamma) \geq C$

*Financial support of "Deutsche Forschungsgemeinschaft" for the project "Randwertprobleme für Willmoreflächen - Analysis, Numerik und Numerische Analysis" (DE 611/5.1) is gratefully acknowledged

holds true for all connected and orientable surfaces of regularity class C^2. As shown in [N1], this condition imposes the following restrictions on the parameters

$$\mu \geq 0, \quad 0 \leq \gamma \leq 1, \quad \gamma H_0^2 \leq \mu(1-\gamma).$$

In the present work we study the special case $H_0 = \mu = 0$, where $\mathcal{F}$ takes the form

$$\mathcal{W}_\gamma(\Gamma) := \int_\Gamma H^2\, dA - \gamma \int_\Gamma K\, dA, \quad 0 \leq \gamma \leq 1. \tag{1.2}$$

This functional models the elastic energy of thin shells. Willmore in [W] studied and popularised the functional $\mathcal{W}_0$, by now called *Willmore functional.*

Note that for $\gamma \in [0,1]$, the functional $\mathcal{W}_\gamma$ is positive semidefinite. To see this, let $\kappa_1, \kappa_2 \in \mathbb{R}$ denote the principal curvatures of the surface. Then we compute

$$4(H^2 - \gamma K) = (\kappa_1 + \kappa_2)^2 - 4\gamma\kappa_1\kappa_2 = (1-\gamma)(\kappa_1+\kappa_2)^2 + \gamma(\kappa_1 - \kappa_2)^2 \geq 0 \quad \text{for } \gamma \in [0,1]$$

which gives the semi-definiteness. Moreover, strict inequality $\mathcal{W}_\gamma(\Gamma) > 0$ holds for every non-planar surface Γ if $0 < \gamma < 1$.

We are mainly interested in minima or critical points of $\mathcal{W}_\gamma$. Such critical points $\Gamma \subset \mathbb{R}^3$ have to satisfy the *Willmore equation*

$$\Delta_\Gamma H + 2H(H^2 - K) = 0 \quad \text{on } \Gamma, \tag{1.3}$$

where Δ_Γ denotes the Laplace-Beltrami operator on Γ, see i.e. [W]. A solution of this non-linear fourth-order differential equation is called *Willmore surface.* Note that the Euler-Lagrange equation is independent of the value of γ since the integral over the Gauss curvature only contributes to the boundary terms on account of the theorem of Gauss-Bonnet.

Existence and regularity results for closed Willmore surfaces of prescribed genus are extensively studied in the literature (see e.g., [BK], [KS1], [KS2], [LPP], [Sn] and [R]), while existence of Willmore surfaces *with prescribed boundaries* is by far less studied. In the presence of boundaries the partial differential equation (1.3) has to be accompanied by appropriate boundary conditions. Possible choices for them are presented in [N1] and [N2] along with corresponding existence results. Nitsche's results are based on perturbation arguments and require certain smallness conditions on the boundary data. On the other hand, Schätzle in [Sch] recently proved existence and regularity of branched Willmore immersions in $\mathbb{S}^n$ satisfying prescribed boundary conditions. By working in $\mathbb{S}^n$ some compactness problems could be overcome.

To present a complete analysis of at least special Willmore surfaces satisfying prescribed boundary conditions, we restrict ourselves to surfaces of revolution generated by rotating a symmetric graph in the $[x,y]$-plane about the x-axis. Existence and classical regularity of those axially symmetric Willmore surfaces with arbitrary symmetric Dirichlet boundary conditions were recently proved in [DDG] and [DFGS]. With the paper at hand we continue these studies, and we solve the existence problem for Willmore surfaces of revolution with position prescribed at the boundary and the second boundary condition they satisfy is the natural one when considering critical points of the Willmore functional in the class of surfaces of revolution generated by symmetric graphs where only the position at the boundary is fixed.

1.1 Main result

We consider surfaces of revolution $\Gamma \subset \mathbb{R}^3$ generated by rotating the graph of a smooth symmetric function $u\colon [-1,1] \to (0,\infty)$ about the x-axis. Within this class of surfaces we look for solutions of the Willmore equation (1.3) under the boundary conditions

$$u(\pm 1) = \alpha > 0 \quad \text{and} \quad H(\pm 1) = \frac{\gamma}{\alpha\sqrt{1+u'(\pm 1)^2}} \quad \text{for } \gamma \in [0,1].$$

Our main result is the following.

Theorem 1.1 (Existence and regularity)**.** *For each $\alpha > 0$ and for each $\gamma \in [0,1]$, there exists a positive and symmetric function $u \in C^\infty([-1,1],(0,\infty))$, i.e. $u(x) > 0$ and $u(x) = u(-x)$, such that the corresponding surface of revolution $\Gamma \subset \mathbb{R}^3$ solves*

$$\begin{cases} \triangle_\Gamma H + 2H(H^2 - K) = 0 \quad \text{on } \Gamma, \\ u(\pm 1) = \alpha \;\; \text{and} \;\; H(\pm 1) = \dfrac{\gamma}{\alpha\sqrt{1+u'(\pm 1)^2}}. \end{cases} \tag{1.4}$$

This fourth-order system along with its natural boundary conditions can be found i.e. in [N1], [N2], or von der Mosel [vM]. In Appendix A we recall how the second boundary condition in (1.4) arises as natural boundary condition for the functional $\mathcal{W}_\gamma$.

For special values of α and γ, explicit solutions of problem (1.4) are known. For example, if $\gamma = 1$ then the circular arc $u(x) = \sqrt{\alpha^2 + 1 - x^2}$ provides an explicit solution of (1.4) for arbitrary $\alpha > 0$. Next, let us define some real number α^* by

$$\alpha^* := \min_{y>0} \frac{\cosh(y)}{y} = \frac{1}{b^*}\cosh(b^*) \approx 1.5088795\ldots \tag{1.5}$$

$$\text{with} \quad b^* \approx 1.1996786\ldots \quad \text{solving} \quad b^* \tanh(b^*) = 1. \tag{1.6}$$

In case $\gamma = 0$ and $\alpha > \alpha^*$, there exist two catenoid solutions of (1.4) of the form $u(x) = \cosh(bx)/b$, $b > 0$ suitably chosen. These two solutions yield surfaces with vanishing mean curvature, i.e. minimal surfaces. Moreover, these explicit examples show that the solutions of problem (1.4) are, in general, not unique. Theorem 1.1 becomes particularly interesting for $\gamma = 0$ and $\alpha < \alpha^*$, as catenoid solutions do no longer exist under this assumption. For $\gamma = 0$ and $\alpha = 1$ there still exists an explicit solution given by $u(x) = 2 - \sqrt{2 - x^2}$, a piece of the well-known Clifford torus.

So we have the following rough picture:
→ non-minimal solutions for $\alpha < \alpha^*$;
→ exactly one minimal surface solution for $\alpha = \alpha^*$;
→ two minimal surface solutions for $\alpha > \alpha^*$;

Existence of rotationally symmetric Willmore surfaces solution of (1.4) for $\gamma = 0$ and for all values of α was observed numerically by Fröhlich [F] in 2004, and by Kastian [Ka] as well as Grunau and Deckelnick [DG]. Moreover, in [Ka] the presence of a third solution for $\alpha > \alpha^*$ was numerically observed, suggesting that α^* is a bifurcation point on the branch of minimal surface solutions. Recently, in [DG] Deckelnick and Grunau prove that α^* is indeed a bifurcation point and so, at least locally, the existence also of a non-minimal solution for $\alpha > \alpha^*$ is settled. In the same paper, by a linearisation around the Clifford torus they prove existence of a solution to (1.4) for $\gamma = 0$ and α near to 1. Here we extend this result by proving existence of solutions for all $\alpha \in (0, \alpha^*)$.

Also the case $\gamma = 1$ is special. Up to some constant, $\mathcal{W}_1(u)$ equals the total elastic energy of u considered as a curve in the hyperbolic half-plane $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 : y > 0\}$ equipped with the metric $ds_h^2 := \frac{1}{y^2}(dx^2 + dy^2)$ (see i.e. [BG], [DDG]). Thus, varying γ within $[0,1]$, we interpolate between the "Euclidean" Willmore functional with $\gamma = 0$, and the "hyperbolic" Willmore functional for $\gamma = 1$.

The proof of Theorem 1.1 is based on the existence results from [DFGS] for symmetric Willmore surfaces of revolution satisfying Dirichlet boundary conditions $u(\pm 1) = \alpha$ and $\mp u'(\pm 1) = \beta$ for $\alpha > 0$ and $\beta \in \mathbb{R}$ arbitrary. We construct a solution of (1.4) by minimising the Willmore energy for fixed α and variable β. Essential tools are the continuity and the monotonicity of the Willmore energy in β.

2 Notations. Dirichlet boundary value problem

2.1 Surfaces of revolution

We consider functions $u \in C^4([-1,1],(0,\infty))$. Rotating the curve $(x,u(x)) \subset \mathbb{R}^2$ about the x-axis generates a *surface of revolution* $\Gamma \subset \mathbb{R}^3$ which can be parametrised by

$$\Gamma \,:\, f(x,\varphi) = \big(x, u(x)\cos\varphi, u(x)\sin\varphi\big) \in \mathbb{R}^3\,, \quad x \in [-1,1],\ \varphi \in [0,2\pi). \tag{2.1}$$

The term "surface" always refers to the mapping f as well as to the set Γ. The condition $u > 0$ implies that f is embedded in $\mathbb{R}^3$ and in particular immersed.

Let κ_1 and κ_2 denote the principal curvatures of $\Gamma \subset \mathbb{R}^3$, i.e. $\kappa_1 = -u''(x)(1+u'(x)^2)^{-\frac{3}{2}}$ and $\kappa_2 = (u(x)\sqrt{1+u'(x)^2})^{-1}$. Its *mean curvature* H and *Gaussian curvature* K are

$$\begin{aligned} H &= \frac{\kappa_1+\kappa_2}{2} = -\frac{u''(x)}{2(1+u'(x)^2)^{3/2}} + \frac{1}{2u(x)\sqrt{1+u'(x)^2}}\,, \\ K &= \kappa_1\kappa_2 = -\frac{u''(x)}{u(1+u'(x)^2)^2}\,. \end{aligned}$$

For the total Gauss curvature we have

$$\int_\Gamma K\, dA = -2\pi \int_{-1}^{1} \frac{u''(x)}{(1+u'(x)^2)^{\frac{3}{2}}}\, dx = -2\pi \left.\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right|_{-1}^{1}\,, \tag{2.2}$$

i.e. the Gauss-Bonnet theorem in our special situation. The integral is already determined by the boundary values $u'(-1), u'(1)$. This fact will become essential for the Dirichlet problem discussed in Section 2.3. Furthermore, $\mathcal{W}_\gamma(\Gamma)$ takes the form

$$\begin{aligned} \mathcal{W}_\gamma(u) := \mathcal{W}_\gamma(\Gamma) &= \frac{\pi}{2}\int_{-1}^{1} \left(\frac{u''(x)}{(1+u'(x)^2)^{3/2}} - \frac{1}{u(x)\sqrt{1+u'(x)^2}}\right)^2 u(x)\sqrt{1+u'(x)^2}\, dx \\ &\quad +2\pi\gamma \left.\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right|_{-1}^{1}\,. \end{aligned} \tag{2.3}$$

Remark 2.1. An important property of the energy $\mathcal{W}_\gamma$ is its rescaling invariance, i.e. given a positive function $u \in C^{1,1}([-r,r],(0,\infty))$ for some $r > 0$, then the rescaled function $v(x) = u(rx)/r \in C^{1,1}([-1,1],(0,\infty))$ has the same energy as u, that is,

$$\mathcal{W}_\gamma(v) = \frac{\pi}{2}\int_{-r}^{r} \left(\frac{u''(x)}{(1+u'(x)^2)^{3/2}} - \frac{1}{u(x)\sqrt{1+u'(x)^2}}\right)^2 u(x)\sqrt{1+u'(x)^2}\, dx + 2\pi\gamma \left.\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right|_{-r}^{r}\,.$$

2.2 Notation

For $\alpha > \alpha^*$, α^* defined in (1.5), the following two numbers

$$b_1(\alpha) := \inf\left\{b > 0 \ : \ \frac{\cosh b}{b} \le \alpha\right\} \quad \text{and} \quad b_2(\alpha) := \sup\left\{b > 0 \ : \ \frac{\cosh b}{b} \le \alpha\right\} \tag{2.4}$$

are well-defined and satisfy the inequality $0 < b_1(\alpha) < b^* < b_2(\alpha) < +\infty$ with b^* from (1.6).

Definition 2.2. For $\alpha > 0$ and $\beta \in \mathbb{R}$ we introduce the space of functions

$$\overline{N}_{\alpha,\beta} := \left\{u \in H^2([-1,1]) \ : \ u(x) > 0 \ , \ u(x) = u(-x) \ , \ u(\pm 1) = \alpha \text{ and } u'(-1) = \beta\right\}$$

along with

$$\overline{T}_{\gamma,(\alpha,\beta)} := \inf \left\{\mathcal{W}_\gamma(u) \ : \ u \in \overline{N}_{\alpha,\beta}\right\} \text{ for } \gamma \in [0,1] \ .$$

Due to technical reasons we shall not work within $\overline{N}_{\alpha,\beta}$, but within the smaller space

$$N_{\alpha,\beta} := \left\{u \in \overline{N}_{\alpha,\beta} \ : \text{if } \alpha > \alpha^* \text{ and } -\alpha < \beta \quad \text{then} \quad u'(x) < \alpha \text{ in } [0,1]\right\} \tag{2.5}$$

with

$$T_{\gamma,(\alpha,\beta)} := \inf \left\{\mathcal{W}_\gamma(u) \ : \ u \in N_{\alpha,\beta}\right\} \text{ for } \gamma \in [0,1] \ . \tag{2.6}$$

One easily sees that the space $N_{\alpha,\beta}$ is never empty and hence $T_{\gamma,(\alpha,\beta)}$ well-defined. That the energy $T_{\gamma,(\alpha,\beta)}$ is attained for all $\alpha > 0$, $\beta \in \mathbb{R}$ and $\gamma \in [0,1]$ is a consequence of the results in [DFGS].

Remark 2.3. If $\alpha \le \alpha^*$ or $-\alpha \ge \beta$ then the spaces $N_{\alpha,\beta}$ and $\overline{N}_{\alpha,\beta}$ coincide and hence $T_{\gamma,(\alpha,\beta)} = \overline{T}_{\gamma,(\alpha,\beta)}$. Moreover, if $\alpha > \alpha^*$ and $-\sinh(b_1(\alpha)) \le \beta$ then the equality $\overline{T}_{\gamma,(\alpha,\beta)} = T_{\gamma,(\alpha,\beta)}$ again holds (compare [DFGS]) even though the space $N_{\alpha,\beta}$ is now a proper subspace of $\overline{N}_{\alpha,\beta}$. In case of $-\alpha < \beta < -\sinh(b_1(\alpha))$, the number $T_{\gamma,(\alpha,\beta)}$ may be strictly bigger than $\overline{T}_{\gamma,(\alpha,\beta)}$.

2.3 The Dirichlet boundary value problem

In this section we recall the existence result for the Dirichlet boundary value problem (2.7) below from [DFGS]. First, this result holds true for $\gamma = 0$ and $\gamma = 1$. At the same time a solution to this problem is a critical point for $\mathcal{W}_\gamma$ independently of γ because, on account of (2.2), the total Gauss curvature is a constant depending only on β. Furthermore, we state monotonicity properties of the minimal energy $T_{\gamma,(\alpha,\beta)}$ in α.

Theorem 2.4. ([DFGS, Th.1.1]) *For each $\alpha > 0$ and for each $\beta \in \mathbb{R}$, there exists a positive and symmetric function $u \in C^\infty([-1,1],(0,\infty))$ such that the corresponding surface of revolution $\Gamma \subset \mathbb{R}^3$ solves*

$$\begin{cases} \triangle_\Gamma H + 2H(H^2 - K) = 0 & \text{on } \Gamma, \\ u(\pm 1) = \alpha, & u'(-1) = -u'(1) = \beta \end{cases} \tag{2.7}$$

and satisfies $\mathcal{W}_0(u) = T_{0,(\alpha,\beta)}$. Moreover, u has the following properties:

1. *If $\beta \ge 0$, then $u' > 0$ in $(-1,0)$.*
2. *If $\beta < 0$, then u' has at most three critical points in $[-1,1]$.*

In [DFGS] the monotonicity behaviour of $T_{1,(\alpha,\beta)}$ in α for fixed β was studied. Those values of α and β, for which a catenoid or an arc of a circle solve (2.7), mark points where the monotonicity of this optimal energy w.r.t. α changes qualitatively. In particular, for $\beta > 0$ and $\alpha = \beta^{-1}$, a solution to (2.7) is an arc of the circle with center at the origin and going through $(1, \alpha)$, while for $\beta < 0$ and $\alpha = \alpha_\beta$ with

$$\alpha_\beta := \frac{\sqrt{1+\beta^2}}{\operatorname{arsinh}(-\beta)} \tag{2.8}$$

the catenoid $u(x) = \cosh(bx)/b$, $b = \operatorname{arsinh}(-\beta)$ is a minimal surface solution to (2.7). Since

$$T_{\gamma,(\alpha,\beta)} = T_{1,(\alpha,\beta)} + 4\pi(1-\gamma)\,\frac{\beta}{\sqrt{1+\beta^2}},$$

the minimal energies $T_{\gamma,(\alpha,\beta)}$ and $T_{1,(\alpha,\beta)}$ show the same monotonicity behaviour w.r.t. α as long as we keep γ and β fixed. Thus, the monotonicity results from [DFGS] on $T_{1,(\alpha,\beta)}$ yield directly

Proposition 2.5. *Let $\gamma \in [0,1]$ be fixed.*

1. *For $\beta > 0$ and $\alpha > \alpha' \geq \frac{1}{\beta}$ it holds $T_{\gamma,(\alpha,\beta)} > T_{\gamma,(\alpha',\beta)}$.*
2. *For $\beta > 0$ and $0 < \alpha' < \alpha \leq \frac{1}{\beta}$ it holds $T_{\gamma,(\alpha,\beta)} < T_{\gamma,(\alpha',\beta)}$.*
3. *For $\beta = 0$ and $0 < \alpha' < \alpha$ it holds $T_{\gamma,(\alpha,\beta)} < T_{\gamma,(\alpha',\beta)}$.*
4. *For $\beta < 0$ and $0 < \alpha' < \alpha \leq \alpha_\beta$ it holds $T_{\gamma,(\alpha,\beta)} < T_{\gamma,(\alpha',\beta)}$.*
5. *For $\beta < 0$ and $\alpha > \alpha' \geq \alpha_\beta$ it holds $T_{\gamma,(\alpha,\beta)} > T_{\gamma,(\alpha',\beta)}$.*

To prove Theorem 1.1 we make use of various important a priori estimates for solutions to (2.7) established in [DFGS]. Let us recall them here. The real numbers b_1, b_2 are defined in (2.4) and α^* is defined in (1.5).

Proposition 2.6. *Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $u \in C^\infty([-1,1],(0,\infty))$ be the function from Theorem 2.4 such that the corresponding surface of revolution $\Gamma \subset \mathbb{R}^3$ solves (2.7). Then u has the following qualitative properties:*

1. *If $\alpha \leq \alpha^*$ then*

$$|u'(x)| \leq \max\left\{|\beta|, \alpha^*, \frac{\sqrt{1+\beta^2}}{\alpha}\right\},$$

$$\sqrt{(\alpha + \max\{1,|\beta|\})^2 - x^2} \geq u(x) \geq \min\left\{\alpha, \frac{1}{2}\frac{\alpha}{\sqrt{1+\beta^2}}, \frac{\max\{|\beta|,\alpha^*\}}{e^{C_2}-1}, \frac{1}{b^*}\right\}$$

with $C_2 = 8(1+\max\{|\beta|,\alpha^*\}^2)$.

2. *If $\alpha > \alpha^*$ then*

$$|u'(x)| \leq \max\left\{\sinh(b_2(\alpha)), |\beta|, \frac{\sqrt{1+\beta^2}}{\alpha}\right\},$$

$$\sqrt{(\alpha + \max\{1,|\beta|\})^2 - x^2} \geq u(x) \geq \min\left\{\alpha, \frac{1}{2}\frac{\alpha}{\sqrt{1+\beta^2}}, \frac{\sinh(b_2(\alpha))}{e^{C_1}-1}, \frac{\max\{|\beta|,\alpha^*\}}{e^{C_2}-1}, \frac{1}{b_2}\right\}$$

with $C_2 = 8(1+\max\{|\beta|,\alpha^*\}^2)$ and $C_1 = 2\cosh(2b_2)(1+\operatorname{arsinh}(|\beta|)(\alpha-\alpha^*))$.

(i) If $-\sinh(b_1(\alpha)) \geq \beta > -\alpha$, *we have*

$$0 \leq u'(x) \leq -\beta \quad \textit{in } [0,1].$$

(ii) If $\beta > -\sinh(b_1(\alpha))$, *we have*

$$-\frac{1}{\alpha^*} \leq u'(x) \leq \sinh(b_1) \ \textit{in } [0,1] \ \textit{and } \sqrt{\alpha^2+1-x^2} \geq u(x) \geq \frac{1}{b_1}\cosh(b_1 x).$$

Combining the results of Theorem 2.4 and Proposition 2.6 we obtain

Corollary 2.7. *Given any* $\gamma \in [0,1]$, $\alpha > 0$ *and* $\beta \in \mathbb{R}$, *there exists some function* $u \in C^\infty([-1,1],\mathbb{R}) \cap N_{\alpha,\beta}$ *such that the corresponding surface of revolution solves* (2.7) *and moreover* $\mathcal{W}_\gamma(u) = T_{\gamma,(\alpha,\beta)}$ *holds.*

3 Continuity and monotonicity of the energy in β

In this section we analyse the behaviour of the optimal energy $T_{\gamma,(\alpha,\beta)}$ w.r.t. β if α and γ are fixed. The results we obtain are the main ingredients for the proof of Theorem 1.1.

3.1 Continuity in β

Lemma 3.1. *Let* $\gamma \in [0,1]$ *be fixed. If* $\alpha \leq \alpha^*$, *then* $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ *is upper semi-continuous for* $\beta \in \mathbb{R}$. *If* $\alpha > \alpha^*$, *then* $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ *is upper semi-continuous for* $\beta \in \mathbb{R}\backslash\{-\alpha\}$.

Proof. Given $u \in N_{\alpha,\beta}$ and $\varepsilon \in \mathbb{R}$ consider the symmetric function $u_\varepsilon(x) := u(x) + \frac{\varepsilon}{2}(1-x^2)$ with the properties $u_\varepsilon(\pm 1) = \alpha$, $u'_\varepsilon(-1) = \beta + \epsilon$. Then $u_\varepsilon \in \overline{N}_{\alpha,\beta+\varepsilon}$ will hold for $|\varepsilon| < \varepsilon_0$, $\varepsilon_0 > 0$ sufficiently small (to have $u_\varepsilon(x) > 0$ in $[-1,1]$). If either $\alpha \leq \alpha^*$ or $-\beta > \alpha$ this implies $u_\varepsilon \in N_{\alpha,\beta+\varepsilon}$ (compare Remark 2.3). If, on the other hand, $\alpha > \alpha^*$ and $\beta > -\alpha$ then $u \in N_{\alpha,\beta}$ implies $u'(x) < \alpha$ in $[0,1]$ by Definition 2.2. This implies $u'_\varepsilon(x) < \alpha$ in $[0,1]$ for $|\varepsilon| \leq \varepsilon_1$ and thus $u_\varepsilon \in N_{\alpha,\beta+\epsilon}$, if $0 < \varepsilon_1 \leq \varepsilon_0$ is chosen sufficiently small. The continuity of the mapping $\varepsilon \mapsto \mathcal{W}_\gamma(u_\varepsilon)$ gives

$$T_{\gamma,(\alpha,\beta)} = \inf_{u \in N_{\alpha,\beta}} \left[\lim_{\varepsilon \to 0} \mathcal{W}_\gamma(u_\varepsilon)\right] \geq \inf_{u \in N_{\alpha,\beta}} \left[\limsup_{\varepsilon \to 0} T_{\gamma,(\alpha,\beta+\varepsilon)}\right] = \limsup_{\varepsilon \to 0} T_{\gamma,(\alpha,\beta+\varepsilon)},$$

which just means that $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ is upper semi-continuous. □

The proof of lower semi-continuity of $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ is more involved and requires the a priori estimates from Proposition 2.6.

Lemma 3.2. *Let* $\gamma \in [0,1]$ *be fixed. If* $\alpha \leq \alpha^*$, *then* $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ *is lower semi-continuous for* $\beta \in \mathbb{R}$. *If* $\alpha > \alpha^*$, *then* $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ *is lower semi-continuous for* $\beta \in \mathbb{R}\backslash\{-\alpha\}$.

Proof. Because of

$$T_{\gamma',(\alpha,\beta)} = T_{\gamma,(\alpha,\beta)} + 4\pi(\gamma-\gamma')\frac{\beta}{\sqrt{1+\beta^2}}$$

it suffices to prove the result for one particular γ, we take $\gamma_0 := \frac{1}{2}$. Let $(\beta_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be some sequence converging to some $\beta \in \mathbb{R}$. Moreover, let u_k be the function from Corollary 2.7 satisfying

$$u_k \in N_{\alpha,\beta_k} \ \text{ and } \ \mathcal{W}_{\gamma_0}(u_k) = T_{\gamma_0,(\alpha,\beta_k)}.$$

Since $(\beta_k)_{k\in\mathbb{N}}$ is uniformly bounded, Proposition 2.6 yields positive constants c_i, $i = 1, 2, 3$, depending only on α such that

$$0 < c_1 \leq u_k(x) \leq c_2 \text{ and } |u_k'(x)| \leq c_3 \quad \text{in } [-1, 1] \tag{3.9}$$

holds true for all $k \in \mathbb{N}$. If moreover $\alpha > \alpha^*$ and $-\alpha < \beta$, then Proposition 2.6 yields additionally

$$u_k'(x) \leq \max\big\{-\beta_k, \sinh(b_1(\alpha))\big\} \quad \text{in } [0, 1]. \tag{3.10}$$

From the upper semi-continuity of Lemma 3.1 we deduce that $T_{\gamma_0,(\alpha,\beta_k)} = \mathcal{W}_{\gamma_0}(u_k) \leq c_4$ holds with some constant c_4. This is true since upper semi-continuous functions achieve a maximum on compact sets. These estimates imply

$$\begin{aligned} c_4 \geq \mathcal{W}_{\gamma_0}(u_k) &= \frac{\pi}{2}\int_{-1}^{1}\left(\frac{u_k''(x)^2 u_k(x)}{(1+u_k'(x)^2)^{\frac{5}{2}}} + \frac{1}{u_k(x)\sqrt{1+u_k'(x)^2}}\right) dx \\ &\geq \frac{\pi}{2}\frac{c_1}{(1+c_3^2)^{\frac{5}{2}}}\int_{-1}^{1} u_k''(x)^2\, dx. \end{aligned} \tag{3.11}$$

Notice that due to the choice $\gamma_0 = \frac{1}{2}$ there are no boundary terms. From (3.11) we obtain uniform boundness of the sequence in $H^2([-1,1])$, and, after passing to a subsequence, Rellich's embedding theorem ensures the existence of $u \in H^2([-1,1])$ such that

$$u_k \rightharpoonup u \text{ in } H^2([-1,1]) \text{ and } u_k \to u \text{ in } C^1([-1,1],\mathbb{R}).$$

The convergence in $C^1([-1,1])$ ensures that u satisfies also the bounds in (3.9), in particular $u(x) > 0$ in $[-1,1]$. Moreover, if $\alpha > \alpha^*$ and $-\alpha < \beta$ then estimate (3.10) yields

$$u'(x) \leq \max\big\{-\beta, \sinh(b_1(\alpha))\big\} < \alpha \quad \text{in } [0, 1]$$

and hence $u \in N_{\alpha,\beta}$. The strong convergence in $C^1([-1,1])$ and the weak convergence in $H^2([-1,1])$ yield

$$\begin{aligned} \mathcal{W}_{\gamma_0}(u_k) &= \frac{\pi}{2}\int_{-1}^{1}\left(\frac{u_k''(x)^2 u(x)}{(1+u'(x)^2)^{\frac{5}{2}}} + \frac{1}{u(x)\sqrt{1+u'(x)^2}}\right) dx + o(1) \\ &\geq \frac{\pi}{2}\int_{-1}^{1}\left(\frac{u''(x)^2 u(x)}{(1+u'(x)^2)^{\frac{5}{2}}} + \frac{1}{u(x)\sqrt{1+u'(x)^2}}\right) dx + o(1) = \mathcal{W}_{\gamma_0}(u) + o(1). \end{aligned}$$

Together with $u \in N_{\alpha,\beta}$ this shows

$$T_{\gamma_0,(\alpha,\beta)} \leq \mathcal{W}_{\gamma_0}(u) \leq \liminf_{k\to\infty} \mathcal{W}_{\gamma_0}(u_k) = \liminf_{k\to\infty} T_{\gamma_0,(\alpha,\beta_k)}$$

proving the claimed lower semi-continuity. □

The combination of the above two results now yields

Corollary 3.3. *Let $\gamma \in [0,1]$, $\alpha > 0$ be fixed. Then $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ is continuous in $\mathbb{R}$ if $\alpha \leq \alpha^*$ while for $\alpha > \alpha^*$ it is continuous in $\mathbb{R}\backslash\{-\alpha\}$.*

3.2 Monotonicity results for large and small β

In this section we show that $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ is an increasing function for sufficiently large positive values of β and a decreasing function for sufficiently small negative values of β. This allows us to restrict to 'bounded' values of β when looking for the absolute minimiser.

Lemma 3.4. *If $\gamma \in [0,1]$ and $\beta > \beta' \geq \alpha^{-1}$, then $T_{\gamma,(\alpha,\beta)} > T_{\gamma,(\alpha,\beta')}$.*

Proof. By Corollary 2.7 there exists some $u \in N_{\alpha,\beta}$ such that $\mathcal{W}_\gamma(u) = T_{\gamma,(\alpha,\beta)}$. Since $u'(-1) = \beta > \beta'$, $u'(0) = 0$, u' is continuous and $u' \geq 0$ in $[-1,0]$ (see Theorem 2.4), there exists $x^* \in (-1,0)$ such that $u'(x^*) = \beta'$ and $u(x^*) > \alpha|x^*|$. We then consider the function $w \in C^{1,1}([-1,1])$ which is equal to $u|_{[x^*,-x^*]}$ rescaled to $[-1,1]$. By construction, $w'(-1) = \beta'$ and $w(\pm1) > \alpha$. By the rescaling invariance of the energy (Remark 2.1) and Proposition 2.5 (note $w(\pm1) > \alpha \geq \beta'^{-1}$) we finally get

$$T_{\gamma,(\alpha,\beta)} = \mathcal{W}_\gamma(u) \geq \mathcal{W}_\gamma(w) \geq T_{\gamma,(w(\pm1),\beta')} > T_{\gamma,(\alpha,\beta')}.$$

Here we have used $\mathcal{W}_\gamma(u) \geq \mathcal{W}_\gamma(w)$ which follows from $\gamma \in [0,1]$. ☐

Remark 3.5. Actually, this result can be generalised to the case $-\infty < \gamma \leq 1$. Using Corollary 2.7, (2.3) and Lemma 3.4 we find

$$T_{\gamma,(\alpha,\beta)} = T_{1,(\alpha,\beta)} + 4\pi(1-\gamma)\frac{\beta}{\sqrt{1+\beta^2}} > T_{1,(\alpha,\beta')} + 4\pi(1-\gamma)\frac{\beta'}{\sqrt{1+\beta'^2}} = T_{\gamma,(\alpha,\beta')},$$

since $1-\gamma \geq 0$ and $\beta > \beta' \geq \alpha^{-1} > 0$.

In the following, we set

$$\beta_2(\alpha) := \begin{cases} 0 & \text{if } \alpha < \alpha^*, \\ -\sinh(b_2) & \text{if } \alpha \geq \alpha^*, \end{cases} \quad \text{with } b_2 = b_2(\alpha) \text{ defined in (2.4).} \tag{3.12}$$

A simple computation shows that, for $\beta < 0$, $\alpha > \alpha_\beta$ implies $\beta > \beta_2(\alpha)$.

Lemma 3.6. *Let $\gamma \in [0,1]$ be fixed. If $\beta' < \beta \leq \min\{-\alpha, \beta_2(\alpha)\}$, then $T_{\gamma,(\alpha,\beta)} < T_{\gamma,(\alpha,\beta')}$.*

Proof. By Corollary 2.7 there exists some $u \in N_{\alpha,\beta'}$ such that $\mathcal{W}_\gamma(u) = T_{\gamma,(\alpha,\beta')}$. Because of $\beta' < -\alpha$ there exists $\overline{x} \in (-1,0)$ the smallest element such that $u(\overline{x}) = -\alpha\overline{x}$. Since $u'(-1) = \beta' < \beta \leq -\alpha$, $u'(\overline{x}) \geq -\alpha$, and u' is continuous, there exists $x^* \in (-1,\overline{x})$ such that $u'(x^*) = \beta$ and $u(x^*) < \alpha|x^*|$. We consider the function $w \in C^{1,1}([-1,1])$ equal to $u|_{[x^*,-x^*]}$ rescaled to $[-1,1]$. By construction there hold $w'(-1) = \beta$ and $w(\pm1) < \alpha$. If $\alpha > \alpha^*$ the assumption $\beta \leq \beta_2(\alpha)$ gives $\alpha \leq \alpha_\beta$ with α_β defined in (2.8). While if $\alpha \leq \alpha^*$ then clearly also $\alpha \leq \alpha_\beta$. In both cases, Remark 2.1 and Proposition 2.5 (noting that $w(\pm1) < \alpha \leq \alpha_\beta$) yield

$$T_{\gamma,(\alpha,\beta')} = \mathcal{W}_\gamma(u) \geq \mathcal{W}_\gamma(w) \geq T_{\gamma,(w(\pm1),\beta)} > T_{\gamma,(\alpha,\beta)}.$$

Here we have used $\mathcal{W}_\gamma(u) \geq \mathcal{W}_\gamma(w)$ due to $\gamma \in [0,1]$. ☐

Remark 3.7. This result still holds for all $\gamma \geq 0$ because

$$T_{\gamma,(\alpha,\beta')} = T_{0,(\alpha,\beta')} - 4\pi\gamma\frac{\beta'}{\sqrt{1+\beta'^2}} > T_{0,(\alpha,\beta)} - 4\pi\gamma\frac{\beta}{\sqrt{1+\beta^2}} = T_{\gamma,(\alpha,\beta)},$$

which holds for $\gamma \geq 0$ and $\beta' < \beta \leq \min\{-\alpha, \beta_2(\alpha)\}$.

3.3 The case $\gamma = 0$

For this case we give a complete description of the monotonicity behaviour of $\beta \mapsto T_{0,(\alpha,\beta)}$ for all values of β. For $\alpha \leq \alpha^*$ this mapping is decreasing on $(-\infty, -\alpha]$ while it is increasing on $[-\alpha, \infty)$. For $\alpha > \alpha^*$ the behaviour is more complicated due to the presence of the two catenoid solutions whose energy $\mathcal{W}_0$ is zero.

Similarly to $\beta_2(\alpha)$, let us introduce

$$\beta_1(\alpha) := \begin{cases} -\alpha^* & \text{if } \alpha < \alpha^*, \\ -\sinh(b_1) & \text{if } \alpha \geq \alpha^*, \end{cases} \quad \text{with } b_1 = b_1(\alpha) \text{ defined in (2.4).} \tag{3.13}$$

Lemma 3.8. *If $\alpha^{-1} \geq \beta > \beta' \geq \max\{-\alpha, \beta_1(\alpha)\}$, then $T_{0,(\alpha,\beta)} > T_{0,(\alpha,\beta')}$.*

Proof. Given $\beta > \beta'$ we first define $b := -\operatorname{arsinh}(\beta)$, $b' := -\operatorname{arsinh}(\beta')$ and note $b < b'$. By Corollary 2.7 there exists some $u \in N_{\alpha,\beta}$ such that $\mathcal{W}_0(u) = T_{0,(\alpha,\beta)}$. Let $f(x)$ be the catenary with initial data $f(-1) = \alpha$, $f'(-1) = \beta$, i.e. the function

$$f(x) = \frac{\alpha}{\cosh b} \cosh\left(\frac{\cosh b}{\alpha}(x+1) - b\right). \tag{3.14}$$

At the point

$$x^* := -1 + \frac{\alpha}{\cosh b}(b - b') \tag{3.15}$$

we have $f'(x^*) = \beta'$. Note that $x^* < -1$ since $b < b'$. Let $v \in C^{1,1}([x^*, -x^*])$ be the symmetric function equal to f on $[x^*, -1]$ and equal to u in $(-1, 0]$. Furthermore, let $w \in C^{1,1}([-1,1])$ be v rescaled to $[-1,1]$. By construction it holds $w'(-1) = \beta'$. In order to apply the monotonicity property of the energy in α we need to show $w(\pm 1) < \alpha$. Since $\beta > \beta' \geq -\alpha$ we have $\sinh(x) < \alpha$ for all $x \in (b, b')$, and hence

$$\frac{\cosh(b') - \cosh(b)}{b' - b} < \alpha \quad \text{or equivalently} \quad \frac{\cosh(b')}{\cosh(b) - \alpha(b - b')} < 1.$$

From this inequality we see

$$w(\pm 1) = \frac{v(x^*)}{|x^*|} = \frac{\alpha \cosh(b')}{\cosh(b)} \frac{1}{1 - \frac{\alpha}{\cosh(b)}(b - b')} = \alpha \frac{\cosh(b')}{\cosh(b) - \alpha(b - b')} < \alpha. \tag{3.16}$$

Since the piece of catenoid has zero energy for $\gamma = 0$, and the energy $\mathcal{W}_\gamma$ is invariant under rescaling (Remark 2.1), we obtain

$$T_{0,(\alpha,\beta)} = \mathcal{W}_0(u) = \mathcal{W}_0(w) \geq T_{0,(w(\pm 1), w'(-1))} = T_{0,(w(-1),\beta')}.$$

Now, if $\beta' \geq 0$, then, using $w(\pm 1) < \alpha < \beta'^{-1}$, Proposition 2.5 yields $T_{0,(w(\pm1),\beta')} > T_{0,(\alpha,\beta')}$. On the other hand, if $\beta' < 0$ we claim $\alpha \leq \alpha_{\beta'}$. This is clear if $\alpha \leq \alpha^*$, while if $\alpha > \alpha^*$ it is assured by the assumption $\beta' \geq -\sinh(b_1)$ which follows from the definition of $\beta_1(\alpha)$. Proposition 2.5 now yields again $T_{0,(w(\pm1),\beta')} > T_{0,(\alpha,\beta')}$. In both cases we obtain $T_{0,(\alpha,\beta)} > T_{0,(\alpha,\beta')}$. □

Combining our Lemmata 3.4, 3.6 and 3.8 with Corollary 3.3 for $\alpha \leq \alpha^*$ we obtain:

Corollary 3.9. *If $0 < \alpha \leq \alpha^*$, then $T_{0,(\alpha,\beta)}$ is increasing in β for $\beta \in [-\alpha, \infty)$ and decreasing for $\beta \in (-\infty, -\alpha]$. The mapping $\beta \mapsto T_{0,(\alpha,\beta)}$ achieves its global minimum at $\beta = -\alpha$.*

We still have to discuss the case $\alpha > \alpha^*$ and $\beta \in [\beta_2(\alpha), \beta_1(\alpha)]$. Here, the monotonicity behaviour becomes more involved due to the presence of the two catenoids corresponding to the two values $\beta_1(\alpha)$ and $\beta_2(\alpha)$ for the boundary datum β.

Lemma 3.10. *If $\alpha > \alpha^*$ and $-\alpha \geq \beta > \beta' \geq \beta_2(\alpha)$, then $T_{0,(\alpha,\beta)} > T_{0,(\alpha,\beta')}$.*

Proof. Note that the assumption $-\alpha > \beta' \geq \beta_2(\alpha)$ yields $\alpha \geq \alpha_{\beta'}$. The claim is proven quite similarly as Lemma 3.8. By Corollary 2.7 there exists some $u \in N_{\alpha,\beta}$ such that $\mathcal{W}_0(u) = T_{0,(\alpha,\beta)}$. Let $f(x)$ be the function from (3.14). At x^* defined as in (3.15), we have $f'(x^*) = \beta'$. Now consider the symmetric function $v \in C^{1,1}([x^*, -x^*])$ which is equal to f in $[x^*, -1]$ and to u in $(-1, 0]$. Furthermore, let $w \in C^{1,1}([-1, 1])$ be its rescaling to $[-1, 1]$. By construction, $w'(-1) = \beta'$ and $w(\pm 1) > \alpha$. Indeed, since $-\alpha \geq \beta > \beta'$ we have $\sinh(x) > \alpha$ for all $x \in (b, b')$ and also

$$\frac{\cosh(b') - \cosh(b)}{b' - b} > \alpha \quad \text{or equivalently} \quad \frac{\cosh(b')}{\cosh(b) - \alpha(b - b')} > 1.$$

This inequality implies $w(\pm 1) > \alpha$ which is proven just like (3.16). We then obtain the inequality

$$T_{0,(\alpha,\beta)} = \mathcal{W}_0(u) = \mathcal{W}_0(w) \geq T_{0,(w(\pm 1),\beta')} > T_{0,(\alpha,\beta')},$$

using Proposition 2.5 together with $w(\pm 1) > \alpha > \alpha_{\beta'}$. □

For $\alpha > \alpha^*$ and $\beta \in (-\alpha, \beta_1(\alpha)]$ the elements $u \in N_{\alpha,\beta}$ have the additional restriction $u'(x) < \alpha$ in $[0, 1]$ (compare Definition 2.2). This property shall be used now in order to prove monotonicity in $\beta \in (-\alpha, \beta_1(\alpha)]$.

Lemma 3.11. *If $\alpha > \alpha^*$ and $\beta_1(\alpha) \geq \beta > \beta' > -\alpha$, then $T_{0,(\alpha,\beta)} < T_{0,(\alpha,\beta')}$.*

Proof. Note that the assumption $-\alpha < \beta \leq \beta_1(\alpha)$ yields $\alpha \geq \alpha_\beta$. By Corollary 2.7 there exists some $u \in N_{\alpha,\beta'}$ such that $\mathcal{W}_0(u) = T_{0,(\alpha,\beta')}$. Moreover, $\beta' \leq u'(x) \leq 0$ in $[-1, 0]$ by Proposition 2.6. From this estimate together with $\beta' > -\alpha$ it follows that $u(x) > \alpha|x|$ for $x \in (-1, 1)$. Since $u'(-1) = \beta' < \beta < 0$, $u'(0) = 0$ and u' is continuous, there exists $x^* \in (-1, 0)$ such that $u'(x^*) = \beta$ and $u(x^*) > \alpha|x^*|$. We consider then the function $w \in C^{1,1}([-1, 1])$ that is equal to $u|_{[x^*,-x^*]}$ rescaled to $[-1, 1]$. By construction, $w'(-1) = \beta$ and $w(\pm 1) > \alpha$. Proposition 2.5 yields

$$T_{0,(\alpha,\beta')} = \mathcal{W}_0(u) \geq \mathcal{W}_0(w) \geq T_{0,(w(\pm 1),\beta)} > T_{0,(\alpha,\beta)}.$$

□

Combining the previous results we have:

Corollary 3.12. *For fixed $\alpha > \alpha^*$, the function $\beta \mapsto T_{0,(\alpha,\beta)}$ is decreasing on the intervals $(-\infty, \beta_2(\alpha)]$ and $(-\alpha, \beta_1(\alpha)]$ while it is increasing on the intervals $[\beta_2(\alpha), -\alpha]$ and $[\beta_1(\alpha), +\infty)$.*

3.4 The case $\gamma \in [0, 1]$

A combination of Lemmata 3.4 and 3.6 yields:

Corollary 3.13. *For $\gamma \in [0, 1]$ and $\alpha \leq \alpha^*$ the mapping $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ is decreasing for $-\infty < \beta \leq -\alpha$ and increasing for $\alpha^{-1} \leq \beta < +\infty$.*

This result does not give us any information about the monotonicity if $-\alpha < \beta < \alpha^{-1}$. We may expect that there exists a unique $\tilde{\beta} = \tilde{\beta}(\alpha, \gamma) \in [-\alpha, \alpha^{-1}]$ such that $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ is decreasing on $(-\infty, \tilde{\beta}]$ and increasing on $[\tilde{\beta}, +\infty)$. In fact, this claim is true for $\gamma = 0$ with $\tilde{\beta} = -\alpha$, due to Corollary 3.9. It is also true for $\gamma = 1$ where we can take $\tilde{\beta} = \alpha^{-1}$.

Corollary 3.14. *For $\gamma \in [0, 1]$ and $\alpha > \alpha^*$ the mapping $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ is decreasing on $(-\infty, \beta_2(\alpha)]$ and $(-\alpha, \beta_1(\alpha)]$, and increasing on $[\alpha^{-1}, \infty)$.*

Proof. For $\beta \geq \alpha^{-1}$ and $\beta \leq \beta_2(\alpha)$ the claim follows from Lemmata 3.4 and 3.6. For $\beta \in (-\alpha, \beta_1(\alpha)]$, we note

$$T_{\gamma,(\alpha,\beta)} = T_{0,(\alpha,\beta)} - 4\pi\gamma \frac{\beta}{\sqrt{1+\beta^2}}.$$

Since $T_{0,(\alpha,\beta)}$ is decreasing by Corollary 3.12 on $(-\alpha, \beta_1(\alpha)]$ and $x \mapsto -x/\sqrt{1+x^2}$ is also decreasing for $x < 0$, for $\beta_1(\alpha) \geq \beta > \beta' > -\alpha$ we obtain

$$T_{\gamma,(\alpha,\beta')} = T_{0,(\alpha,\beta')} - 4\pi\gamma \frac{\beta'}{\sqrt{1+\beta'^2}} > T_{0,(\alpha,\beta)} - 4\pi\gamma \frac{\beta}{\sqrt{1+\beta^2}} = T_{\gamma,(\alpha,\beta)}.$$

The claim follows. □

Similarly to Corollary 3.14, this result does not yield information on the monotonicity if $\beta_1(\alpha) < \beta < \alpha^{-1}$. One may conjecture that there exists some $\tilde{\beta} = \tilde{\beta}(\alpha, \gamma) \in [\beta_1(\alpha), \alpha^{-1}]$ such that $\beta \mapsto T_{\gamma,(\alpha,\beta)}$ is decreasing on $(-\alpha, \tilde{\beta}]$ and increasing on $[\tilde{\beta}, +\infty)$.

Remark 3.15. Similarly to the case $\gamma = 0$ treated in in Section 3.3, we can completely discuss the monotonicity behaviour of $\beta \mapsto T_{1,(\alpha,\beta)}$ for $\gamma = 1$. It is decreasing on $(-\infty, \alpha^{-1}]$ and increasing on $[\alpha^{-1}, +\infty)$. The minimum $\beta = \alpha^{-1}$ corresponds to the circular arc $u(x) = \sqrt{\alpha^2 + 1 - x^2}$, $u \in N_{\alpha,\alpha^{-1}}$ which has zero energy $\mathcal{W}_1(u) = 0$.

The proof is quite similar to the case $\gamma = 0$. Instead of adding a piece of a catenoid, as done in the proof of Lemmata 3.8 and 3.10, we now add a circular arc. While for $\gamma = 0$ adding a piece of catenoid does not change the energy $\mathcal{W}_0$, in the case of $\gamma = 1$ adding a circular arc does not change the energy $\mathcal{W}_1$. We point out that the procedure of adding 'pieces' with zero energy cannot be used for $0 < \gamma < 1$ since for this range of γ the energy $\mathcal{W}_\gamma$ is always larger than zero.

4 Proof of Theorem 1.1

We first study the case $\alpha \leq \alpha^*$. Setting $\beta^- := \min\{-\alpha, \beta_2(\alpha)\}$, $\beta^+ := \alpha^{-1}$, Corollary 3.13 implies

$$T_{\gamma,\alpha} := \inf_{\beta\in\mathbb{R}} T_{\gamma,(\alpha,\beta)} = \inf_{\beta^-\leq\beta\leq\beta^+} T_{\gamma,(\alpha,\beta)}.$$

The continuity of the energy in β, proven in Corollary 3.3, yields some $\beta^* \in [\beta^-, \beta^+]$ such that $T_{\gamma,\alpha} = T_{\gamma,(\alpha,\beta^*)}$. By Corollary 2.7 there exists some $u \in N_{\alpha,\beta^*} \cap C^\infty([-1,1],(0,\infty))$ such that $\mathcal{W}_\gamma(u) = T_{\gamma,(\alpha,\beta^*)}$. Since u minimises the energy $\mathcal{W}_\gamma$ within the class $\cup_{\beta\in\mathbb{R}} N_{\alpha,\beta}$, it solves boundary value problem (1.4).

Let us now study the case $\alpha > \alpha^*$. Here we set $\beta^- := \beta_1(\alpha) > -\alpha$, $\beta^+ := \alpha^{-1}$. Corollary 3.14 yields

$$T_{\gamma,\alpha} := \inf_{-\alpha<\beta<+\infty} T_{\gamma,(\alpha,\beta)} = \inf_{\beta^-\leq\beta\leq\beta^+} T_{\gamma,(\alpha,\beta)}.$$

Again, Corollary 3.3 gives some $\beta^* \in [\beta^-, \beta^+]$ such that $T_{\gamma,\alpha} = T_{\gamma,(\alpha,\beta^*)}$. Corollary 2.7 yields some $u \in N_{\alpha,\beta^*} \cap C^\infty([-1,1],\mathbb{R})$ such that $\mathcal{W}_\gamma(u) = T_{\gamma,(\alpha,\beta^*)}$. This function u minimises the energy $\mathcal{W}_\gamma$ within the class $\cup_{-\alpha<\beta<+\infty} N_{\alpha,\beta}$ and hence is solution of the boundary value problem (1.4). Here it is crucial that $\beta^* > -\alpha$.

Remark 4.1. In the particular case $\alpha \leq \alpha^*$ and $\gamma = 0$ the monotonicity property of the energy in β (Corollary 3.9) yields that the constructed solution of (1.4) satisfies $u'(-1) = -\alpha$. One can verify this for the values of α for which an explicit solution to (1.4) is known. For $\alpha = 1$ this solution is a piece of the Clifford torus, i.e. the surface of revolution corresponding to $f(x) := 2 - \sqrt{2 - x^2}$. One sees that $f(-1) = 1$ and $f'(-1) = -1$. Another explicit solution is the catenoid $x \mapsto g(x) := \cosh(b^*x)/b^*$ with b^* defined in (1.6). This function has boundary value $g(-1) = \alpha^*$ with α^* defined in (1.5) and $g'(-1) = -\sinh(b^*) = -\alpha^*$ by definition of b^* and α^*.

A Natural boundary conditions

The following lemma yields the first variation of the functional $\mathcal{W}_\gamma$.

Lemma A.1. *Let $u \in C^4([-1,1],(0,\infty))$. Then for all $\varphi \in H^2([-1,1]) \cap H_0^1([-1,1])$, we have*

$$\frac{d}{dt}\int_\Gamma H(u+t\varphi)^2\, dA[u+t\varphi]\Big|_{t=0} = -2\pi\left[H(x)\frac{u(x)\varphi'(x)}{1+u'(x)^2}\right]_{-1}^{1} -2\pi\int_{-1}^{1} u(x)\varphi(x)\left(\Delta_\Gamma H(x) + 2H(H^2-K)\right)dx,$$

and

$$\frac{d}{dt}\int_\Gamma K[u+t\varphi]\, dA[u+t\varphi]\Big|_{t=0} = -2\pi\left[\frac{\varphi'(x)}{(1+u'(x)^2)^{3/2}}\right]_{-1}^{1},$$

Γ being the surface of revolution generated by $u+t\varphi$.

The first statement was proved in [DG, Lemma 6]. The second identity follows directly if we write the Gauss curvature K in coordinates. Thus, the first variation of the composed functional $\mathcal{W}_\gamma(u)$ can be written as

$$\frac{d}{dt}\mathcal{W}_\gamma(u+t\varphi)\Big|_{t=0} = -2\pi\left[\left(H(x) - \frac{\gamma}{u(x)\sqrt{1+u'(x)^2}}\right)\frac{u(x)\varphi'(x)}{1+u'(x)^2}\right]_{-1}^{1} -2\pi\int_{-1}^{1} u\varphi\left(\Delta_\Gamma H + 2H^3 - 2HK\right)dx$$

for all $\varphi \in H^2([-1,1]) \cap H_0^1([-1,1])$. The corresponding boundary value problem is given by (1.4). In order to provide a geometric interpretation of this boundary value problem, we observe

Lemma A.2. *Let $x \in (-1,1)$ be fixed, and consider the curve $\varphi \mapsto X(x,\varphi)$ on Γ. Then it holds*

$$\kappa_n(x) = \frac{1}{u(x)\sqrt{1+u'(x)^2}}$$

for its normal curvature w.r.t. the surface unit normal vector

$$\nu(x,\varphi) = \left(u'(x), -\cos\varphi, -\sin\varphi\right)\frac{1}{\sqrt{1+u'(x)^2}}.$$

Proof. Parametrising the curve by arclength $s \in [0, 2\pi u(x)]$ gives

$$X(s) = \left(x, u(x)\cos\frac{s}{u(x)}, u(x)\sin\frac{s}{u(x)}\right),$$

$$X'(s) = \left(0, -\sin\frac{s}{u(x)}, \cos\frac{s}{u(x)}\right),$$

$$X''(s) = \left(0, -\frac{1}{u(x)}\cos\frac{s}{u(x)}, -\frac{1}{u(x)}\sin\frac{s}{u(x)}\right).$$

Thus, the normal curvature w.r.t. ν is given by

$$\kappa_n(x) = \langle X''(s), \nu(x, \frac{s}{u(x)})\rangle = \frac{1}{\sqrt{1+u'(x)^2}}\frac{1}{u(x)}.$$

$\square$

On account of this lemma, the natural boundary data can be expressed in terms of the geometric quantity κ_n at the boundary of the surface: We can write (1.4) in the form

$$\begin{cases} \Delta_\Gamma H + 2H^3 - 2HK = 0 & \text{on } \Gamma, \\ u(\pm 1) = \alpha, \qquad H(\pm 1) = \gamma \kappa_n(\pm 1). \end{cases}$$

A detailed computation of natural boundary conditions even for Helfrich's functional can be found in the literature. We want to mention i.e. Nitsche [N1], [N2], and von der Mosel [vM].

B Estimates

Lemma B.1. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the solution $u \in C^\infty([-1,1],(0,\infty))$ of (2.7), constructed in the proof of Theorem 2.4, satisfies*

$$u(x) \le \sqrt{(\alpha + \max\{1, |\beta|\})^2 - x^2}, \quad x \in [-1,1].$$

Proof. If $\alpha\beta \le 1$, the function u satisfies $x + u(x)u'(x) \ge 0$ in $[0,1]$ (see [DFGS, Lemma 3.16] for $\beta \ge 0$, and with a similar reasoning for $\beta < 0$). Thus,

$$u(x) \le \sqrt{1 + \alpha^2 - x^2} \le \sqrt{(1+\alpha)^2 - x^2} \quad \text{in } [-1,1].$$

If $\alpha\beta > 1$, u satisfies $x + u(x)u'(x) \le 0$ in $[0,1]$ (see [DFGS, Lemma 3.9]), and integrating from 0 to $x \in (0,1]$ gives

$$u(x) \le \sqrt{u(0)^2 - x^2} \quad \text{with } u(0) \le \alpha + \beta$$

where the last inequality follows from $|u'(x)| \le \beta$ for all $x \in [-1,1]$ ([DFGS, Theorem 3.11]). □

Lemma B.2. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Let $u \in C^\infty([-1,1],(0,\infty))$ denote the solution of (2.7), constructed in the proof of Theorem 2.4. Then u restricted on $[0,1]$ has the following properties:*

1. *If $\beta \ge 0$, then $-\max\{\beta, \alpha^{-1}\} \le u'(x) \le 0$.*
2. *If $\beta < 0$, $-\beta < \alpha$ and $\alpha \ge \alpha_\beta$, then $0 \le u'(x) \le -\beta$.*
3. *If $\beta < 0$, $-\beta \ge \alpha$ and $\alpha \ge \alpha_\beta$, then $0 \le u'(x) \le \sinh(b_2)$ with $b_2 = b_2(\alpha)$ from (2.4).*
4. *If $\beta < 0$ and $\alpha < \alpha_\beta$, then*

$$-\frac{\sqrt{1+\beta^2}}{\alpha} \le u'(x) \le \max\{-\beta, \alpha^*\}.$$

In the particular case $\alpha_\beta > \alpha \ge \alpha^*$ and $\beta > -\sinh(b_1(\alpha))$, we have

$$\sinh(b_1 x) \ge u'(x) \ge -(\alpha^*)^{-1}$$

with $b_1 = b_1(\alpha)$ as defined in (2.4).

Proof. 1. Let $\beta \ge 0$. If $\alpha\beta \le 1$, u satisfies $x + u(x)u'(x) \ge 0$ and $u'(x) \le 0$ in $[0,1]$ ([DFGS, Lemma 3.16]). It follows $|u'(x)| \le \alpha^{-1}$. If $\alpha\beta > 1$, u satisfies $u'(x) \le 0$ and $|u'(x)| \le \beta$ for all $x \in [-1,1]$ ([DFGS, Theorem 3.11]).

2. See Theorem 4.24 from [DFGS].

3. See Theorem 4.17 from [DFGS].

4. Following [DFGS, Lemma 4.27], u hast at most three critical points in $[-1,1]$, and it holds $u'(x) \leq \max\{-\beta, \alpha^*\}$ ([DFGS, Th. 4.39] if $\alpha \geq \alpha^*$, [DFGS, Th. 4.48] if $\alpha < \alpha^*$). If u has exactly one critical point in $[-1,1]$, then $u' \geq 0$ in $[0,1]$. Otherwise, there is $x_0 \in (0,1)$ so that $u'(x_0) = 0$, $u' > 0$ in $(x_0, 1]$, $u' < 0$ in $(0, x_0)$. Since $x + u(x)u'(x) \geq 0$ in $[0,1]$ and

$$u(x_0) \geq \frac{\alpha}{\operatorname{arsinh}(-\beta)(\alpha_\beta - \alpha)} x_0 \tag{2.17}$$

by Lemma 4.29 from [DFGS], we get by definition of α_β

$$u'(x) \geq -\frac{\operatorname{arsinh}(-\beta)(\alpha_\beta - \alpha)}{\alpha} \geq -\frac{\cosh(\operatorname{arsinh}(-\beta))}{\alpha} = -\frac{\sqrt{1+\beta^2}}{\alpha}.$$

In the special case $\alpha_\beta > \alpha \geq \alpha^*$ and $\beta > -\sinh(b_1(\alpha))$, Lemma 4.36 from [DFGS] gives

$$u(x) \geq \frac{1}{b_1}\cosh(b_1 x) \quad \text{and} \quad u'(x) \leq \sinh(b_1 x).$$

The estimate follows using $x + u(x)u'(x) \geq 0$ in $[0,1]$.

□

For $\alpha > \alpha^*$ we defined $b_1 = b_1(\alpha) < b^* < b_2 = b_2(\alpha)$ via $\alpha = \cosh(b_i)/b_i$ for $i = 1, 2$.

Lemma B.3. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Let $u \in C^\infty([-1,1],(0,\infty))$ be the solution of (2.7) constructed in Theorem 2.4. Then u restricted on $[0,1]$ has the following properties:*

1. *If $\beta \geq 0$, then $u(x) \geq \alpha$.*
2. *If $\beta < 0$ and $\alpha > \alpha_\beta$, then*

$$\min_{x\in[0,1]} u(x) \geq \frac{\sinh(b_2)}{e^{C_1} - 1} \quad \textit{with } C_1 = 2\cosh(2b_2)(1 + \operatorname{arsinh}(-\beta)(\alpha - \alpha^*)). \tag{2.18}$$

3. *If $\beta < 0$ and $\alpha = \alpha_\beta$, then $u(x) \geq b_2^{-1}$.*
4. *If $\beta < 0$ and $\alpha < \alpha_\beta$, then*

$$u(x) \geq \min\left\{\frac{1}{2}\frac{\alpha}{\sqrt{1+\beta^2}}, \frac{\max\{-\beta,\alpha^*\}}{e^{C_2} - 1}\right\} \quad \textit{with } C_2 = 8(1 + \max\{-\beta, \alpha^*\}^2).$$

   In the particular case $\alpha_\beta > \alpha \geq \alpha^*$ and $\beta > -\sinh(b_1(\alpha))$, this estimate can be improved with $u(x) \geq b_1^{-1}$.

Proof. 1. By Lemma B.2 we know that if $\beta \geq 0$, then $u' \geq 0$ and hence $u(x) \geq \alpha$.

For the other cases we recall Lemma 4.9 from [DFGS]: Let $\nu := \max_{x\in[0,1]}\{u'(x)\}$ and $x_0 \geq 0$ so that $u'(x_0) = 0$ and $u' > 0$ in $(x_0, 1]$. Then there hold

$$\min_{x\in[0,1]} u(x) = u(x_0) \geq \nu\frac{1 - x_0}{e^C - 1} \quad \text{with } C = \frac{1}{2}\nu\sqrt{1+\nu^2}\left(\mathcal{W}_1(u) + \frac{4\beta}{\sqrt{1+\beta^2}}\right). \tag{2.19}$$

2. If $\beta < 0$ and $\alpha > \alpha_\beta$, we have $x_0 = 0$ and $u'(x) \leq \sinh(b_2(\alpha))$ for $x \in [0,1]$ by Lemma B.2. Here, use that $-\beta < \sinh(b_2(\alpha))$ if $-\beta < \alpha$. The estimate (2.18) follows from (2.19) using the energy estimate from ([DFGS, Prop.6.8]):

$$\mathcal{W}_1(u) \leq \frac{-8\beta}{\sqrt{1+\beta^2}}(1 + \operatorname{arsinh}(-\beta)(\alpha - \alpha_\beta)) \leq 8(1 + \operatorname{arsinh}(-\beta)(\alpha - \alpha^*)).$$

3. If $\beta < 0$ and $\alpha = \alpha_\beta$, the solution is $u(x) = \cosh(b_1 x)/b_1$ or $u(x) = \cosh(b_2 x)/b_2$ with $b_1 \leq b_2$ and hence, the estimate follows directly.

4. If $\beta < 0$ and $\alpha < \alpha_\beta$, we distinguish between $x_0 \leq 1/2$ and $x_0 > 1/2$. In the first case, we proceed as we did for $\alpha > \alpha_\beta$. By the estimate on u' from Lemma B.2, and taking the following energy estimate into account (see [DFGS, Prop.6.10])

$$\mathcal{W}_1(u) \leq \frac{-8\beta}{\sqrt{1+\beta^2}} + 8\tanh\left(\operatorname{arsinh}(-\beta)\frac{\alpha - \alpha_\beta}{\alpha}\right) \leq 16,$$

inequality (2.19) gives us

$$\min_{x\in[0,1]} u(x) \geq \frac{\max\{-\beta, \alpha^*\}}{e^{C_2} - 1} \text{ with } C_2 = 8(1 + \max\{-\beta, \alpha^*\}^2).$$

But if $x_0 \geq 1/2$, it follows from Lemma 4.29 in [DFGS] (see (2.17)) that

$$u(x) \geq u(x_0) \geq \frac{1}{2}\frac{\alpha}{\operatorname{arsinh}(-\beta)(\alpha_\beta - \alpha)} \geq \frac{1}{2}\frac{\alpha}{\cosh(\operatorname{arsinh}(-\beta))} = \frac{1}{2}\frac{\alpha}{\sqrt{1+\beta^2}}\,.$$

In the special case $\alpha_\beta > \alpha \geq \alpha^*$ and $\beta > -\sinh(b_1(\alpha))$, Lemma 4.36 from [DFGS] gives us $u(x) \geq \cosh(b_1 x)/b_1$, and therefore $u(x) \geq 1/b_1$.

□

Acknowledgment. The second author thanks Klaus Deckelnick and Hans-Christoph Grunau for useful discussions.

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