Willmore surfaces of revolution bounding two prescribed circles

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March 30, 2011

Abstract

We consider the family of smooth embedded rotationally symmetric annular type surfaces in $\mathbb{R}^3$ having two concentric circles contained in two parallel planes of $\mathbb{R}^3$ as boundary. Minimising the Willmore functional within this class of surfaces we prove the existence of smooth axi-symmetric Willmore surfaces having these circles as boundary. When the radii of the circles tend to zero we prove convergence of these solutions to the round sphere.

Keywords. Natural boundary conditions, Willmore surface, surface of revolution.
AMS classification. 49Q10; 53C42, 35J65, 34L30.

1 Introduction and main results

A smooth, immersed two-dimensional surface $\Gamma \subset \mathbb{R}^3$ is a Willmore surface if it is stationary with respect to compactly supported variations for the Willmore functional

$$\mathcal{W}(\Gamma) := \int_\Gamma H^2 \, dA \,. \tag{1}$$

Here H is the mean curvature of Γ. The Willmore functional is a special case of the more general Helfrich functional. These functionals are of geometric interest. They appear, in particular, in the theory of elasticity as models for the elastic energy of thin planes (see [7], [12] and [13]). The Euler-Lagrange equation (called Willmore equation) associated to (1) is

$$\triangle H + 2H(H^2 - K) = 0 \quad \text{on } \Gamma \tag{2}$$

where $\triangle$ denotes the Laplace-Beltrami operator on the surface Γ.

Many results concerning existence and regularity of closed Willmore surfaces are present in the literature, see for instance [16], [1], [9] and [14]. We are interested in studying existence of Willmore surfaces with boundary and satisfying prescribed boundary conditions. Even though already in 1993 Nitsche in [12] attracted the attention to this problem not much is yet known. One of the main difficulties is that equation (2) is of fourth order and not uniformly elliptic. Moreover, the Willmore functional is not convex. Schätzle in [15] proved existence of Willmore immersions in $\mathbb{S}^n$ satisfying Dirichlet boundary conditions. Working on $\mathbb{S}^n$ some compactness problems could be overcome. Another approach to the problem is to study existence of solutions to (2) with boundary conditions under certain symmetry assumptions. This leads to the study of Willmore surfaces of revolution. Existence of Willmore surfaces of revolution generated by symmetric graphs satisfying arbitrary symmetric Dirichlet boundary conditions has been proven in [5] (see also [4])

*Financial support of "Deutsche Forschungsgemeinschaft" for the project "Randwertprobleme für Willmoreflächen - Analysis, Numerik und Numerische Analysis" (DE 611/5.1) is gratefully acknowledged

by solving a minimisation problem. Scholtes in [17] studied the functional obtained by adding an additional area term to the Willmore functional. He could prove existence of minimisers in the class of surfaces of revolution generated by graphs satisfying prescribed (but not arbitrary) Dirichlet boundary data.

Another challenging boundary value problem is obtained by fixing only the boundary of the surfaces among which to vary. Since the problem is of fourth order, a second boundary condition 'arises' , the so-called 'natural' boundary condition. In the considered case the natural boundary condition is that the mean curvature has to be zero at the boundary (see [2, App.A] or [18]). This boundary value problem for surfaces of revolution generated by symmetric graphs has been studied in [2] and [6]. In this paper we extend the results from [2]. Here we consider surfaces of revolution generated by rotating a regular smooth curve along the x-axis. The boundary consists of two circles on planes parallel to the y, z-plane and centered at $(-1, 0, 0)$ and $(1, 0, 0)$ respectively. The radii are arbitrary, in particular the two circles do not necessarily have the same radius. Moreover, we do not restrict ourselves to graphs and neither to symmetric curves.

Before stating the main theorem we introduce for some parameter $\alpha_l > 0$ the number

$$\alpha_r^*(\alpha_l) := \inf_{\gamma\in\mathbb{R}} \frac{\alpha_l}{\cosh(\gamma)} \cosh\left(\frac{2\cosh(\gamma)}{\alpha_l} + \gamma\right) > 0\,. \tag{3}$$

Denoting by $S_r := \{re^{i\varphi} : \varphi \in \mathbb{R}\}$ the circle of radius r centered at the origin, our main result is the following:

Theorem 1.1. *Let $C_{\alpha_l} := \{-1\} \times S_{\alpha_l}$, $C_{\alpha_r} := \{1\} \times S_{\alpha_r}$ denote two concentric circles in parallel planes of $\mathbb{R}^3$ with radii $\alpha_l, \alpha_r > 0$. Then there exists some smooth, annular type Willmore surface $\Gamma \subset \mathbb{R}^3$ minimising the Willmore energy among all rotationally symmetric, annular type surfaces with boundary $C_{\alpha_l} \cup C_{\alpha_r}$. The surface Γ is embedded into $\mathbb{R}^3$ and admits the representation*

$$\Gamma = \{(x, u(x)\cos\varphi, u(x)\sin\varphi) : x \in [-1, 1]\,,\ \varphi \in \mathbb{R}\} \tag{4}$$

with some function $u \in C^\infty([-1, 1], (0, +\infty))$. The surface Γ is solution of the following boundary value problem

$$\begin{cases} \triangle H + 2H(H^2 - K) = 0 & \text{on } \Gamma, \\ \partial\Gamma = C_{\alpha_l} \cup C_{\alpha_r}, & H = 0 \ \text{ on } \partial\Gamma\,. \end{cases} \tag{5}$$

Finally, one of the following three alternatives holds:

a) If $\alpha_r > \alpha_r^*(\alpha_l)$, there exist precisely two such solutions Γ, both being catenoids with $H \equiv 0$.

b) If $\alpha_r = \alpha_r^*(\alpha_l)$, there exists precisely one such solution Γ, a catenoid with $H \equiv 0$.

c) If $\alpha_r < \alpha_r^*(\alpha_l)$, there exists at least one such solution Γ. Its mean curvature satisfies $H = 0$ on $C_{\alpha_l} \cup C_{\alpha_r}$ and $H \neq 0$ on $\Gamma\backslash(C_{\alpha_l} \cup C_{\alpha_r})$.

Naturally, alternative $c)$ is the most interesting part of this result as the constructed Willmore surface is not a minimal surface. Alternative $c)$ corresponds precisely to the case where no annular type minimal surface spanning the two concentric circles exists (see Proposition 2.1). Also note that the solution from part $c)$ minimises under axi-symmetric variations but is only stationary under general (i.e. not necessarily axi-symmetric) variations. Presently, we do not know whether there exists some non-rotationally symmetric, annular type surface spanning $C_{\alpha_l} \cup C_{\alpha_r}$ with smaller Willmore energy than the one constructed in Theorem 1.1.

Our second result concerns the limit case when both α_l and α_r converge to zero, i.e. the bounding circles C_{α_l} and C_{α_r} collapse to the points $(-1, 0, 0)$ and $(1, 0, 0)$ respectively.

Theorem 1.2. *For α_l, $\alpha_r > 0$ let $\Gamma = \Gamma_{\alpha_l,\alpha_r}$ be the surface from Theorem 1.1 above and u_{α_l,α_r} be the positive function generating the surface $\Gamma_{\alpha_l,\alpha_r}$ as in (4). Then $\Gamma_{\alpha_l,\alpha_r}$ converges to the round sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ as both $\alpha_l, \alpha_r \to 0$ in the sense that the functions u_{α_l,α_r} converge uniformly to the function $\sqrt{1-x^2}$ in $[-1,1]$.*

The asymptotic behavior of the minimisers in case of Willmore surfaces of revolution generated by symmetric graphs with prescribed Dirichlet boundary conditions is studied in [5]. In that paper it is proven that the functions generating the minimisers converge to the function $\sqrt{1-x^2}$ in $C^m([-1+\delta_0, 1-\delta_0])$ for any $\delta_0 > 0$. More precisely, in case of symmetric Dirichlet boundary conditions one has two parameters. One prescribes the same radius $\alpha > 0$ for both boundary circles. Another parameter $\beta \in \mathbb{R}$ describes the contact angle between the surface and the two planes containing the bounding circles. In the limit procedure in [5], β is kept fixed while α is converges to zero. A similar result is proven in [8] in case of symmetric natural boundary conditions.

1.1 Notation and structure of the paper

For $a, b \in \mathbb{R}$, $a < b$, let $c(t) = (x(t), y(t)) : [a,b] \to \mathbb{R} \times (0, +\infty)$ be some smooth regular curve and

$$\Gamma = \{(x(t), y(t)\cos\varphi, y(t)\sin\varphi) \,:\, t \in [a,b]\,,\ \varphi \in [0, 2\pi)\}$$

be the surface of revolution corresponding to c. The Willmore energy of Γ is given by

$$\mathcal{W}(c) = \frac{\pi}{2}\int_a^b \Big(\frac{x'y'' - x''y'}{(x'^2+y'^2)^{3/2}} - \frac{x'}{y(x'^2+y'^2)^{1/2}}\Big)^2 y(x'^2+y'^2)^{1/2}\, dt\,.$$

If the curve c is in fact a graph over the x-axis, i.e. $c(t) = (t, u(t))$, then we obtain

$$\mathcal{W}(c) = \mathcal{W}(u) = \frac{\pi}{2}\int_a^b \Big(\frac{u''}{(1+u'^2)^{3/2}} - \frac{1}{u(1+u'^2)^{1/2}}\Big)^2 u(1+u'^2)^{1/2}\, dx\,. \tag{6}$$

Definition 1.1. Let $\widetilde{T}_{\alpha_l,\alpha_r}$ denote the set of all regular curves $c \in W^{2,2}([-1,1], \mathbb{R} \times (0,+\infty))$ connecting the points $(-1, \alpha_l)$ and $(1, \alpha_r)$, i.e. $c(-1) = (-1, \alpha_l)$, $c(1) = (1, \alpha_r)$. Moreover, let T_{α_l,α_r} denote the set of all functions $u \in W^{2,2}([-1,1], (0,+\infty))$ with boundary conditions $u(-1) = \alpha_l$, $u(1) = \alpha_r$. Finally, we define

$$\widetilde{M}_{\alpha_l,\alpha_r} = \inf_{c \in \widetilde{T}_{\alpha_l,\alpha_r}} \mathcal{W}(c) \quad \text{and} \quad M_{\alpha_l,\alpha_r} = \inf_{u \in T_{\alpha_l,\alpha_r}} \mathcal{W}(u)\,.$$

In order to show that M_{α_l,α_r} is attained it is convenient to work in a smaller class than T_{α_l,α_r}.

Definition 1.2. Given parameters $\alpha_l, \alpha_r > 0$, $L > 0$ we define the space

$$T_{\alpha_l,\alpha_r,L} := \big\{u \in T_{\alpha_l,\alpha_r} \ :\ u(x) \geq L^{-1} \text{ and } |u'(x)| \leq L \text{ in } [-1,1]\big\}$$

as well as the numbers

$$M_{\alpha_l,\alpha_r,L} := \inf_{u \in T_{\alpha_l,\alpha_r,L}} \mathcal{W}(u)\,.$$

Remark. The set $T_{\alpha_l,\alpha_r,L}$ is empty if $L > 0$ is too small. However, $T_{\alpha_l,\alpha_r,L}$ is non-empty and hence $M_{\alpha_l,\alpha_r,L}$ well-defined for sufficiently large L, see Lemma 3.1 below.

The reason for working within the smaller class $T_{\alpha_l,\alpha_r,L}$ is that it is relatively simple to construct minimisers $u = u_L$ in this class. The main task consists in proving a priori estimates for these minimisers u_L independent of L.

The paper is organised as follows. In Section 2 we prove the equality $\widetilde{M}_{\alpha_l,\alpha_r} = M_{\alpha_l,\alpha_r}$. Hence it is sufficient to study the minimisation problem in the class of graphs. In Section 3 we show a priori estimates for minimisers in the smaller class $T_{\alpha_l,\alpha_r,L}$ and prove that these estimates are independent of L for L sufficiently large. This is the key point in the proof of Theorem 1.1 presented in Section 4. Finally in Section 5 we study the behavior of minimisers for $\alpha_l, \alpha_r \to 0$ and prove Theorem 1.2.

2 Reduction to the case of graphs and monotonicity property of the energy

Identifying some function $u \in T_{\alpha_l,\alpha_r}$ with its graph parametrisation $c(t) := (t, u(t)) \in \widetilde{T}_{\alpha_l,\alpha_r}$ we obtain the inclusion $T_{\alpha_l,\alpha_r} \subset \widetilde{T}_{\alpha_l,\alpha_r}$ and hence $\widetilde{M}_{\alpha_l,\alpha_r} \leq M_{\alpha_l,\alpha_r}$. The goal of this section is to prove the equality $\widetilde{M}_{\alpha_l,\alpha_r} = M_{\alpha_l,\alpha_r}$.

We start by determining for which data $\alpha_l, \alpha_r > 0$ a minimal surface actually is a solution of the boundary value problem (5). For this purpose we consider the catenaries through $(-1, \alpha_l)$, i.e. the one-parameter family

$$u_\gamma(x) := \frac{\alpha_l}{\cosh(\gamma)} \cosh\Big(\frac{\cosh(\gamma)}{\alpha_l}(x+1) + \gamma\Big), \quad x \in \mathbb{R} \tag{7}$$

with a parameter $\gamma \in \mathbb{R}$. We have $u_\gamma(-1) = \alpha_l$, $u_\gamma'(-1) = \sinh(\gamma)$ and vanishing Willmore energy $\mathcal{W}(u_\gamma) = 0$. The surface of revolution corresponding to u_γ is a minimal surface, called catenoid. A catenary belongs to T_{α_l,α_r} whenever there is a $\gamma \in \mathbb{R}$ such that $u_\gamma(1) = \alpha_r$. One can see that this is equivalent to $\alpha_r \geq \alpha_r^*(\alpha_l)$ with $\alpha_r^*(\alpha_l)$ defined in (3) in the introduction. We first prove the following result, which we already mentioned in the introduction.

Proposition 2.1. *For $\alpha_l, \alpha_r > 0$ let C_{α_l}, C_{α_r} denote the two circles from Theorem 1.1. Then one of the following three alternatives holds:*

a) If $\alpha_r > \alpha_r^*(\alpha_l)$, then there are precisely two annular type minimal surfaces spanning $C_{\alpha_l} \cup C_{\alpha_r}$, both being catenoids.

b) If $\alpha_r = \alpha_r^*(\alpha_l)$, there exists precisely one annular type minimal surface spanning $C_{\alpha_l} \cup C_{\alpha_r}$, a catenoid.

c) If $\alpha_r < \alpha_r^*(\alpha_l)$, no annular type minimal surface spanning $C_{\alpha_l} \cup C_{\alpha_r}$ exists.

Proof. Due to [11, Theorem 1.1] there exist at most two such minimal surfaces. In particular, all annular type minimal surfaces spanning $C_{\alpha_l} \cup C_{\alpha_r}$ must be surfaces of revolution, since otherwise one might produce infinitely many of them simply by rotation. However, catenoids (and planes) are the only minimal surfaces of revolution and the claim follows from definition of $\alpha_r^*(\alpha_l)$. □

A simple consequence is

Lemma 2.2. *Given $\alpha_l > 0$ let $\alpha_r^*(\alpha_l)$ be defined as in* (3)*. A catenary belongs to the space T_{α_l,α_r} whenever $\alpha_r \geq \alpha_r^*(\alpha_l)$. In particular, $M_{\alpha_l,\alpha_r} = \widetilde{M}_{\alpha_l,\alpha_r} = 0$ is satisfied for any $\alpha_r \geq \alpha_r^*(\alpha_l)$.*

This lemma and a simple study of the function u_γ defined in (7) immediately yield part a) and b) of Theorem 1.1. Our next result describes a construction to replace a regular curve by a curve admitting a non-parametric representation with almost the same Willmore energy but lower boundary values.

Lemma 2.3. *Given $\alpha_l, \alpha_r > 0$, any curve $c \in \widetilde{T}_{\alpha_l,\alpha_r}$ and $\delta > 0$ there exist $\alpha_l' \in (0, \alpha_l)$, $\alpha_r' \in (0, \alpha_r)$ and some function $u \in T_{\alpha_l',\alpha_r'}$ with $\mathcal{W}(u) \le \mathcal{W}(c) + \delta$.*

Proof. For $\varepsilon > 0$ we define the curve $c_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t))$ by

$$y_\varepsilon(t) := \frac{1}{\varrho} y(t) \quad , \quad x_\varepsilon(t) := -1 + \frac{1}{\varrho}\int_{-1}^{t} \big(|x'(\tau)| + \varepsilon\big) d\tau \quad \text{for } t \in [-1, 1]$$

with the rescaling factor

$$\varrho = \varrho(\varepsilon) := \frac{1}{2}\int_{-1}^{1} \big(|x'(\tau)| + \varepsilon\big) d\tau \ge 1 + \varepsilon > 1 \, .$$

Note that c_ε is a regular curve, $c_\varepsilon \in W^{2,2}([-1, 1], \mathbb{R} \times (0, +\infty))$, $x_\varepsilon(-1) = -1$ and $x_\varepsilon(1) = 1$ are satisfied. Due to the conformal invariance of the Willmore functional, in particular the invariance with respect to translations and reflections, one finds $\mathcal{W}(c) = \mathcal{W}(c_0)$. Together with the continuity of $\varepsilon \mapsto \mathcal{W}(c_\varepsilon)$ one deduces the convergence $\mathcal{W}(c_\varepsilon) \to \mathcal{W}(c)$ as $\varepsilon \to 0$. Because of $x_\varepsilon'(t) \ge \frac{\varepsilon}{\varrho} > 0$ in $[-1, 1]$ the curve c_ε has a non-parametric representation $u_\varepsilon := y_\varepsilon \circ x_\varepsilon^{-1} \in W^{2,2}([-1, 1], (0, +\infty))$. The claim follows noting that $u_\varepsilon(-1) = \frac{y(-1)}{\varrho} < y(-1) = \alpha_l$, $u_\varepsilon(1) = \frac{y(1)}{\varrho} < y(1) = \alpha_r$ (since $\varrho > 1$) as well as $\mathcal{W}(u_\varepsilon) = \mathcal{W}(c_\varepsilon) \to \mathcal{W}(c)$ for $\varepsilon \to 0$. ☐

Lemma 2.4. *Given $\alpha_l, \alpha_r > 0$, any $u \in T_{\alpha_l,\alpha_r}$ and $\alpha_r' \ge \alpha_r$ there exists some $v \in T_{\alpha_l,\alpha_r'}$ with $\mathcal{W}(v) \le \mathcal{W}(u)$.*

Proof. We need to study only the case $\alpha_r' < \alpha_r^*(\alpha_l)$ since otherwise there exists some catenary $v \in T_{\alpha_l,\alpha_r'}$ with $\mathcal{W}(v) = 0$. For some parameter $\tau \in [-1, 1]$ consider the function

$$v_\tau(x) := \begin{cases} u(x) & \text{for } x \in [-1, \tau] \\ w(x) & \text{for } x \in (\tau, 1] \end{cases}$$

where $w(x)$ denotes the catenary with initial data $w(\tau) = u(\tau)$, $w'(\tau) = u'(\tau)$. Note that v_τ depends continuously on τ and satisfies $v_\tau(-1) = u(-1) = \alpha_l$. The function v_{-1} coincides with some catenary and thus $v_{-1}(1) \ge \alpha_r^*(\alpha_l) > \alpha_r' \ge \alpha_r$ must hold. The function v_1 coincides with $u \in T_{\alpha_l,\alpha_r}$ and hence $v_1(1) = u(1) = \alpha_r$. The intermediate value theorem yields some $\tau = \tau(\alpha_r') \in [-1, 1]$ such that $v_\tau(1) = \alpha_r'$, i.e. $v_\tau \in T_{\alpha_l,\alpha_r'}$. From the estimate

$$\mathcal{W}(u) \ge \mathcal{W}(u|_{[-1,\tau]}) = \mathcal{W}(v_\tau)$$

the claim follows with $v = v_\tau$. ☐

Lemmas 2.3 and 2.4 yield $\widetilde{M}_{\alpha_l,\alpha_r} = M_{\alpha_l,\alpha_r}$ (see Definition 1.1) and the monotonicity of the energy.

Corollary 2.5 ($\widetilde{M}_{\alpha_l,\alpha_r} = M_{\alpha_l,\alpha_r}$). *The equality $\widetilde{M}_{\alpha_l,\alpha_r} = M_{\alpha_l,\alpha_r}$ holds for any $\alpha_l, \alpha_r > 0$, i.e. any minimiser within the small class T_{α_l,α_r} is also a minimiser in the larger class $\widetilde{T}_{\alpha_l,\alpha_r}$.*

Proof. Because of $T_{\alpha_l,\alpha_r} \subset \widetilde{T}_{\alpha_l,\alpha_r}$ we only need to prove $\widetilde{M}_{\alpha_l,\alpha_r} \geq M_{\alpha_l,\alpha_r}$. Given any $c \in \widetilde{T}_{\alpha_l,\alpha_r}$ and $\delta > 0$ Lemma 2.3 yields some $\alpha_l' \in (0, \alpha_l)$, $\alpha_r' \in (0, \alpha_r)$ and $u \in T_{\alpha_l',\alpha_r'}$ such that $\mathcal{W}(u) \leq \mathcal{W}(c) + \delta$. Applying Lemma 2.4 twice one obtains some $v \in T_{\alpha_l,\alpha_r}$ with $\mathcal{W}(v) \leq \mathcal{W}(u)$. We conclude $\mathcal{W}(v) \leq \mathcal{W}(c) + \delta$ and hence $M_{\alpha_l,\alpha_r} \leq \mathcal{W}(c) + \delta$ for any $c \in \widetilde{T}_{\alpha_l,\alpha_r}$ and $\delta > 0$. This yields $M_{\alpha_l,\alpha_r} \leq \widetilde{M}_{\alpha_l,\alpha_r}$. $\square$

Corollary 2.6 (Monotonicity of the energy)**.** *Let M_{α_l,α_r} be defined as in Definition 1.1 for $\alpha_l, \alpha_r > 0$. Then M_{α_l,α_r} is monotonically decreasing in α_l for each fixed α_r, and monotonically decreasing in α_r for each fixed α_l.*

By Corollary 2.5 the above result is also valid for $\widetilde{M}_{\alpha_l,\alpha_r}$.

3 A priori estimates for the constrained minimisers

In this section we prove a priori estimates for the minimisers in $T_{\alpha_l,\alpha_r,L}$ (see Definition 1.2).

We start with establishing an upper bound on the energy $M_{\alpha_l,\alpha_r,L}$ from Definition 1.2, assuming L to be sufficiently large.

Lemma 3.1. *For $\alpha_l, \alpha_r > 0$ there exists a constant L_0 depending only on α_l, α_r such that the energy satisfies $M_{\alpha_l,\alpha_r,L} < 4\pi$ whenever $L \geq L_0$.*

Proof. Consider the circular arc

$$v(x) := \sqrt{\frac{\alpha_r^2 + \alpha_l^2}{2} + 1 - x^2 + \frac{\alpha_r^2 - \alpha_l^2}{2} x} \quad \text{for } x \in [-1, 1] \tag{8}$$

which belongs to T_{α_l,α_r} and hence also to $T_{\alpha_l,\alpha_r,L}$, provided $L \geq L_0$ with

$$L_0 = L_0(\alpha_l, \alpha_r) := \max_{x \in [-1,1]} \left(v(x)^{-1} + |v'(x)| \right). \tag{9}$$

The surface of revolution corresponding to v is a piece of a sphere and hence $\mathcal{W}(v) < 4\pi$, as the Willmore energy of a sphere is 4π. We conclude $M_{\alpha_l,\alpha_r,L} < 4\pi$ whenever $L \geq L_0$. $\square$

3.1 Estimates on the hyperbolic curvature of the minimisers

As already observed by Bryant and Griffiths [3] and Langer and Singer [10], there is an interesting relation between the Willmore energy of surfaces of revolution and the elastic energy of curves in the hyperbolic half-plane. Indeed, for $u \in T_{\alpha_l,\alpha_r}$ and $a, b \in [-1, 1]$, $a < b$, one has

$$\mathcal{W}(u|_{[a,b]}) = \frac{\pi}{2} \int_a^b \kappa^2(x) \frac{\sqrt{1 + u'^2}}{u} \, dx - 2\pi \Big[\frac{u'}{\sqrt{1 + u'^2}} \Big]_a^b \tag{10}$$

where

$$\kappa(x) := \frac{u u''}{(1 + u'^2)^{3/2}} + \frac{1}{(1 + u'^2)^{1/2}} = \frac{u u'' + 1 + u'^2}{(1 + u'^2)^{3/2}} \tag{11}$$

denotes the curvature of the planar curve $x \mapsto (x, u(x))$ with respect to the hyperbolic half-plane metric. Curves with $\kappa(x) \equiv 0$ are precisely the geodesics of the hyperbolic half-plane. These are semi-circles whose center lie on the x-axis or semi-lines parallel to the y-axis. These curves play an essential role in studying Willmore surfaces of revolution (see [4], [5] and [2]).

Using circles as barriers from below and catenaries as barriers from above we prove pointwise bounds on the hyperbolic curvature of any minimiser in $T_{\alpha_l,\alpha_r,L}$.

Lemma 3.2. *For $\alpha_l, \alpha_r > 0$ let L_0 be the constant defined in* (9)*. Then the hyperbolic curvature of any minimiser $u \in T_{\alpha_l,\alpha_r,L}$, $L \geq L_0$, satisfies $0 \leq \kappa(x) \leq 2$, $x \in [-1,1]$. Moreover, the inequality $u(x) \leq v(x)$ holds in $[-1,1]$, v denoting the circular arc from (8).*

Proof.

1) We first prove the lower bound $\kappa(x) \geq 0$. For parameters $z \in \mathbb{R}$, $\varrho > 0$ define the function

$$s_{\varrho,z}(x) := \sqrt{\max\left\{\varrho^2 - (x-z)^2, 0\right\}} \quad , \; x \in \mathbb{R}.$$

Note that $s_{\varrho,z}$ restricted to $[z-\varrho, z+\varrho]$ is simply a semi-circle centered at $(z,0)$ of radius ϱ. For $z \in \mathbb{R}$ we next define

$$\begin{aligned} r(z) &:= \sup\{\varrho > 0 \;:\; s_{\varrho,z}(x) \leq u(x) \text{ for all } x \in [-1,1]\} \quad \text{and} \\ g(z) &:= \{x \in [-1,1] \;:\; u(x) = s_{r(z),z}(x)\}\;. \end{aligned}$$

Then $g(z)$ is a nonempty, closed subset of $[-1,1]$ for any $z \in \mathbb{R}$. We prove that $g(z)$ is actually a closed interval. Setting $x_1 := \inf g(z)$, $x_2 := \sup g(z)$ and $I := [x_1, x_2]$ we have $g(z) \subset I$. We are done if $x_1 = x_2$ or $g(z) = I$. Otherwise, let v be the function equal to u on $[-1,1]\backslash I$ and equal to $s_{r(z),z}$ on I. We first observe that $v \in T_{\alpha_l,\alpha_r,L}$. Indeed since $u = v$ on ∂I and $v|_I$ is a piece of a semicircle, $v(x) \geq \inf u(x) \geq L^{-1}$ holds on $[-1,1]$. Moreover, $v \in W^{2,2}([-1,1],(0,+\infty))$ and $|v'(x)| \leq L$ since, by construction, $u'(x_1) \geq v'(x_1)$ with equality if $x_1 \in (-1,1)$ and $u'(x_2) \leq v'(x_2)$ with equality if $x_2 \in (-1,1)$. Now, we compare the Willmore energies of u and v. Since $v|_I$ is a piece of a semicircle, its hyperbolic curvature vanishes there. Using formula (10) we estimate

$$\mathcal{W}(v) - \mathcal{W}(u) = \mathcal{W}\big(v|_I\big) - \mathcal{W}\big(u|_I\big) \leq 2\pi\Big[\frac{u'}{\sqrt{1+u'^2}}\Big]_{x_1}^{x_2} - 2\pi\Big[\frac{v'}{\sqrt{1+v'^2}}\Big]_{x_1}^{x_2} \leq 0\;,$$

using once again that $u'(x_2) \leq v'(x_2)$ and $u'(x_1) \geq v'(x_1)$ which follows from $u \geq v$ in I and $u = v$ on $\partial I = \{x_1, x_2\}$. This shows $\mathcal{W}(v) \leq \mathcal{W}(u)$. Furthermore, $u|_I \equiv v|_I$ must hold since otherwise we would obtain the strict inequality $\mathcal{W}(v) < \mathcal{W}(u)$, contradicting the assumption of u being a minimiser in $T_{\alpha_l,\alpha_r,L}$. This proves $I = g(z)$ for all $z \in \mathbb{R}$.

Now we can find some constant $M > 0$ such that $g(z) = \{-1\}$ for all $z \leq -M$ and $g(z) = \{1\}$ for all $z \geq M$. By continuity of the radius $r(z)$ in z and of the function u in x, the graph of the multi-mapping g is closed. It follows that, writing $g(z) = [x_1(z), x_2(z)]$ for $z \in [-M, M]$, the function x_1 is lower semi-continuous ($x_1 : \mathbb{R} \to \mathbb{R}$), while x_2 is upper semi-continuous. Then, given any $x_* \in [-1,1]$, an intermediate value argument (i.e. a bisection argument) yields some $z_* \in \mathbb{R}$ such that $x_* \in g(z_*)$. This means $s_{r(z_*),z_*}(x) \leq u(x)$ in $[-1,1]$ and $s_{r(z_*),z_*}(x_*) = u(x_*)$, i.e. the graph of u lies above the circle $s_{r(z_*),z_*}$ while it touches the circle at the point $(x_*, u(x_*))$. The circle $s_{r(z_*),z_*}$ has vanishing hyperbolic curvature everywhere and hence $\kappa(x_*) \geq 0$.

2) To prove that $u(x) \leq v(x)$ in $[-1,1]$, let us write v given in (8) as $v(x) = \sqrt{r^2 - (x-x_0)^2}$ for $r > 0$ and $x_0 \in \mathbb{R}$ choosen appropriately. We recall that $v(-1) = \alpha_l$ and $v(1) = \alpha_r$. The inequality $\kappa(x) \geq 0$ proven in part 1) together with (11) imply $0 \leq 2(1 + u'^2 + uu'') = \big[(x-x_0)^2 + u^2(x)\big]''$. Thus the mapping $x \mapsto \varphi(x) := (x-x_0)^2 + u^2(x)$ is convex. Noting $\varphi(-1) = \varphi(1) = r^2$, we deduce $\varphi(x) \leq r^2$ in $[-1,1]$ or equivalently $u(x) \leq \sqrt{r^2 - (x-x_0)^2} = v(x)$ in $[-1,1]$.

3) We now derive the upper bound $\kappa(x) \leq 2$. The idea is similar to part 1). Instead of using semicircles from below, we approach the graph of u from above by suitable catenaries. Choose some function $v \in T_{\alpha_l,\alpha_r}$ such that $u(x) < v(x)$ holds for $x \in (-1,1)$, for example $v(x) := u(x) + 1 - x^2$. For parameters $\gamma \in \mathbb{R}$, $z \in [-1,1]$ let

$$c(x) = c_{\gamma,z}(x) := \frac{v(z)}{\cosh(\gamma)} \cosh\Big(\frac{\cosh(\gamma)}{v(z)}(x-z) + \gamma\Big) \quad \text{for } x \in \mathbb{R}$$

denote the catenary with initial data $c(z) = v(z)$, $c'(z) = \sinh(\gamma)$. For $z \in \mathbb{R}$ we also define

$$\begin{aligned} \gamma(z) &:= \sup\big\{\gamma \in \mathbb{R} \,:\, u(x) \leq c_{\gamma',z}(x) \text{ for all } x \in [-1,z] \text{ and } \gamma' \leq \gamma\big\} \quad \text{and} \\ g(z) &:= \big\{x \in [-1,z] \,:\, u(x) = c_{\gamma(z),z}(x)\big\}\,. \end{aligned}$$

As in part 1), we prove that $g(z)$ is some closed interval by setting $x_1 := \inf g(z)$, $x_2 := \sup g(z)$ and $I := [x_1, x_2]$. If $x_1 < x_2$, then let $w \in T_{\alpha_l,\alpha_r,L}$ denote the function equal to u on $[-1,1]\backslash I$ and equal to $c_{\gamma(z),z}$ on I. Here we note $x_2 < z$, $u'(x_2) = w'(x_2)$ and $u'(x_1) \leq w'(x_1)$. Then the equation

$$\mathcal{W}(w) = \mathcal{W}(w|_{[-1,1]\backslash I}) = \mathcal{W}(u|_{[-1,1]\backslash I}) = \mathcal{W}(u) - \mathcal{W}(u|_I)$$

together with $\mathcal{W}(u) \leq \mathcal{W}(w)$ imply $\mathcal{W}(u|_I) = 0$. However, this is only possible if $u|_I \equiv v|_I$ proving $I = g(z)$. We have $1 \in g(1)$ and $g(-1) = \{-1\}$. The continuity of the function $\gamma(z)$ in z and the continuity of the function u in x give that the graph of the multi-mapping g is closed. An intermediate value argument (as in part 1)) yields for any $x_* \in (-1,1)$ some $z_* \in (x_*, 1)$ with the properties $u(x) \leq c_{\gamma(z_*),z_*}(x)$ in $[-1, z_*]$ and $u(x_*) = c_{\gamma(z_*),z_*}(x_*)$. The graph of u lies locally below the catenary $c_{\gamma(z_*),z_*}$ while it touches the catenary at the point $(x_*, u(x_*))$. The hyperbolic curvature of the catenary $c_{\gamma(z_*),z_*}$ is bounded from above by 2 and we obtain $\kappa(x_*) \leq 2$.

□

Thanks to the pointwise estimates on the hyperbolic curvature of the minimiser we find that it is sufficient to get estimates on the minimiser from below and of the derivative at the boundary in order to get pointwise estimates of the first and second order derivative of the function in the interior of the interval.

Corollary 3.3. *For $\alpha_r, \alpha_l > 0$ let L_0 be the constant defined in* (9). *Let $u \in T_{\alpha_l,\alpha_r,L}$, $L \geq L_0$, be a minimiser for the Willmore energy in this class. Let $K > 0$ and $\varepsilon > 0$ be such that $u'(-1) \geq -K$, $u'(1) \leq K$ as well as $u(x) \geq \varepsilon > 0$ in $[-1,1]$. Then u satisfies the estimates*

$$|u'(x)| \leq C \text{ in } [-1,1] \quad \text{and} \quad |u''(x)| \leq 2(1+C^2)^{3/2}\varepsilon^{-1} \quad \text{a.e. in } [-1,1]$$

with the constant $C = C(K,\varepsilon) = (2 + \max\{\alpha_l, \alpha_r\}K)\varepsilon^{-1}$. In particular, $u \in W^{2,\infty}([-1,1],(0,+\infty))$ is true.

Proof. The inequality $\kappa(x) \geq 0$ from Lemma 3.2 together with (11) imply $1 + u'^2 + uu'' = (x + uu')' \geq 0$. Therefore the mapping $x \mapsto x + u(x)u'(x)$ is increasing. In particular

$$-1 + \alpha_l u'(-1) \leq x + u(x)u'(x) \leq 1 + \alpha_r u'(1) \;\text{ for all } x \in [-1,1]\,, \tag{12}$$

and that gives

$$|u'(x)| \leq (2 + \max\{\alpha_l, \alpha_r\}K)\varepsilon^{-1} = C \;\text{ for all } x \in [-1,1]\,.$$

The inequality $0 \le \kappa(x) \le 2$ from Lemma 3.2 together with (11) also yield

$$-1 \le \frac{uu''}{(1+u'^2)^{3/2}} \le 2 \quad \text{a.e. in } [-1,1]$$

and we conclude

$$|u''(x)| \le 2(1+C^2)^{3/2}\varepsilon^{-1} \text{ a.e. in } [-1,1] \text{ and } u \in W^{2,\infty}([-1,1],(0,+\infty))\,.$$

□

In order to prove Theorem 1.1 it remains to show the a priori estimates $u'(-1) \ge -K$, $u'(1) \le K$ and $u(x) \ge \varepsilon$ with constants K, ε only depending on α_l and α_r but not on L. These estimates are proved in the following section.

3.2 The remaining a priori estimates

We start by proving some estimates on the Willmore energy from below. This yields (see Corollary 3.6 below) a bound on the length of the interval where a function with Willmore energy bounded by 4π is allowed to become small.

Lemma 3.4. *Consider $a, b \in [-1,1]$, $a < b$. The Willmore energy of $u \in W^{2,2}([a,b],(0,+\infty))$ satisfies the two lower bounds*

$$\mathcal{W}(u) \ge -2\pi\Big[\frac{u'}{\sqrt{1+u'^2}}\Big]_a^b \quad \text{and} \quad \mathcal{W}(u) \ge \frac{\pi}{2}\int_a^b \frac{1}{u\sqrt{1+u'^2}}dx - \pi\Big[\frac{u'}{\sqrt{1+u'^2}}\Big]_a^b\,.$$

Proof. Starting from (6) and using the inequality $(p-q)^2 \ge -4pq$ one gets

$$\mathcal{W}(u) \ge -2\pi\int_a^b \frac{u''}{(1+u'^2)^{3/2}}dx = -2\pi\Big[\frac{u'}{\sqrt{1+u'^2}}\Big]_a^b\,.$$

Similarly, we get another estimate from below on the energy starting again from formula (6) and using the inequality $(p-q)^2 \ge q^2 - 2pq$:

$$\mathcal{W}(u) \ge \frac{\pi}{2}\int_a^b \frac{1}{u\sqrt{1+u'^2}}dx - \pi\int_a^b \frac{u''}{(1+u'^2)^{3/2}}dx = \frac{\pi}{2}\int_a^b \frac{1}{u\sqrt{1+u'^2}}dx - \pi\Big[\frac{u'}{\sqrt{1+u'^2}}\Big]_a^b\,.$$

□

Lemma 3.5. *Let $u \in W^{2,2}([0,1],(0,+\infty))$ satisfy $0 < u(x) \le \frac{1}{20}$. Then $\mathcal{W}(u) > \pi$.*

Proof. Set $I := [\frac{1}{4}, \frac{3}{4}]$ and $\varepsilon := \frac{1}{20}$. One of the following three cases will apply.

a) If $|u'(x)| \le 1$ for all $x \in I$, Lemma 3.4 then yields

$$\mathcal{W}(u) \ge \frac{\pi}{2}\int_I \frac{1}{u\sqrt{1+u'^2}} - \pi\Big[\frac{u'}{\sqrt{1+u'^2}}\Big]_{1/4}^{3/4} \ge \frac{\pi}{4\varepsilon\sqrt{2}} - \frac{2\pi}{\sqrt{2}} = \frac{3\pi}{\sqrt{2}} > \pi\,.$$

b) If $u'(x_1) > 1$ for some $x_1 \in I$, then the mean value theorem yields some $x_2 \in (3/4, 1)$ such $u'(x_2) \le 4\varepsilon$ (since $0 < u(x) \le \varepsilon$ in $[0,1]$). Together with Lemma 3.4 we deduce

$$\mathcal{W}(u) \ge \mathcal{W}(u|_{[x_1,x_2]}) \ge 2\pi\Big[\frac{-u'}{\sqrt{1+u'^2}}\Big]_{x_1}^{x_2} \ge 2\pi\Big[\frac{1}{\sqrt{2}} - \frac{4\varepsilon}{\sqrt{1+16\varepsilon^2}}\Big] > \pi\,.$$

c) The remaining case $u'(x_1) < -1$ for some $x_1 \in I$ can be treated as case b).

$\square$

Remark. The smallness condition $0 < u(x) \le \frac{1}{20}$ is surely not optimal. However, note that the constant function $u(x) \equiv \frac{1}{2}$ has Willmore energy $\mathcal{W}(u) = \pi$. Lemma 3.5 will be false if one only requires $0 < u(x) \le \frac{1}{2}$ instead.

Corollary 3.6. *Let $u \in W^{2,2}([a,b],(0,+\infty))$, $a,b \in [-1,1]$ with $a < b$, satisfy $\mathcal{W}(u) < 4\pi$ and $0 < u(x) \le \varepsilon$ in $[a,b]$ for some $\varepsilon > 0$. Then $b - a < 80\varepsilon$ must hold.*

Proof. The claim is proved by contradiction. Let us assume that $\frac{b-a}{80} \ge \varepsilon$. Then the functions

$$u_k(x) := \frac{4}{b-a}\, u\Big(a + \frac{b-a}{4}(x+k)\Big) \quad \text{for } x \in [0,1]\ ,\ k = 0,\dots,3\ ,$$

satisfy $0 < u_k(x) \le \frac{1}{20}$ in $[0,1]$. Lemma 3.5 yields $\mathcal{W}(u_k) \ge \pi$ and together with the invariance of the Willmore energy under translations and rescaling one obtains

$$\mathcal{W}(u) = \sum_{k=0}^{3} \mathcal{W}(u_k) \ge \sum_{k=0}^{3} \pi = 4\pi$$

contradicting the assumption $\mathcal{W}(u) < 4\pi$. $\square$

Comparing the minimisers with catenaries from above, we now obtain an estimate on the derivative at the boundary of the interval and then an estimate from below independent of L. The following lemma gives a bound on the slope of the catenaries that lie completely above the minimiser. The idea is that if the slopes of these catenaries become very large, the catenaries get arbitrarily close to the x-axis and so does the graph of u, lying completely below all these catenaries. Applying Corollary 3.6, we show that this costs too much Willmore energy.

Lemma 3.7. *For $a \in [0,1]$ and $\lambda > 0$ let $u \in W^{2,2}([-1,a],(0,+\infty))$ satisfy $\mathcal{W}(u) < 4\pi$ and $u(a) \le \lambda$. Assume furthermore that there exists $\gamma_* \in \mathbb{R}$ such that*

$$u(x) \le c_\gamma(x) \quad \textit{for all } x \in [-1,a] \quad \textit{and} \quad \gamma \le \gamma_*$$

where c_γ denotes the catenary with initial data $c_\gamma(a) = \lambda$ and $c'_\gamma(a) = \sinh(\gamma)$, i.e.

$$c_\gamma(x) := \frac{\lambda}{\cosh(\gamma)} \cosh\Big(\frac{\cosh(\gamma)}{\lambda}(x-a) + \gamma\Big)\ . \tag{13}$$

Then $\gamma_* \le \max\{162, \lambda\}$ must hold.

Proof. Denote $\bar\gamma := \max\{160, \lambda - 2\}$. We may assume $\gamma_* > \bar\gamma$ since otherwise we are done. For arbitrary $\gamma \in [\bar\gamma, \gamma_*]$ we define $x_\gamma := a - \frac{\lambda\gamma}{\cosh(\gamma)} \in (-1,a)$ with the property

$$u(x_\gamma) \le c_\gamma(x_\gamma) = \frac{\lambda}{\cosh(\gamma)}\ .$$

For all $x \in [a - \lambda\bar\gamma/\cosh(\bar\gamma), a - \lambda\gamma_*/\cosh(\gamma_*)]$ there exists $\gamma \in [\bar\gamma, \gamma_*]$ such that $x = x_\gamma$. We conclude

$$u(x) \le \frac{\lambda}{\cosh(\bar\gamma)} \le \frac{\lambda\bar\gamma}{160\cosh(\bar\gamma)} \quad \text{for all } x \in \Big[a - \frac{\lambda\bar\gamma}{\cosh(\bar\gamma)}, a - \frac{\lambda\gamma_*}{\cosh(\gamma_*)}\Big]\ .$$

Then Corollary 3.6 yields

$$\frac{\lambda\bar{\gamma}}{\cosh(\bar{\gamma})} - \frac{\lambda\gamma_*}{\cosh(\gamma_*)} < \frac{\lambda\bar{\gamma}}{2\cosh(\bar{\gamma})} \, .$$

We conclude

$$\frac{\bar{\gamma}}{2\cosh(\bar{\gamma})} < \frac{\gamma_*}{\cosh(\gamma_*)} \le \frac{\gamma_* - 2}{2\cosh(\gamma_* - 2)}$$

and hence $\gamma_* \le \bar{\gamma} + 2$, proving the claim. □

Theorem 3.8 (Boundary gradient estimate)**.** *Consider $\alpha_r, \alpha_l > 0$ and let L_0 be the constant defined in* (9)*. Any minimiser u for the Willmore energy in the class $T_{\alpha_l,\alpha_r,L}$, $L \ge L_0$, satisfies the estimates*

$$u'(-1) \ge -\sinh\big(\max\{162, \alpha_l\}\big) \quad \textit{and} \quad u'(1) \le \sinh\big(\max\{162, \alpha_r\}\big) \, .$$

Proof. We only prove the upper bound for $u'(1)$ as the proof of the lower bound for $u'(-1)$ is similar. For $\gamma \in \mathbb{R}$ let $c_\gamma(x)$ denote the catenary with initial data $c_\gamma(1) = \alpha_r$, $c'_\gamma(1) = \sinh(\gamma)$ (i.e. c_γ is the catenary given in (13) with $\lambda = \alpha_r$ and $a = 1$). We define the number

$$\gamma_* := \sup\big\{\gamma \in \mathbb{R} \, : \, u(x) \le c_{\gamma'}(x) \text{ for all } x \in [-1,1] \text{ and } \gamma' \le \gamma\big\} \, .$$

From $c_{\gamma_*}(1) = u(1)$ one easily deduces $u'(1) \ge c'_{\gamma_*}(1) = \sinh(\gamma_*)$. We proceed by proving the equality $u'(1) = c'_{\gamma_*}(1)$. It is convenient to distinguish two cases. If the function u satisfies $u(x) < c_{\gamma_*}(x)$ for all $x \in [-1,1)$, the definition of γ_* yields $u'(1) \le c'_{\gamma_*}(1)$, proving the equality. If instead $u(x_*) = c_{\gamma_*}(x_*)$ for some $x_* \in [-1,1)$, we consider the function $v \in T_{\alpha_l,\alpha_r,L}$ that is equal to u on $[-1, x_*]$ and equal to c_{γ_*} on $[x_*, 1]$. The inequality

$$\mathcal{W}(u) \le \mathcal{W}(v) = \mathcal{W}(v|_{[-1,x_*]}) = \mathcal{W}(u|_{[-1,x_*]}) = \mathcal{W}(u) - \mathcal{W}(u|_{[x_*,1]})$$

implies $\mathcal{W}(u|_{[x_*,1]}) = 0$. However, this is only possible if $u|_{[x_*,1]} \equiv v|_{[x_*,1]}$ holds, proving $u'(1) = v'(1) = c'_{\gamma_*}(1) = \sinh(\gamma_*)$ also in the second case. Now Lemma 3.7 with $a = 1$ and $\lambda = u(1) = \alpha_r$ yields $\gamma_* \le \max\{162, \alpha_r\}$ and hence $u'(1) = \sinh(\gamma_*) \le \sinh\big(\max\{162, \alpha_r\}\big)$. □

In the next result we construct at every point $x \in [-1,1]$ a catenary lying completely above the graph of u, while touching the graph at the point $(x, u(x))$. Using Lemma 3.7, we can control the slope of this catenary and hence also the distance of the catenary to the x-axis (see inequality (14) below). The catenaries are constructed with the same idea as in Lemma 3.2.

Theorem 3.9 (Estimate from below)**.** *For $\alpha_r, \alpha_l > 0$ let L_0 be the constant defined in* (9)*. Any minimiser u of the Willmore energy in $T_{\alpha_l,\alpha_r,L}$, with $L \ge L_0$ large enough, satisfies*

$$u(x) \ge \frac{\min\{\alpha_l, \alpha_r\}}{\cosh\left(\max\{162, \alpha_l + 2, \alpha_r + 2\}\right)} \quad \textit{for all } x \in [-1,1] \, .$$

Proof. We proceed as in the proof of Lemma 3.2 part 3). Let $v \in T_{\alpha_l,\alpha_r,L}$ denote the circular arc defined by (8). Consider $\tilde{v} \in T_{\alpha_l,\alpha_r,L}$ defined by $\tilde{v}(x) = v(x) + 1 - x^2$, $x \in [-1,1]$. We have $\min\{\alpha_l, \alpha_r\} \le \tilde{v}(x) \le \max\{\alpha_l, \alpha_r\} + 2$ in $[-1,1]$. It follows from Lemma 3.2 that $u(x) < \tilde{v}(x)$ holds in $(-1,1)$ and $u(\pm 1) = \tilde{v}(\pm 1)$. For parameters $\gamma \in \mathbb{R}$, $z \in [0,1]$ let

$$c(x) = c_{\gamma,z}(x) := \frac{\tilde{v}(z)}{\cosh(\gamma)} \cosh\Big(\frac{\cosh(\gamma)}{\tilde{v}(z)}(x - z) + \gamma\Big) \quad \text{for } x \in \mathbb{R}$$

denote the catenary with initial data $c(z) = \tilde{v}(z)$ and $c'(z) = \sinh(\gamma)$. Next we define

$$\gamma_*(z) := \sup\big\{\gamma \in \mathbb{R} \, : \, u(x) \le c_{\gamma',z}(x) \text{ for all } x \in [-1,z] \text{ and } \gamma' \le \gamma\big\} \, .$$

Lemma 3.7, applied to $u|_{[-1,z]}$ with $\lambda = \tilde{v}(z)$, yields the upper bound

$$\gamma_*(z) \le \max\{162, \tilde{v}(z)\} \le \max\left\{162, \max\{\alpha_l, \alpha_r\} + 2\right\} = \max\{162, \alpha_l + 2, \alpha_r + 2\}$$

and hence

$$c_{\gamma_*(z),z}(x) \ge \frac{\tilde{v}(z)}{\cosh(\gamma_*)} \ge \frac{\min\{\alpha_l, \alpha_r\}}{\cosh\left(\max\{162, \alpha_l + 2, \alpha_r + 2\}\right)} \quad \text{for all } x \in [-1, z] \text{ and } z \in [0, 1] \,. \quad (14)$$

To finish the proof, we show that for any $x \in [0,1]$ we can find $z \in [0,1]$ such that $u(x) = c_{\gamma_*(z),z}(x)$. This together with (14) yields the claim. For $z \in [-1,1]$ we define the set valued function

$$g(z) := \left\{x \in [-1, z] \,:\, u(x) = c_{\gamma_*(z),z}(x)\right\}$$

and note that $g(z)$ is non-empty and closed. Moreover, proceeding as in the proof of Lemma 3.2 part 3) we find that $g(z)$ is a closed interval for any $z \in [-1,1]$ and since $1 \in g(1)$, $g(-1) = \{-1\}$, an intermediate value argument yields for any $x_* \in [-1,1]$ some $z_* \in [x_*, 1]$ such that $x_* \in g(z_*)$, i.e. $u(x_*) = c_{\gamma_*(z_*),z_*}(x_*)$, holds. Together with (14) we conclude

$$u(x) \ge \frac{\min\{\alpha_l, \alpha_r\}}{\cosh\left(\max\{162, \alpha_l + 2, \alpha_r + 2\}\right)} \quad \text{for all } x \in [0, 1] \,.$$

Note that (14) is valid only for $z \in [0,1]$ so that the above reasoning only works for $x \in [0,1]$. However, by considering the reflection $\tilde{u}(x) := u(-x)$, which is a minimiser in the class $T_{\alpha_r,\alpha_l,L}$, one obtains the same estimate also for $x \in [-1,0]$. $\square$

Combining Corollary 3.3, Theorems 3.8 and 3.9 we obtain the desired estimates.

Theorem 3.10. *Given $\alpha_l, \alpha_r > 0$ there exists some constant $C = C(\alpha_l, \alpha_r) > 0$ such that any minimiser u for the Willmore energy in the class $T_{\alpha_l,\alpha_r,L}$, $L \ge C$, satisfies the estimates*

$$u(x) \ge \frac{1}{C} \quad \textit{and} \quad |u'(x)| \le C \quad \textit{in } [-1,1] \,.$$

Remark. It is important to note that the constant C of this result depends only on α_l and α_r but is independent of L.

4 Construction of a minimiser

Proof of Theorem 1.1. Lemma 2.2 and a simple study of the function u_γ defined in (7) yield immediately part a) and b) of the claim. We divide the proof of part c) into three steps.

1) Let L_0 be the constant defined in (9). For $L \ge L_0$ the set $T_{\alpha_l,\alpha_r,L}$ from Definition 1.2 is non-empty and $M_{\alpha_l,\alpha_r,L} < 4\pi$ holds. We now prove the existence of a minimiser for the Willmore energy in $T_{\alpha_l,\alpha_r,L}$, $L \ge L_0$. Fix some $u \in T_{\alpha_l,\alpha_r,L}$. Starting from (6) and using the inequality $(p-q)^2 \ge \frac{1}{2}p^2 - q^2$ we compute

$$\begin{aligned} \mathcal{W}(u) \;&\ge\; \frac{\pi}{2}\int_{-1}^{1}\Big(\frac{(u'')^2 u}{2(1+u'^2)^{5/2}} - \frac{1}{u(1+u'^2)^{1/2}}\Big)dx \\ &\ge\; \frac{\pi}{4L(1+L^2)^{5/2}}\int_{-1}^{1}(u'')^2 dx - \pi L \,. \end{aligned}$$

In particular, a bound on $\mathcal{W}(u)$ implies a bound on u'' in $L^2([-1,1])$ and hence a bound on u in the space $W^{2,2}([-1,1],(0,+\infty))$. Now let $\{u_k\}_{k\in\mathbb{N}}$ be a minimising sequence for the

Willmore energy in $T_{\alpha_l,\alpha_r,L}$, i.e. $\mathcal{W}(u_k) \to M_{\alpha_l,\alpha_r,L}$ for $k \to \infty$. By the argument above u_k is then uniformly bounded in $W^{2,2}([-1,1],(0,+\infty))$. A subsequence u_k converges weakly in $W^{2,2}([-1,1],(0,+\infty))$ and, by compact embedding, also strongly in $C^1([-1,1],(0,+\infty))$ to some limit function $u \in W^{2,2}([-1,1],(0,+\infty))$. From the strong convergence in C^1 we deduce $u(-1) = \alpha_l$, $u(1) = \alpha_r$, $u(x) \geq L^{-1}$, $|u'(x)| \leq L$ in $[-1,1]$ and hence $u \in T_{\alpha_l,\alpha_r,L}$. A lower semi-continuity argument yields

$$\mathcal{W}(u) \leq \liminf_{k\to\infty} \mathcal{W}(u_k) = M_{\alpha_l,\alpha_r,L} \,.$$

On the other hand, $u \in T_{\alpha_l,\alpha_r,L}$ implies $\mathcal{W}(u) \geq M_{\alpha_l,\alpha_r,L}$ and hence $\mathcal{W}(u) = M_{\alpha_l,\alpha_r,L}$. Thus, u is indeed a minimiser of the Willmore energy in the class $T_{\alpha_l,\alpha_r,L}$. Moreover, Corollary 3.3 yields $u \in W^{2,\infty}([-1,1],(0,+\infty)) = C^{1,1}([-1,1],(0,+\infty))$.

2) Let $C = C(\alpha_l,\alpha_r)$ denote the constant from Theorem 3.10 and $u = u_C$ be a minimiser for the Willmore energy in the class $T_{\alpha_l,\alpha_r,C}$. We prove that this u is a minimiser in the large class T_{α_l,α_r}. Given $v \in T_{\alpha_l,\alpha_r}$, choose some constant $L \geq C$ large enough such that $v \in T_{\alpha_l,\alpha_r,L}$. If $w \in T_{\alpha_l,\alpha_r,L}$ denotes a minimiser in the class $T_{\alpha_l,\alpha_r,L}$, then Theorem 3.10 shows in fact $w \in T_{\alpha_l,\alpha_r,C}$. Because of $L \geq C$, w is also a minimiser in the class $T_{\alpha_l,\alpha_r,C}$ and we obtain $\mathcal{W}(u) = \mathcal{W}(w) \leq \mathcal{W}(v)$. Since $v \in T_{\alpha_l,\alpha_r}$ is arbitrary, u must be a minimiser in the class T_{α_l,α_r}, proving the claim. Moreover, u also provides a minimiser in the even larger space $\widetilde{T}_{\alpha_l,\alpha_r}$ of immersed regular curves by Corollary 2.5.

3) With the same arguments as in [4, Thm.3.9 Step 2] one can prove $u \in C^\infty([-1,1])$. Let $\Gamma = \Gamma(u)$ denote the surface of revolution corresponding to u. The Euler-Lagrange equation satisfied by Γ is given by $\triangle H + 2H(H^2 - K) = 0$ on Γ, where $\triangle$ denotes the Laplace-Beltrami operator on the surface Γ. Moreover, $H = 0$ on $C_{\alpha_l} \cup C_{\alpha_r}$ arises as the natural boundary condition for our variational problem (see [2, App.A] or [18]). In Lemma 3.2 we have proven that the solution Γ lies locally on one side of the catenoid, in particular $H \geq 0$ or $H \leq 0$ everywhere on Γ, the sign depending on the choice of the normal vector. From the strong maximum principle, applied to the second order elliptic equation $\triangle H + 2H(H^2 - K) = 0$ we deduce either $H \equiv 0$ on Γ or $H \neq 0$ on $\Gamma \backslash (C_{\alpha_l} \cup C_{\alpha_r})$. The case $H \equiv 0$ corresponds to $\alpha_r \geq \alpha_r^*$ when the minimiser is a minimal surface of revolution, i.e. some catenoid.

□

Corollary 4.1. *For $\alpha_l, \alpha_r > 0$ let M_{α_l,α_r} be defined as in Definition 1.1 and $\alpha_r^*(\alpha_l)$ be defined as in (3). Then $\alpha_r \mapsto M_{\alpha_l,\alpha_r}$ is strictly monotonically decreasing in $(0, \alpha_r^*(\alpha_l))$.*

Proof. Let α_r, α_r' satisfy $0 < \alpha_r < \alpha_r' < \alpha_r^*(\alpha_l)$. Let $u \in T_{\alpha_l,\alpha_r}$ be a minimiser for the Willmore energy in T_{α_l,α_r}. Then u solves the corresponding Euler-Lagrange equation, that is u is solution of the fourth order ordinary differential equation given in [5, Lemma 2.2]. In particular, u does not coincide locally with a catenary. With the construction in Lemma 2.4 we find a $v \in T_{\alpha_l,\alpha_r'}$ such that $\mathcal{W}(v) \leq \mathcal{W}(u)$ and v coincides with a catenary on $[x_*, 1]$ for some $x_* \in (-1,1)$. Hence $\mathcal{W}(v) < \mathcal{W}(u)$ and also $M_{\alpha_l,\alpha_r'} < M_{\alpha_l,\alpha_r}$. □

In [2] we studied the case of symmetric boundary conditions $\alpha = \alpha_l = \alpha_r$, minimising there only within the class symmetric graphs. We could prove that for $\alpha < \alpha^* = \inf_{\gamma\in\mathbb{R}} \frac{\cosh(\gamma)}{\gamma} \approx 1.5089$ the minimisers satisfy $u'(-1) = -u'(1) = -\alpha$. We cannot expect the same behavior in the more general case studied in this paper, but still we can show the following.

Lemma 4.2. *Given $\alpha_l, \alpha_r > 0$, let u be a minimiser for the Willmore energy in T_{α_l,α_r}. Then $u'(-1) < 0$ and $u'(1) > 0$ must hold.*

Proof. We prove only that $u'(1) > 0$ since the proof of $u'(-1) < 0$ is similar. We proceed by contradiction. If $u'(1) < 0$, let c be the catenary such that $c(1) = u(1) = \alpha_r$ and $c'(1) = u'(1)$ and $x_0 > 1$ be such that $c(x_0) = \alpha_r$. We take the function $\tilde{w} \in W^{2,2}([-1, x_0], (0, +\infty))$ that is equal to u on $[-1, 1]$ and equal to c on $(1, x_0]$. We denote by w the function defined on $[-1, 1]$ obtained from $\tilde{w}$ by appropriate translation and rescaling. By construction, $w(-1) < \alpha_l$ and $w(1) < \alpha_r$. Lemma 2.4 applied twice yields a function $v \in T_{\alpha_l,\alpha_r}$ such that

$$\mathcal{W}(v) \leq \mathcal{W}(w) = \mathcal{W}(\tilde{w}) = \mathcal{W}(u)\,.$$

Hence v is also a minimiser in T_{α_l,α_r}. By construction v coincides with a catenary on an interval of positive length and therefore v is equal to a catenary on the entire interval $[-1, 1]$, since both v and the catenary are solutions of the fourth order Euler-Lagrange equation with the same initial values. We obtain $\mathcal{W}(v) = 0$, a contradiction to the assumption $\alpha_r < \alpha_r^*(\alpha_l)$.

In the case $u'(1) = 0$ the construction is the same as above with the only difference that the point x_0 is chosen so that $\cosh((x_0 - 1)/\alpha_r) < (x_0 + 1)/2$. □

5 Convergence to a sphere for $\alpha_l, \alpha_r \to 0$

Here we study the behavior of the minimisers, which admits a representation as in (4), as both α_l and α_r converge to zero. In this situation the two circles defining the boundary of the surface $\Gamma_{\alpha_l,\alpha_r}$ collapse to points. We will show that $\Gamma_{\alpha_l,\alpha_r}$, the surface of revolution generated by the graph of the positive function u_{α_l,α_r}, converges to the round sphere $\mathbb{S}^2$ in the sense that the functions u_{α_l,α_r} converge uniformly to the function $\sqrt{1-x^2}$ in $[-1, 1]$ as $\alpha_l, \alpha_r \to 0$. In the case of symmetric boundary conditions $\alpha_l = \alpha_r$ this result was proved in [8].

We start by proving that the energy of Γ converges to the energy of a round sphere, i.e. to 4π.

Lemma 5.1. *For $a, b \in [-1, 1]$, $a < b$, let $u \in W^{2,2}([a, b], (0, +\infty))$ satisfy $\mathcal{W}(u) < 4\pi$ and $\max\{u(a), u(b)\} \leq \varepsilon^2$ for $\varepsilon < \min\{\frac{b-a}{2}, \frac{1}{80}e^{-12}\}$. Then the following estimates are satisfied*

$$\mathcal{W}(u) \geq 4\pi\left(1 - \delta(\varepsilon)\right) \quad \text{and} \quad \int\limits_{a+\varepsilon}^{b-\varepsilon} \kappa^2(x)\frac{\sqrt{1+u'(x)^2}}{u(x)}dx \leq 8\,\delta(\varepsilon)$$

with κ the hyperbolic curvature of u as defined in (11) *and*

$$\delta(\varepsilon) := 1 - \sqrt{1 + \frac{12}{\log(80\varepsilon)}} > 0\,. \tag{15}$$

Proof. We first prove the following: Close to each boundary point there is a point with large derivative (in absolute value). Applying Corollary 3.6 to u restricted to the interval $[a, a+\varepsilon]$ we get that there exist some $x_* \in [a, a+\varepsilon]$ such that $u(x_*) \geq \frac{\varepsilon}{80}$. We set $L := \sup\{u'(x) : x \in [a, a+\varepsilon]\}$ and $I := \{x \in [a, a+\varepsilon] : u'(x) \geq 0\}$. Notice that, due to the assumption on ε, L is strictly positive. Moreover, I is not necessarily an interval but it is a closed set, by the continuity of u'. We choose some $x_1 \in [a, a+\varepsilon]$ with $u'(x_1) = L$. From Lemma 3.4 we first deduce

$$4\pi > \mathcal{W}(u) \geq \frac{\pi}{2}\int_a^b \frac{1}{u\sqrt{1+u'^2}}dx - 2\pi$$

and continue by estimating

$$\begin{aligned} 12 \;&\geq\; \int_a^b \frac{1}{u\sqrt{1+u'^2}}dx \geq \int_I \frac{1}{u\sqrt{1+u'^2}}dx \geq \int_I \frac{u'}{uL\sqrt{1+L^2}}dx \geq \frac{1}{1+L^2}\int_a^{x_*}\frac{u'}{u}dx \\ &=\; \frac{1}{1+L^2}\log\frac{u(x_*)}{u(a)} \geq \frac{1}{1+L^2}\log\frac{\varepsilon}{80\varepsilon^2} = \frac{-\log(80\varepsilon)}{1+L^2}\,. \end{aligned}$$

After suitably rearranging one obtains

$$\frac{u'(x_1)}{\sqrt{1+u'(x_1)^2}} = \frac{L}{\sqrt{1+L^2}} \geq \sqrt{1+\frac{12}{\log(80\varepsilon)}}\ .$$

where we use $0<\varepsilon<\frac{1}{80}e^{-12}$. In a similar way we can find some $x_2 \in [b-\varepsilon, b]$ such that

$$\frac{u'(x_2)}{\sqrt{1+u'(x_2)^2}} \leq -\sqrt{1+\frac{12}{\log(80\varepsilon)}}\ .$$

From Lemma 3.4 applied to $u|_{[x_1,x_2]}$ we obtain

$$\mathcal{W}(u) \geq \mathcal{W}(u|_{[x_1,x_2]}) \geq -2\pi\Big[\frac{u'}{\sqrt{1+u'^2}}\Big]_{x_1}^{x_2} \geq 4\pi\big(1-\delta(\varepsilon)\big)\ ,$$

with $\delta(\varepsilon)$ defined in (15). Moreover, using $x_1 \leq a+\varepsilon$, $x_2 \geq b-\varepsilon$ together with formula (10) for the Willmore energy one deduces

$$\frac{\pi}{2}\int\limits_{a+\varepsilon}^{b-\varepsilon} \kappa^2 \frac{\sqrt{1+u'^2}}{u} dx \leq \frac{\pi}{2}\int_{x_1}^{x_2} \kappa^2 \frac{\sqrt{1+u'^2}}{u} dx = \mathcal{W}(u|_{[x_1,x_2]}) + 2\pi\Big[\frac{u'}{\sqrt{1+u'^2}}\Big]_{x_1}^{x_2} < 4\pi\delta(\varepsilon)\ ,$$

proving the second estimate in the claim. □

An immediate consequence is the convergence of the energy to the one of the sphere.

Corollary 5.2. *The energy M_{α_l,α_r} from Definition 1.1 converges to 4π as $\alpha_l, \alpha_r \to 0$.*

Lemma 5.3. *Let $\varepsilon_0 > 0$ be such that $\delta(\varepsilon_0) = \frac{1}{2}$ with $\delta(\varepsilon)$ defined in* (15). *For $\alpha_l, \alpha_r > 0$ such that $\min\{\alpha_l, \alpha_r\} \leq \varepsilon^2$ with $\varepsilon < \min\{\varepsilon_0, \frac{1}{80}e^{-12}\}$ let u be a minimiser for the Willmore energy in T_{α_l,α_r}. Then u satisfies*

$$u(x) \geq \varepsilon^2 \quad \textit{and} \quad |u'(x)| \leq (2+\varepsilon^2\sinh(162))\varepsilon^{-2} \quad \textit{for all}\, x \in [-1+3\varepsilon, 1-3\varepsilon]\ .$$

Proof. We have $\mathcal{W}(u) = M_{\alpha_l,\alpha_r} < 4\pi$ by Lemma 3.1. We prove the first claim by contradiction. Let us assume that there exist $\varepsilon < \min\{\varepsilon_0, \frac{1}{80}e^{-12}\}$ and some $x_* \in [-1+3\varepsilon, 1-3\varepsilon]$ with $u(x_*) < \varepsilon^2$. Then Lemma 5.1, applied on the intervals $[-1, x_*]$ and $[x_*, 1]$, proves $\mathcal{W}(u|_{[-1,x_*]}) \geq 4\pi(1-\delta(\varepsilon)) > 2\pi$ as well as $\mathcal{W}(u|_{[x_*,1]}) > 2\pi$. This implies $\mathcal{W}(u) > 2\pi + 2\pi = 4\pi$, contradicting $\mathcal{W}(u) < 4\pi$ and proving the first claim. To prove the second inequality we first deduce from Theorem 3.8 that $u'(-1) \geq -\sinh(162)$ and $u'(1) \leq \sinh(162)$ must hold. These estimates and $u(x) \geq \varepsilon^2$ in $[-1+3\varepsilon, 1-3\varepsilon]$, just proved, combined with the estimate (12) in the proof of Corollary 3.3 give

$$|u'(x)| \leq (2+\max\{\alpha_l,\alpha_r\}\sinh(162))\varepsilon^{-2} \leq (2+\varepsilon^2\sinh(162))\varepsilon^{-2}\ .$$

□

We are now ready to prove Theorem 1.2.

Theorem 5.4. *Let $\{\alpha_{l,n}\}_{n\in\mathbb{N}}, \{\alpha_{r,n}\}_{n\in\mathbb{N}}$ be two strictly positive sequences converging to zero. For each $n \in \mathbb{N}$ let u_n be a minimiser for the Willmore energy in $T_{\alpha_{l,n},\alpha_{r,n}}$. Then u_n converges uniformly on $[-1, 1]$ to the function $u_0(x) := \sqrt{1-x^2}$.*

Proof. Without loss of generality we may assume that $\alpha_{l,n}, \alpha_{r,n} \leq 1$ for all $n \in \mathbb{N}$. Defining the sequence $\varphi_n(x) := x^2 + u_n^2(x)$, it suffices to show that φ_n converge uniformly to $\varphi_0 \equiv 1$. From Lemma 3.2 it follows that $u_n(x) \leq \max\{\alpha_{l,n}, \alpha_{r,n}\} + 1$ for all n, and hence φ_n is uniformly bounded from above. If κ_n denotes the hyperbolic curvature of u_n, then we have the relation

$$\kappa_n = \frac{u_n u_n'' + 1 + u_n'^2}{(1 + u_n'^2)^{3/2}} = \frac{\varphi_n''}{2(1 + u_n'^2)^{3/2}} \, . \tag{16}$$

Lemma 3.2 implies $\kappa_n(x) \geq 0$ and hence $\varphi_n'' \geq 0$ in $[-1, 1]$. From (12) together with Theorem 3.8 we conclude for all $n \in \mathbb{N}$

$$-1 - \alpha_{l,n} \sinh(162) \leq x + u_n(x) u_n'(x) = \frac{1}{2} \varphi_n'(x) \leq 1 + \alpha_{r,n} \sinh(162) \quad \text{for } x \in [-1, 1] \, .$$

Hence $\varphi_n'(x)$ is uniformly bounded in $[-1, 1]$ and, after passing to some subsequence, φ_n converges uniformly in $[-1, 1]$ to some limit function $\varphi_0 \in C^{0,1}([-1, 1], \mathbb{R})$. From $\varphi_n(-1) = 1 + \alpha_{l,n}^2$, $\varphi_n(1) = 1 + \alpha_{r,n}^2$ we deduce $\varphi_0(-1) = 1 = \varphi_0(1)$. We prove now that φ_0 is a linear function. Fixing $\delta > 0$, we first observe that Lemma 5.1 yields

$$0 = \lim_{n \to \infty} \int_{-1+\delta}^{1-\delta} \kappa_n^2 \frac{\sqrt{1 + u_n'^2}}{u_n} \, dx \, ,$$

while Lemma 5.3 shows

$$\inf_{\substack{x \in [-1+\delta, 1-\delta] \\ n \in \mathbb{N}}} u_n(x) = m > 0 \quad , \quad \sup_{\substack{x \in [-1+\delta, 1-\delta] \\ n \in \mathbb{N}}} |u_n'(x)| = L < +\infty \, .$$

From (16), $u_n \leq 2$ for all $n \in \mathbb{N}$ together with the estimate above we reach

$$0 = \lim_{n \to \infty} \int_{-1+\delta}^{1-\delta} \kappa_n^2 \frac{\sqrt{1 + u_n'^2}}{u_n} \, dx \geq \frac{1}{8(1 + L^2)^{5/2}} \lim_{n \to \infty} \int_{-1+\delta}^{1-\delta} (\varphi_n'')^2 \, dx \, .$$

The sequence φ_n'' converges to zero in $L^2(-1+\delta, 1-\delta)$ and we obtain $\varphi_0 \in W^{2,2}([-1+\delta, 1-\delta], (0, \infty))$ with $\varphi_0'' \equiv 0$ in $(-1 + \delta, 1 - \delta)$ for any $\delta > 0$. Thus, φ_0 is a linear function and because of $\varphi_0(-1) = 1 = \varphi_0(1)$ we finally obtain $\varphi_0 \equiv 1$, as claimed. $\square$

Acknowledgment The authors wish to thank the referee for useful suggestions.

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