

Separating positivity and regularity for fourth order Dirichlet problems in 2d-domains

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Abstract

The main result in this paper is that the solution operator for the bi-laplace problem with zero Dirichlet boundary conditions on a bounded smooth 2d-domain can be split in a positive part that contains the singular part and a smooth sign-changing part. Such a splitting allows one to find a priori estimates for fourth order problems similar to the one proved via the maximum principle in second order elliptic boundary value problems. The proof depends on a careful approximative fill-up of the domain by a finite collection of limacons. The limacons involved are such that the Green function for the Dirichlet bi-laplacian on each of these domains is strictly positive.

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1 Introduction and main results

A mayor tool for second order elliptic equations is the maximum principle. The maximum principle not only implies that a positive source will give a positive solution but it helps to obtain a priori estimates and hence to find regularity results. Especially in nonlinear equations such a priori estimates play a crucial role. Several results are referred to by the name maximum principle but the result that we want to refer to is the local result that reads for the laplacian as $\Delta u \geq 0$ in a neighborhood of a implies that u cannot have a strict maximum in a . A serious obstruction for higher order elliptic equations is that one cannot expect a similar result as functions like $\pm x^2$ clearly show.

The situation becomes more complicated when considering a positivity preserving property which is often also named “maximum principle”. For the laplacian that is: $-\Delta u \geq 0$ in Ω and $u \geq 0$ on $\partial\Omega$ implies $u \geq 0$ in Ω (with Ω a bounded domain in \mathbb{R}^n). This “global maximum principle” also holds for some special higher dimensional problems. Indeed, $\Delta^2 u \geq 0$ in B and $-\frac{\partial}{\partial|x|}u \geq 0$, $u \geq 0$ on ∂B implies $u \geq 0$ in B . Here B is a ball in \mathbb{R}^n with $n \leq 4$. For this special result see [16]. With $\frac{\partial}{\partial|x|}u = u = 0$ on ∂B the result holds for B in any \mathbb{R}^n and goes back 100 years to Boggio ([3]). The restriction to the ball is rather crucial. Since Duffin’s counterexample ([10]) it has become well known that on most domains such a positivity preserving property fails (see [15]).

In [19] Nehari looks for subdomains of Ω , characterized by the position of the points x and y and by simple geometric properties of Ω , in which the Green function for the biharmonic problem with Dirichlet boundary condition on Ω may be shown to be positive.

In order to find a priori estimates it is however not necessary to have such a sign preserving result; it is sufficient that the singularity of the solution operator has a fixed sign. This separation of the solution operator in a smooth but sign changing part and a singular part of fixed sign is the main result of the present paper. However, since we are using conformal mappings, our present result is restricted to two dimensional domains. Note that in two dimensions the singularity of the solution operator for the bilaplacian appears in the second derivative. Indeed the fundamental solution is $\frac{-1}{8\pi} |x|^2 \ln |x|$.

Let us be more precise. For Ω an open bounded $C^{4,\alpha}$ domain in \mathbb{R}^2 we will show that the solution operator for

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

can be split in the way we just mentioned. Crucial is that we find a uniform behavior of such a splitting even near the boundary. Such a result away from the boundary, that is in compact subsets of Ω , was proven in [14].

We proceed as follows. We recall from [7] that for Ω taken from some family of limacons the Green function for (1) is positive. Secondly, one may show that small perturbations of those limacons do not destroy the positivity of the corresponding Green function. Thirdly, one may construct a finite number of such slightly perturbed limacons $\{E_j \subset \mathbb{R}^2\}$ that are such that the boundary of Ω is covered by the boundaries of those perturbed limacons while these limacons cover a neighborhood of the boundary of Ω . Together with a covering of the interior one is able to construct the desired splitting of the solution through a separation of unity related with that covering. Roughly explained, for each $x \in \Omega$ there is an element E_j in this finite covering such that the Green function for (1) can be decomposed as the sum of $G_{E_j}(x, y)$ and a remainder term $G_j^{\text{rest}}(x, y)$ where $G_{E_j}(x, y)$ is positive and $G_j^{\text{rest}}(x, y)$ is without singularity. Note that the choice of E_j depends on x . Since the extension of $G_{E_j}(x, y)$ from E_j^2 to Ω^2 by 0 is not smooth one may guess that the just mentioned decomposition is more

involved than just this simple sum.

1.1 Main results

In this section we state the two main results of the paper. First we fix some notation.

The Green function G_Ω is such that the solution of problem (1) for appropriate f can be written as

$$u(x) = \int_{\Omega} G_\Omega(x, y) f(y) dy.$$

In the following $d_\Omega(\cdot)$ denotes the distance to the boundary in the domain Ω :

$$d_\Omega(z) := \inf_{x \in \partial\Omega} |z - x|.$$

Two closely related versions of the main result are the following. The first one is a pointwise description which focusses on the splitting of the solution operator.

Theorem 1.1 *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain with $\partial\Omega \in C^{16}$. Then there exist $G_\Omega^{reg}, G_\Omega^{sing} : \bar{\Omega}^2 \rightarrow \mathbb{R}$ such that the Green function for (1) can be written as*

$$G_\Omega(x, y) = G_\Omega^{reg}(x, y) + G_\Omega^{sing}(x, y)$$

and the following is satisfied:

1. (a) $G_\Omega^{sing}(x, y) \geq 0$ on $\bar{\Omega}^2$;
 (b) $G_\Omega^{sing} \in C^{1,\gamma}(\bar{\Omega}^2) \cap C_0^1(\bar{\Omega}^2)$ for all $\gamma \in (0, 1)$;
 (c) $G_\Omega^{sing} \in C^{15,\gamma}(\{(x, y) \in \bar{\Omega}^2; x \neq y\})$ for all $\gamma \in (0, 1)$;
2. (a) $G_\Omega^{reg} \in C^{15,\gamma}(\bar{\Omega}^2) \cap C_0^1(\bar{\Omega}^2)$ for all $\gamma \in (0, 1)$.

Remark 1.1.1 *For the condition $\partial\Omega \in C^{16}$ see Definition 1.6.*

Remark 1.1.2 *Since G_Ω is symmetric one may assume that both G_Ω^{reg} and G_Ω^{sing} are symmetric. If not yet symmetric, then set $G_{\Omega, new}^{\dots}(x, y) := \frac{1}{2}G_\Omega^{\dots}(x, y) + \frac{1}{2}G_\Omega^{\dots}(y, x)$.*

The next result is a kind of maximum principle, that is, it gives a pointwise bound from above for the solution in terms of the positive part of the right hand side and a weaker norm of the solution itself. Before we state the result let us recall that the space $W^{-m,p}(\Omega)$ is the dual space of $W_0^{m,p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, and its norm can be defined as follows

$$\|u\|_{W^{-m,p}(\Omega)} := \sup \left\{ u(\varphi); \varphi \in W_0^{m,p'}(\Omega), \|\varphi\|_{W^{m,p'}(\Omega)} \leq 1 \right\}.$$

Theorem 1.2 *Let $0 < \alpha < 1$ and $p \in (1, \infty)$. Suppose that Ω is a bounded simply connected domain in \mathbb{R}^2 with $\partial\Omega \in C^{4,\alpha}$. Then for any $q > 2$ and $\varepsilon > 0$ there exists a constant $c_{q,\Omega,\varepsilon} > 0$ such that for $f \in L^p(\Omega)$ the solution $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ of (1) satisfies*

$$u(x) \leq c_{q,\Omega,\varepsilon} \left(\|f^+\|_{L^1(B(x,\varepsilon) \cap \Omega)} + \|u\|_{W^{-1,q}(\Omega)} \right) \text{ for every } x \in \Omega.$$

Here f^+ denotes the positive part of f .

Remark 1.2.1 *More precise information on how $c_{q,\Omega,\varepsilon}$ depends on q, Ω and ε can be found in Theorem 4.1. For those who want to avoid norms for negative Sobolev spaces we recall that $\|u\|_{W^{-1,q}(\Omega)} \leq c(s, q, \Omega) \|u\|_{L^s(\Omega)}$ for $s > 2q(q+2)^{-1}$.*

1.2 Some notations

Let us fix the following (for later use we consider \mathbb{R}^n with general n).

Notation 1.3 *Let Ω be a bounded domain in \mathbb{R}^n and let f and g be functions on $\Omega \times \Omega$.*

- For $\alpha, \beta \in \mathbb{N}^n$ we set $|\alpha| = \sum_{k=1}^n \alpha_k$ and

$$D_x^\alpha D_y^\beta f(x, y) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} \frac{\partial^{|\beta|}}{\partial y_1^{\beta_1} \partial y_2^{\beta_2} \dots \partial y_n^{\beta_n}} f(x, y).$$

- An equivalence relation for f and g which are nonnegative (See [13]):

$$f \sim g \text{ on } \Omega \times \Omega$$

if and only if there are $c_1, c_2 > 0$ such that

$$c_1 f(x, y) \leq g(x, y) \leq c_2 f(x, y) \text{ for all } x, y \in \Omega.$$

- A dominance relation with respect to a nonnegative f :

$$f \preceq g \text{ on } \Omega \times \Omega$$

if and only if there is $c > 0$ such that

$$f(x, y) \leq c g(x, y) \text{ for all } x, y \in \Omega.$$

The Hölder spaces $C^r(\bar{\Omega})$ and $C^{r,\gamma}(\bar{\Omega})$ with $r \in \mathbb{N}$ and $\gamma \in (0, 1]$ are supplied with the norm:

$$\begin{aligned} \|f\|_{C^r(\bar{\Omega})} &:= \sum_{|\alpha| \leq r} \|D^\alpha f\|_\infty, \\ \|f\|_{C^{r,\gamma}(\bar{\Omega})} &:= \|f\|_{C^r(\bar{\Omega})} + \sum_{|\alpha|=r} [D^\alpha f]_\gamma, \end{aligned}$$

where $[f]_\gamma := \sup \left\{ \frac{|f(x)-f(y)|}{|x-y|^\gamma}; x, y \in \bar{\Omega}, x \neq y \right\}$. For convenience we set $C^{r,0}(\bar{\Omega}) := C^r(\bar{\Omega})$. In the following $C_c^r(\Omega)$ denotes the set of all functions in $C^r(\Omega)$ whose supports are compact subsets of Ω .

For $m \in \mathbb{N}$ and $p \geq 1$, $p \in \mathbb{R}$, $W^{m,p}(\Omega)$ denotes the Sobolev space with the norm

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}.$$

We fix the following notation to point out on which quantities the constants depend.

Notation 1.4 For $\alpha, \beta, \gamma \in \mathbb{R}$, $C = C(\alpha, \beta, \gamma)$ means that C depends only on α, β and γ , and that C is bounded for bounded values of these parameters.

Next we will need some notation concerning the domain and its boundary.

Notation 1.5 (relatively open subset of the boundary) For K a subset of $\partial\Omega \subset \mathbb{R}^n$, set

$$K^{\circ, \partial\Omega} := (K \cup (\partial\Omega)^c)^\circ \cap \partial\Omega. \quad (2)$$

In the literature several definitions of $C^{\ell,\alpha}$ -domains appear. To avoid any ambiguity we explicitly give the version that we will use.

Definition 1.6 (uniform $C^{\ell,\alpha}$ regularity condition for Ω) Let $\ell \in \mathbb{N}^+$, $\alpha \in [0, 1]$ and Ω be a bounded domain in \mathbb{R}^n . The domain Ω satisfies the uniform $C^{\ell,\alpha}$ regularity condition (we write $\partial\Omega \in C^{\ell,\alpha}$) if there exist a positive constant M , a finite open covering $\{U_j\}_{j \in J}$ of $\partial\Omega$, a corresponding collection $\{\varphi_j\}_{j \in J}$ of $C^{\ell,\alpha}$ mappings such that for every $j \in J$:

1. $\varphi_j : U_j \rightarrow B = \{y \in \mathbb{R}^n : |y| < 1\}$ is a bijection; set $\psi_j = \varphi_j^{inv}$;
2. with $(\varphi_{j,1}, \dots, \varphi_{j,n})$ and $(\psi_{j,1}, \dots, \psi_{j,n})$ the components of φ_j and ψ_j :

$$\|\varphi_{j,i}\|_{C^{\ell,\alpha}(\bar{U}_j)} \leq M \text{ and } \|\psi_{j,i}\|_{C^{\ell,\alpha}(\bar{B})} \leq M \text{ for all } i;$$

3. $\varphi_j(U_j \cap \Omega) = \{y \in B : y_n > 0\}$;

and there exists $\delta > 0$ such that

$$\{x \in \Omega : d(x, \partial\Omega) < \delta\} \subset \bigcup_{j \in J} \psi_j \left(\left\{ y \in \mathbb{R}^n : |y| < \frac{1}{2} \right\} \right).$$

Definition 1.6 is similar to the uniform C^ℓ regularity condition in [1, Def.4.10 page 84].

It will also be convenient to fix the following numbers.

Notation 1.7 Let Ω be a bounded domain with $\partial\Omega \in C^2$.

1. We write ρ_Ω for the largest number r such that both Ω and $\mathbb{R}^n \setminus \Omega$ can be filled with balls of radius r . To be precise: for $r > 0$ set $\Omega_r := \{z \in \Omega : d(z, \partial\Omega) \geq r\}$, $\bar{\Omega}_r := \{z \in \mathbb{R}^n \setminus \bar{\Omega} : d(z, \partial\Omega) \geq r\}$. Set $\rho_\Omega > 0$ the largest r such that the following holds:

$$\Omega = \bigcup_{z \in \Omega_r} B_r(z) \text{ and } \mathbb{R}^n \setminus \bar{\Omega} = \bigcup_{z \in \bar{\Omega}_r} B_r(z).$$

2. We will also use R_Ω defined as the smallest R such that $\Omega \subset B_R(z)$ for some $z \in \mathbb{R}^2$.

Remark 1.7.1 For most domains we may take $\rho_\Omega = \kappa^{-1}$ where κ denotes the maximal curvature. But notice that ρ_Ω can be strictly smaller than κ^{-1} . For example this happens in the case of a dumb-bell shaped domain with a very narrow passage.

2 Domains with a positive biharmonic Green function

In this section we concentrate on the positivity preserving property of problem (1) in two-dimensional domain. Let first fix what we mean by this.

Definition 2.1 We say that problem (1) on a domain Ω satisfies the positivity preserving property if for any f positive the solution u of (1) is also positive.

Obviously (1) on a domain Ω satisfies the positivity preserving property if and only if the biharmonic Green function associated to problem (1) on Ω is positive.

It is well known that problem (1) is positivity preserving on the disk (see [3]). In the following we first recall a recent result in [7] where a family of domains (limaçons) is given on which the biharmonic Green function associated to problem (1) is positive. Next we will show that small $C^{2,\gamma}$ perturbations of these domains do not destroy this property.

2.1 Limaçon de Pascal

In [7] one finds that on some limaçons the Green function for (1) is strictly positive. Since these limaçons are our starting point we will shortly recall some properties of these domains.

The Limaçon de Pascal Ω_a with $a \in [0, \frac{1}{2}]$ is defined as the image of the unit disk through the conformal map

$$h_a(x_1, x_2) = (x_1 + 2ax_1x_2, x_2 + ax_2^2 - ax_1^2 + 1 - a) \text{ for } a \in [0, \frac{1}{2}]. \quad (3)$$

The result that is proved in [7] is the following:

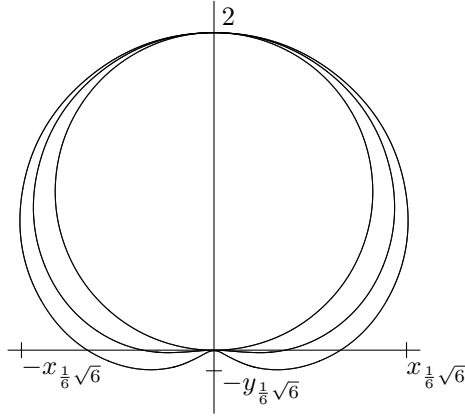


Figure 1: Limaçons Ω_a for respectively $a = 0$, $a = \frac{3}{10}$ and $a = \frac{1}{6}\sqrt{6}$.

Proposition 2.2 *The Green function G_{Ω_a} for (1) with $\Omega = \Omega_a$ and $a \in [0, \frac{1}{2}]$ is positive if and only if $a \in [0, \frac{1}{6}\sqrt{6}]$. Moreover, there exist $c_1, c_2 > 0$ such that for $a \in [0, \frac{1}{6}\sqrt{6}]$ the following estimates hold. Writing for short $d_a(\cdot) = d_{\Omega_a}(\cdot)$:*

$$G_{\Omega_a}(x, y) \leq c_1 d_a(x)d_a(y) \min \left\{ 1, \frac{d_a(x)d_a(y)}{|x-y|^2} \right\},$$

$$G_{\Omega_a}(x, y) \geq c_2 \left(\frac{1}{6}\sqrt{6} - a \right) d_a(x)d_a(y) \min \left\{ 1, \frac{d_a(x)d_a(y)}{|x-y|^2} \right\}.$$

Remark 2.2.1 *In [17] Hadamard was able to compute an explicit formula for the biharmonic Green function on a limaçon. The fact that this Green function is positive for $a \in [0, \frac{1}{6}\sqrt{6}]$ has been proven in [7].*

We will also need scaled limaçons and we will define these for $R > 0$ by

$$\Omega_{a,R} := \{(Rx, Ry) : (x, y) \in h_a(B_1(0))\},$$

with $B_1(0) = \{(\eta, \xi) \in \mathbb{R}^2 : \eta^2 + \xi^2 < 1\}$. In the following Ω_a denotes $\Omega_{a,1}$.

In the present paper we will consider limaçons $\Omega_{a,R}$ for $a \in [0, \bar{a}]$ where \bar{a} is strictly between $\frac{1}{4}$ and $\frac{1}{6}\sqrt{6}$. By taking \bar{a} strictly smaller than $\frac{1}{6}\sqrt{6}$ we will obtain estimates of the Green function $G_{\Omega_a}(\cdot, \cdot)$ which are uniform with respect to a .

Some geometrical facts :

1. For all $a \in [0, \frac{1}{2}]$ the limaçon $\Omega_{a,R}$ is symmetric with respect to the second axis and both $(0, 0)$ and $(0, 2R)$ lie on $\partial\Omega_{a,R}$.

2. Let $[-x_a, x_a] \times [-y_a, 2]$ denote the smallest rectangle that contains $\Omega_{a,1}$. Then

$$a \mapsto x_a \text{ and } a \mapsto y_a \tag{4}$$

are nondecreasing functions for $a \in [0, \frac{1}{2}]$ with $1 \leq x_a \leq 1.3$ and $0 \leq y_a \leq 0.25$.

3. For $a \in [\frac{3}{16}, \frac{5}{16}]$, we will use $k_{a,R} : [-Rx_a, Rx_a] \rightarrow \mathbb{R}$ to describe the lower part of the boundary $\partial\Omega_{a,R}$:

$$k_{a,R}(x) = \inf \{y : (x, y) \in \Omega_{a,R}\}. \tag{5}$$

In particular in the approximation we will use that the following relations hold:

$$k''_{a,R}(0) = \frac{1}{R} \frac{1-4a}{(1-2a)^2} \text{ and} \tag{6}$$

$$\left\| \frac{\partial^i}{\partial x^i} k_{a,R} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq \frac{b_i}{R^{i-1}} \text{ for } i = 1, \dots, 5, \tag{7}$$

with $x_a^* = \frac{1}{2}(1 - \sqrt{3a})$. Notice that $x_a^* \in (\frac{1}{5}x_a, \frac{1}{2}x_a)$ where x_a is defined near (4). The constants b_i can be taken independently of $a \in [\frac{3}{16}, \frac{5}{16}]$.

Special values of the parameter a are the following:

- $a = 0$: $\Omega_{0,R}$ is the disk with radius R and center $(0, R)$;
- $a = \frac{1}{4}$: $\Omega_{a,R}$ is convex if and only if $a \in [0, \frac{1}{4}]$;
- $a = \frac{1}{6}\sqrt{6} \approx .40825$: the Green function associated to (1) for $\Omega = \Omega_{a,R}$ is positive if and only if $a \in [0, \frac{1}{6}\sqrt{6}]$, see [7];
- $a = \frac{1}{2}$: $\Omega_{\frac{1}{2},R}$ is a cardioid.

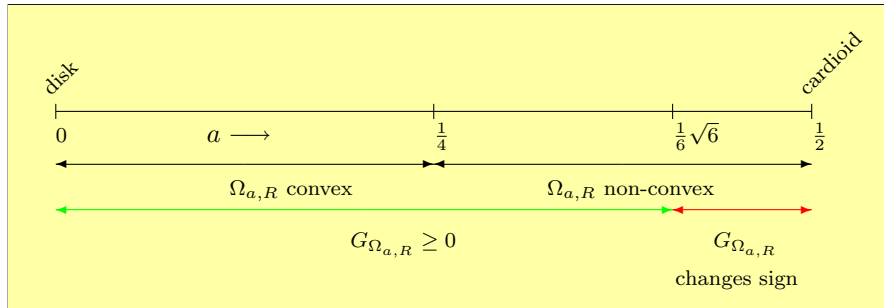


Figure 2: In the graph the critical values of the parameter a for convexity of the limaçons and positivity of the Green function.

2.2 Perturbations from the bilaplacian on a limaçon

In this section we study the positivity preserving property of problem (1) on a domain $\Omega \subset \mathbb{R}^2$ that is ε -close in a $C^{2,\gamma}$ -sense to a limaçon.

The concept of ε -closeness of domains that we use is the one introduced in [12, Def.1.1]. For sake of completeness we recall the definition.

Definition 2.3 *Let $\varepsilon > 0$. We call Ω ε -close in $C^{k,\gamma}$ -sense to Ω^* if there exists a $C^{k,\gamma}$ -mapping $g : \bar{\Omega}^* \rightarrow \bar{\Omega}$ such that $g(\bar{\Omega}^*) = \bar{\Omega}$ and*

$$\|g - Id\|_{C^{k,\gamma}(\bar{\Omega}^*)} \leq \varepsilon.$$

The main result of the section is the following.

Theorem 2.4 (Perturbation of the domain) *Let $\bar{a} \in (\frac{1}{4}, \frac{1}{6}\sqrt{6})$ and $\gamma \in (0, 1)$. Then there exist $\varepsilon_0 > 0$ and $c_1, c_2 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$ and $a \in [0, \bar{a}]$ the following holds.*

If Ω is ε -close in $C^{2,\gamma}$ -sense to Ω_a , then the Green function G_Ω of (1) satisfies

$$0 < c_1 D_\Omega(x, y) \leq G_\Omega(x, y) \leq c_2 D_\Omega(x, y) \text{ for every } x, y \in \Omega,$$

where

$$D_\Omega(x, y) = d_\Omega(x)d_\Omega(y) \min \left\{ 1, \frac{d_\Omega(x)d_\Omega(y)}{|x-y|^2} \right\}. \quad (8)$$

Remark 2.4.1 *In [7] the same estimates from above of G_Ω are given but with more regularity required at the boundary. Thanks to the ε -closeness we get a better estimate from below and the same from above with less assumptions on the boundary.*

The proof consists of several steps and uses similar arguments as in [12] for a disk. For convenience we summarize the main parts here.

We first show that ε -closeness in $C^{2,\gamma}$ -sense of Ω to Ω_a implies the existence of a biholomorphic map $\varphi_a : \Omega_a \rightarrow \Omega$ such that

$$\|\varphi_a - Id\|_{C^{2,\gamma'}(\bar{\Omega}_a)} \leq \delta(\varepsilon) \text{ for } 0 < \gamma' < \gamma. \quad (9)$$

Next, through this conformal mapping φ_a problem (1) on Ω is transformed into the following problem on Ω_a :

$$\begin{cases} (\Delta^2 + A)u = \tilde{f} & \text{in } \Omega_a, \\ u = 0 & \text{on } \partial\Omega_a, \\ \frac{\partial}{\partial\nu}u = 0 & \text{on } \partial\Omega_a, \end{cases} \quad (10)$$

where A is a lower order perturbation of the biharmonic operator. See [13, Remark after Theorem 5.1]. From (9) one also has that there exists a $\delta_1 = \delta_1(\varepsilon) > 0$ such that the coefficients of A in (10) satisfy

$$\sup_{|\alpha| \leq 3} \|A_\alpha\|_\infty \leq \delta_1.$$

We then see that the positivity of the Green function associated to problem (10) implies the positivity of the Green function associated to problem (1) thanks to the properties of conformal maps ([20]). Hence, instead of proving directly Theorem 2.4 we prove the following result.

Theorem 2.5 (Perturbation of Δ^2 by lower order terms) *Let $a \in [0, \bar{a}]$ with \bar{a} as in Theorem 2.4. Consider problem (10) with $A = \sum_{|\alpha| \leq 3} A_\alpha(x) D^\alpha$, $A_\alpha \in C(\bar{\Omega}_a)$ and let $G_{\Omega_a, A}$ the Green function associated to (10).*

Then there exists $\eta_0 > 0$ such that, whenever $\|A_\alpha\|_\infty \leq \eta_0$ for all α with $|\alpha| \leq 3$, the Green function associated to (10) is positive. Moreover, there exist $d_1, d_2 > 0$ such that, with $D_{\Omega_a}(x, y)$ as in (8), the following holds:

$$d_1 D_{\Omega_a}(x, y) \leq G_{\Omega_a, A}(x, y) \leq d_2 D_{\Omega_a}(x, y). \quad (11)$$

Theorem 2.5 says that if the lower order perturbation of the biharmonic operator is small then the positivity preserving property of system (10) in Ω_a follows from the positivity preserving property of problem (1) on the same domain.

A result similar to Theorem 2.5 was proven in [13] for the polyharmonic Dirichlet boundary value problem on the unit disk B . The main ingredient of the proof are appropriate estimates of

$$H_B^k(x, y, z) := \frac{G_B(x, z) |D_z^k G_B(z, y)|}{G_B(x, y)}, \quad (12)$$

which were proved in [12]. Notice that in [12] one considers Ω being a ball. The only place however where that fact is used is in the explicit estimates of H_B^k . Indeed all the other arguments can be applied to any planar smooth domain Ω whose Green function is positive in the strict sense as in the left hand side of (11). Hence to prove Theorem 2.5 we first show that $H_{\Omega_a}^k$ (that is the quotient in (12) calculated for G_{Ω_a}) satisfies the same estimates as H_B^k and then refer to the work in [13].

In the next section we construct the conformal mapping from “ Ω ε -close to Ω_a ” to the limaçon Ω_a and we state the equivalence of Theorem 2.4 and Theorem 2.5. Then we prove the perturbation result of Theorem 2.5.

2.2.1 Conformal transformation

In this subsection we prove that problem (1) on Ω that is ε -close to Ω_a , corresponds to a problem of the type (10) on Ω_a with the coefficients of A , the lower order perturbation of Δ^2 , being small. Or, to be more precise, there is a function $\varepsilon \mapsto \delta(\varepsilon)$ with $\delta(\varepsilon) \rightarrow 0$ when $\varepsilon \downarrow 0$, such that

$$\Omega \text{ } \varepsilon\text{-close in } C^{2,\gamma}\text{-sense to } \Omega_a \implies \sup_{|\alpha| \leq 3} \|A_\alpha\|_\infty \leq \delta(\varepsilon).$$

Or in other words, that Theorem 2.5 implies Theorem 2.4.

The first step consist of proving existence of a biholomorphic map from the limaçon to a domain ε -close to the limaçon which is near the identity in $C^{2,\gamma}$ -sense.

Proposition 2.6 *Let $a \in [0, \bar{a}]$ and $\gamma \in (0, 1)$. Then there exist $\bar{\varepsilon} > 0$ and $c = c(\bar{a}) > 0$ such that for $\varepsilon \in [0, \bar{\varepsilon})$ and every $\gamma' \in (0, 1)$ with $\gamma' < \gamma$ we have the following.*

If Ω is ε -close in $C^{2,\gamma}$ -sense to Ω_a then there is a biholomorphic map $\varphi_a : \bar{\Omega}_a \rightarrow \bar{\Omega}$, with $\varphi_a \in C^{2,\gamma'}(\bar{\Omega}_a)$ and $\varphi_a^{-1} \in C^{2,\gamma'}(\bar{\Omega})$, such that

$$\|\varphi_a - Id\|_{C^{2,\gamma'}(\bar{\Omega}_a)} \leq c \varepsilon^{\gamma-\gamma'}.$$

The proof of Proposition 2.6 consist of the following three lemmas.

Since $a \leq \bar{a} < \frac{1}{6}\sqrt{6} < \frac{1}{2}$ one may check that the map h_a , defined in (3), is conformal and one-to-one on the domain

$$B_{\sqrt{1.5}} := \left\{ x \in \mathbb{R}^2 : \|x\|_2 < \sqrt{1.5} \right\}.$$

We choose the value $\varepsilon_1 \in (0, 1)$ such that, if Ω is ε -close in $C^{2,\gamma}$ -sense to Ω_a for $\varepsilon \in (0, \varepsilon_1)$, then $h_a^{-1}(\Omega) \subset B_{\sqrt{1.5}}$. It follows that h_a^{-1} is a conformal map on any domain Ω which is ε -close in $C^{2,\gamma}$ -sense to the limaçon for $\varepsilon < \varepsilon_1$.

Lemma 2.7 *Let $a \in [0, \bar{a}]$ and $\gamma \in (0, 1)$. There exists $c_1 = c_1(\bar{a}) > 0$ such that the following holds. If Ω is ε -close in $C^{2,\gamma}$ -sense to Ω_a for $\varepsilon \in (0, \varepsilon_1)$, then $\Omega^* := h_a^{-1}(\Omega)$ is $c_1\varepsilon$ -close in $C^{2,\gamma}$ -sense to the disk B .*

Proof. Let g be a $C^{2,\gamma}$ -mapping, $g : \bar{\Omega}_a \rightarrow \bar{\Omega}$, such that $\|g - Id\|_{C^{2,\gamma}(\bar{\Omega}_a)} \leq \varepsilon$. We define the map $f : \bar{B} \rightarrow \bar{\Omega}^*$ by

$$f(x) := (h_a^{-1} \circ g \circ h_a)(x),$$

where $h_a : \bar{B} \rightarrow \bar{\Omega}_a$ and $h_a^{-1} : \bar{\Omega} \rightarrow \bar{\Omega}^*$ are conformal (see Figure 3). Then there exists a positive constant c_1 , depending on $\|h_a\|_{C^4}$ and $\|h_a^{-1}\|_{C^4}$, such that $\|f - Id\|_{C^{2,\gamma}(\bar{B})} \leq c_1\varepsilon$. ■

In the following Ω^* denotes $h_a^{-1}(\Omega)$.

Lemma 2.8 *Let $a \in [0, \bar{a}]$ and $\gamma \in (0, 1)$. Then there exist $\varepsilon_2 > 0$ and $c_2 = c_2(\bar{a}) > 0$ such that for every $\varepsilon \in (0, \varepsilon_2)$ and $\gamma' \in (0, 1)$ with $\gamma' < \gamma$, the following holds. If Ω is ε -close in $C^{2,\gamma}$ -sense to Ω_a , then there exists a biholomorphic mapping $\varphi : \bar{B} \rightarrow \bar{\Omega}^*$ with $\varphi \in C^{2,\gamma'}(\bar{B})$, $\varphi^{-1} \in C^{2,\gamma'}(\bar{\Omega}^*)$ and such that*

$$\|\varphi - Id\|_{C^{2,\gamma'}(\bar{B})} \leq c_2\varepsilon^{\gamma-\gamma'}.$$

Proof. From Lemma 2.7 it follows that Ω^* is $c_1\varepsilon$ -close to B . Applying Proposition A.1 we have that there exists $\varepsilon_0 > 0$ such that “ Ω^* $c_1\varepsilon$ -close to B ” for $c_1\varepsilon \in (0, \varepsilon_0)$ implies the existence of a biholomorphic mapping $\varphi : \bar{B} \rightarrow \bar{\Omega}^*$ with $\varphi \in C^{2,\gamma'}(\bar{B})$, $\varphi^{-1} \in C^{2,\gamma'}(\bar{\Omega}^*)$ and such that it holds

$$\|\varphi - Id\|_{C^{2,\gamma'}(\bar{B})} \leq c_2\varepsilon^{\gamma-\gamma'},$$

for every $\gamma' \in (0, 1)$ with $\gamma' < \gamma$. The claim follows by taking $\varepsilon_2 = \min \{ \varepsilon_1, c_1^{-1}\varepsilon_0 \}$. ■

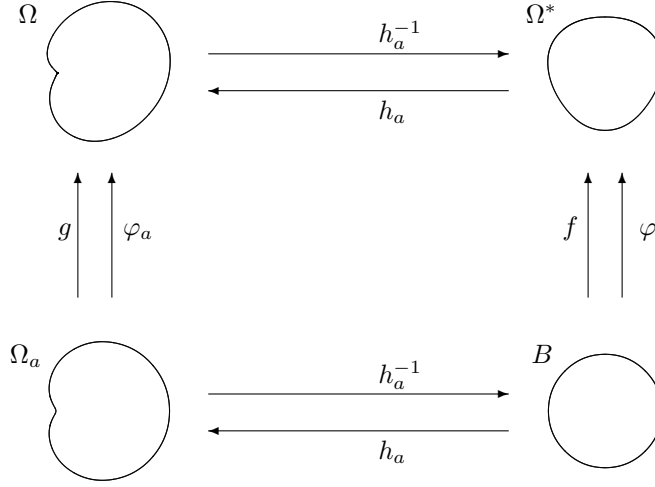


Figure 3: The maps between Ω , Ω_a , B and Ω^* .

Lemma 2.9 *Let $a \in [0, \bar{a}]$ and $\gamma \in (0, 1)$. There exist $\varepsilon_3 > 0$ and $c_3 = c_3(\bar{a}) > 0$ such that for every $\varepsilon \in (0, \varepsilon_3)$ and $\gamma' \in (0, 1)$ with $\gamma' < \gamma$ the following holds.*

If Ω is ε -close in $C^{2,\gamma}$ -sense to Ω_a , then there exists a biholomorphic mapping $\varphi_a : \bar{\Omega}_a \rightarrow \bar{\Omega}$ with $\varphi_a \in C^{2,\gamma'}(\bar{\Omega}_a)$, $\varphi_a^{-1} \in C^{2,\gamma'}(\bar{\Omega})$ and such that

$$\|\varphi_a - Id\|_{C^{2,\gamma'}(\bar{\Omega}_a)} \leq c_3 \varepsilon^{\gamma - \gamma'}.$$

Proof. We denote φ_a the map from $\bar{\Omega}_a$ to $\bar{\Omega}$ given by

$$\varphi_a(x) := (h_a \circ \varphi \circ h_a^{-1})(x).$$

Here φ is the conformal map of Lemma 2.8. The map φ_a is biholomorphic as a composition of biholomorphic maps. Furthermore we have $\varphi_a \in C^{2,\gamma'}(\bar{\Omega}_a)$ and $\varphi_a^{-1} \in C^{2,\gamma'}(\bar{\Omega})$ since $\varphi \in C^{2,\gamma'}(\bar{B})$ and $\varphi^{-1} \in C^{2,\gamma'}(\bar{\Omega}^*)$.

By the way the holomorphic map φ_a is defined one finds that there exists a positive constant K , depending on $\|h_a\|_{C^4}$ and $\|h_a^{-1}\|_{C^4}$, such that

$$\|\varphi_a - Id\|_{C^{2,\gamma'}(\bar{\Omega}_a)} \leq K \|\varphi - Id\|_{C^{2,\gamma'}(\bar{B})}.$$

The claim follows by choosing $c_3 = Kc_2$ and $\varepsilon_3 = \varepsilon_2$ with c_2 and ε_2 as defined in Lemma 2.8. ■

Remark 2.9.1 *Notice that Proposition 2.6 follows from Lemma 2.9.*

We are now ready to prove that the positivity preserving property of problem (10) with a small perturbation of Δ^2 on Ω_a implies the positivity preserving property of problem (1) on Ω ε -close in $C^{2,\gamma}$ -sense to Ω_a .

Corollary 2.10 *Let $\gamma \in (0, 1)$. For every $\delta > 0$ small enough and $a \in [0, \bar{a}]$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in [0, \varepsilon_0)$ the following holds.*

If Ω is ε -close in $C^{2,\gamma}$ -sense to the limaçon Ω_a and the coefficients of the operator A satisfy

$$\sup_{|\alpha| \leq 3} \|A_\alpha\|_\infty \leq \delta, \quad (13)$$

then the positivity of the Green function associated to problem (10) on Ω_a implies the positivity of the Green function associated to problem (1) on Ω .

Proof. To prove the claim we show that problem (1) on Ω ε -close in $C^{2,\gamma}$ -sense to Ω_a can be “transformed” into problem (10) on Ω_a with the coefficients of the lower order operator A satisfying (13).

Let u be solution of problem (1) on Ω . Consider $\delta_0 < \min\{\frac{1}{2}, 2^{-7}\delta\}$. By Proposition 2.6 we know that there exists a $\varepsilon_0 = \varepsilon_0(\delta_0) > 0$ such that for $\varepsilon \in [0, \varepsilon_0)$ and $\gamma' \in (0, 1)$ with $\gamma' < \gamma$ we have the following. If Ω is ε -close to Ω_a in $C^{2,\gamma}$ -sense then there exists a conformal map $\varphi_a : \Omega_a \rightarrow \Omega$ such that

$$\|\varphi_a - Id\|_{C^{2,\gamma'}(\bar{\Omega}_a)} \leq \delta_0.$$

We define the function $v_a(x) := u \circ \varphi_a(x)$ on Ω_a . Clearly $u > 0$ if and only if $v_a > 0$. Since φ_a is a conformal map, the function v_a satisfies

$$\begin{cases} \Delta^2 v_a - 2\nabla|\varphi'_a|^2 \cdot \nabla \frac{\Delta v_a}{|\varphi'_a|^2} - 4|\varphi''_a|^2 \frac{\Delta v_a}{|\varphi'_a|^2} &= |\varphi'_a|^4 f \circ \varphi_a & \text{in } \Omega_a, \\ v_a &= 0 & \text{on } \partial\Omega_a, \\ \frac{\partial}{\partial \nu} v_a &= 0 & \text{on } \partial\Omega_a, \end{cases} \quad (14)$$

where φ'_a denotes the complex derivative of φ_a . Hence v_a is solution of a problem as in (10). The coefficients of the lower order perturbation of Δ^2 in (14) satisfy (13) by the choice of δ_0 . ■

Remark 2.10.1 *Notice that since we are working with conformal mappings it is sufficient to have $C^{2,\gamma}$ -closeness in order to transform problem (1) into problem (10). Working with general transformations fourth order derivatives would appear and $C^{4,\gamma}$ -closeness would be necessary.*

As a consequence of Corollary 2.10, Theorem 2.4 will follow from Theorem 2.5.

2.2.2 Proof of the perturbation theorem

In [13] Theorem 2.5 has been proven in the unit disk (that is Ω_0). We now give a sketch of the proof for Ω_a , $a \in [0, \bar{a}]$ by following similar steps.

First we state some estimates for (12) with G_B replaced by G_{Ω_a} .

Theorem 2.11 *Let $k = (k_1, k_2)$ with $k_1, k_2 \in \mathbb{N}$ and $|k| \leq 3$. The following estimates hold for any $a \in [0, \bar{a}]$ and $x, y, z \in \Omega_a$.*

1. If $|k| = 3$, then

$$\frac{G_{\Omega_a}(x, z) |D_z^k G_{\Omega_a}(z, y)|}{G_{\Omega_a}(x, y)} \asymp \frac{1}{|x - z|} + \frac{1}{|y - z|}.$$

2. If $|k| = 2$, then

$$\frac{G_{\Omega_a}(x, z) |D_z^k G_{\Omega_a}(z, y)|}{G_{\Omega_a}(x, y)} \asymp \log \left(\frac{3}{|z - y|} \right).$$

3. If $|k| \leq 1$, then

$$\frac{G_{\Omega_a}(x, z) |D_z^k G_{\Omega_a}(z, y)|}{G_{\Omega_a}(x, y)} \preceq 1.$$

Proof. With the same method as has been used in [13] the result follows from the optimal estimate from below for G_{Ω_a} , which has been proved in [7] (see Proposition B.1), and from the estimates of the derivatives of the Green function, which have been proved in [8] (see Proposition B.2). ■

Let \mathcal{G}_Ω denote the Green operator associated to problem (1) in Ω , that is

$$\mathcal{G}_\Omega f(x) := \int_\Omega G_\Omega(x, y) f(y) dy.$$

By the estimate in Theorem 2.11 one may observe that the derivatives of the Green function have an integrable singularity. Hence one finds the following two corollaries of Theorem 2.11.

Corollary 2.12 *There exists $M \in \mathbb{R}^+$ such that for any $0 \leq f \in L^p(\Omega_a)$ with $p \geq 1$ and $k = (k_1, k_2) \in \mathbb{N}^2$ with $0 \leq |k| \leq 3$, the following estimate holds for all $a \in [0, \bar{a}]$*

$$|(\mathcal{G}_{\Omega_a} D^k \mathcal{G}_{\Omega_a} f)(x)| \leq M (\mathcal{G}_{\Omega_a} f)(x) \text{ for all } x \in \Omega_a.$$

Corollary 2.13 *Let $a \in [0, \bar{a}]$ and $\eta > 0$ be such that the coefficients of A in (10) satisfy $\|A_\alpha\|_\infty \leq \eta$ for all $|\alpha| \leq 3$. Then for any $0 \leq f \in L^p(\Omega_a)$ with $p \geq 1$*

$$|(\mathcal{G}_{\Omega_a} A \mathcal{G}_{\Omega_a} f)(x)| \leq 10 M \eta (\mathcal{G}_{\Omega_a} f)(x) \text{ for all } x \in \Omega_a,$$

and furthermore

$$\left| \left((\mathcal{G}_{\Omega_a} A)^i \mathcal{G}_{\Omega_a} f \right) (x) \right| \leq (10 M \eta)^i (\mathcal{G}_{\Omega_a} f)(x) \text{ for all } x \in \Omega_a,$$

where M is the constant of Corollary 2.12.

For the proofs we refer to [13, Cor.4.2, Lem.5.4-5.5].

Proof of Theorem 2.5. Let u be a solution of (10). Proceeding as in [13, Lemma 5.3] one finds that there exists a $\eta_1 > 0$ such that $(\mathcal{I} + \mathcal{G}_{\Omega_a} A)^{-1}$ is well

defined when the coefficients of A satisfy $\|A_\alpha\|_\infty \leq \eta_1$ for $|\alpha| \leq 3$. We have $u = -\mathcal{G}_{\Omega_a} Au + \mathcal{G}_{\Omega_a} f$, or $u = (\mathcal{I} + \mathcal{G}_{\Omega_a} A)^{-1} \mathcal{G}_{\Omega_a} f$, and may formally write

$$\begin{aligned} \mathcal{G}_{\Omega_a, A} &= (\mathcal{I} + \mathcal{G}_{\Omega_a} A)^{-1} \mathcal{G}_{\Omega_a} \\ &= \mathcal{G}_{\Omega_a} - \mathcal{G}_{\Omega_a} A \mathcal{G}_{\Omega_a} + (\mathcal{G}_{\Omega_a} A)^2 \mathcal{G}_{\Omega_a} - (\mathcal{G}_{\Omega_a} A)^3 \mathcal{G}_{\Omega_a} + (\mathcal{G}_{\Omega_a} A)^4 \mathcal{G}_{\Omega_a} - \dots \end{aligned} \quad (15)$$

Using Corollary 2.13 from (15) taking $\eta_0 = \min \left\{ \frac{1}{30M}, \eta_1 \right\}$ and $\eta \leq \eta_0$ the series converges and we get

$$\frac{1}{2} \mathcal{G}_{\Omega_a} \leq \mathcal{G}_{\Omega_a, A} \leq \frac{3}{2} \mathcal{G}_{\Omega_a}. \quad (16)$$

The estimate in (11) follows directly from (16) and Proposition B.1. \blacksquare

Remark 2.13.1 *For the problem*

$$\begin{cases} (\Delta^2 + A)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega, \end{cases}$$

with Ω ε -close in $C^{2,\gamma}$ -sense to Ω_a for $a \in [0, \bar{a}]$ and with A the lower order perturbation of the bilaplacian such that $\|A_\alpha\|_\infty \leq \eta$ for $|\alpha| \leq 3$, the result stated in Theorem 2.5 is still valid for ε and η sufficiently small.

3 Filling the domain with perturbed limacons

In this section we prove that a sufficiently smooth bounded two-dimensional domain can be approximated by limaçon-like domains in the sense we want. That is, we will construct a finite number of domains E_j such that:

1. the union of E_j covers Ω near $\partial\Omega$;
2. the union of ∂E_j covers the boundary $\partial\Omega$;
3. each E_j is close in $C^{2,\gamma}$ -sense to a limaçon $\Omega_{a,R}$ with $a \in [0, \bar{a}]$ in a uniform way;
4. the E_j uniformly satisfy the uniform $C^{4,\gamma}$ regularity condition in $a \in [0, \bar{a}]$.

The precise statement is given in Theorem 3.8.

3.1 Local approximation

We first show that for each z_0 on $\partial\Omega$ there exists a domain ε -close to a limaçon which boundary intersects $\partial\Omega$ in a neighborhood of z_0 . In order to do that it will be convenient to use local systems of Cartesian coordinates. The following lemma lists some technical results.

Lemma 3.1 *Let $\ell \geq 2$ and Ω be a domain in \mathbb{R}^2 satisfying the uniform $C^{\ell,\alpha}$ regularity condition, Definition 1.6, with constant M and mappings $\varphi_j \in C^{\ell,\alpha}$, $j \in J$. Let ρ_Ω be as in Notation 1.7 and set $x_{\rho_\Omega} := \frac{\sqrt{3}}{2} \rho_\Omega$.*

Then for every $z_0 \in \partial\Omega$ there exists a local Cartesian coordinates system and a function $g_{z_0} \in C^{\ell,\alpha}$, $g_{z_0} : [-x_{\rho_\Omega}, x_{\rho_\Omega}] \rightarrow \mathbb{R}$, such that:

1. $z_0 = (0, 0)$;
2. the x -axis is tangential to $\partial\Omega$ in z_0 ;
3. the y -axis has the direction of the internal normal to $\partial\Omega$ in z_0 ;
4. $B_{\frac{1}{2}\rho_\Omega}(z_0) \cap \partial\Omega \subset \{(x, y) : x \in [-x_{\rho_\Omega}, x_{\rho_\Omega}] \text{ and } y = g_{z_0}(x)\}$;
5. $\|g_{z_0}\|_{C^{\ell,\alpha}[-x_{\rho_\Omega}, x_{\rho_\Omega}]} \leq 2(\ell + 1)M$.

Remark 3.1.1 *Observe that the function g_{z_0} of Lemma 3.1 satisfies $|g'_{z_0}(x)| \leq \frac{1}{\sqrt{3}}$.*

We skip the rather technical proof of Lemma 3.1.

In the following theorem we will state that for every point of the boundary of a domain satisfying the uniform $C^{4,\alpha}$ regularity condition there exists a limaçon $\Omega_{a,R}$ that approximates $\partial\Omega$ in the point in C^2 -sense. Furthermore we will construct a domain $\tilde{\Omega}$ that is ε -close to the limaçon $\Omega_{a,R}$ and which boundary coincides with $\partial\Omega$ in a neighborhood of that point. By construction $\tilde{\Omega}$ is a domain satisfying the uniform $C^{4,\alpha}$ regularity condition with constant M_1 where M_1 depends only on M and ρ_Ω .

For the purpose of a uniform statement we will have to rescale to limaçons of 'unit' size. In order to do so we define for a given f the scaled function:

$$f^R(x, y) := R^{-1} f(Rx, Ry) \text{ for } R \in \mathbb{R}^+. \quad (17)$$

Theorem 3.2 *Assume that the following holds for some $\alpha, \gamma \in (0, 1)$:*

- i. $\Omega \subset \mathbb{R}^2$ is a simply connected domain satisfying the uniform $C^{4,\alpha}$ regularity condition with constant M ;
- ii. $g_{z_0} \in C^{4,\alpha}$ for $z_0 \in \partial\Omega$ are functions that describe the boundary of Ω as in Lemma 3.1 and fix $R := \min \left\{ \frac{1}{2} \left(\max_{z_0 \in \partial\Omega} \|g''_{z_0}\|_\infty \right)^{-1}, 1 \right\}$;
- iii. $\varepsilon > 0$ is such that for all $\tilde{\Omega}$ which are ε -close to $\Omega_{a,1}$ in $C^{2,\gamma}$ sense with $a \in \left[\frac{3}{16}, \frac{5}{16} \right]$, the Green function associated to problem (1) on $\tilde{\Omega}$ is positive.

Then there is $\delta = \delta(M, \rho_\Omega^{-1}, \varepsilon, \gamma) \in \left(0, \frac{1}{16}R\right)$ such that the following holds. For every $z_0 \in \partial\Omega$ there exist $a \in \left[\frac{3}{16}, \frac{5}{16}\right]$, a limaçon $\Omega_{a,R}$ and a $C^{4,\alpha}$ map $f_{a,R} : \tilde{\Omega}_{a,R} \rightarrow f_{a,R}(\tilde{\Omega}_{a,R})$ such that:

1. $\partial\Omega \cap B_\delta(z_0) = \partial(f_{a,R}(\Omega_{a,R})) \cap B_\delta(z_0)$;
2. the map $f_{a,R}^R := (f_{a,R})^R$ is ε -close in $C^{2,\gamma}$ -sense to the identity on $\bar{\Omega}_{a,1}$:
 $\|f_{a,R}^R - Id\|_{C^{2,\gamma}(\bar{\Omega}_{a,1})} \leq \varepsilon$;
3. the map $f_{a,R}^R$ is $C^{4,\alpha}$ bounded by some $A = A(M, \rho_\Omega^{-1}, \varepsilon, \gamma)$: that is,
 $\|f_{a,R}^R\|_{C^{4,\alpha}(\bar{\Omega}_{a,1})} \leq A$.

Remark 3.2.1 We construct a $C^{4,\alpha}$ mapping $f_{a,R} : \bar{\Omega}_{a,R} \rightarrow f_{a,R}(\bar{\Omega}_{a,R})$ in order that $f_{a,R}(\bar{\Omega}_{a,R})$ is a domain satisfying the uniform $C^{4,\alpha}$ regularity condition with constant M_1 where $M_1 = M_1(M, \rho_\Omega^{-1}, \varepsilon, \gamma)$. Using the result in [9] it should be possible to relax the regularity of the boundary to C^4 .

Remark 3.2.2 In order to approximate $\partial\Omega$ with limacons in $C^{2,\gamma}$ -sense it is sufficient that Ω satisfies the uniform $C^{2,\alpha}$ regularity condition for $\alpha > \gamma$.

Remark 3.2.3 The R defined in Theorem 3.2 depends on Ω via the constant M of the uniform $C^{4,\alpha}$ regularity condition.

Corollary 3.3 Assume that $\Omega, \alpha, \gamma, \varepsilon$ are such that the hypotheses of Theorem 3.2 hold true and let R as defined in that theorem. Then there is $\delta > 0$ such that for every $z_0 \in \partial\Omega$ there exists a domain E_{z_0} that satisfies the following:

1. E_{z_0} satisfies the uniform $C^{4,\alpha}$ regularity condition with constant $M_1 = M_1(M, \rho_\Omega^{-1}, \varepsilon, \gamma) > 0$;
2. E_{z_0} is ε -close in $C^{2,\gamma}$ -sense to a limaçon $\Omega_{a,R}$ with $a \in [\frac{3}{16}, \frac{5}{16}]$;
3. $z_0 \in (\bar{E}_{z_0} \cap \partial\Omega)^{\circ, \partial\Omega}$.

Furthermore, letting K_{z_0} be the component of $(\bar{E}_{z_0} \cap \partial\Omega)^{\circ, \partial\Omega}$ that contains z_0 :

4. $B_\delta(z_0) \cap \partial\Omega = B_\delta(z_0) \cap K_{z_0}$;
5. E_{z_0} and Ω have the same outward normal for any $x \in K_{z_0}$.

The proof of Theorem 3.2 is divided into several steps. We first present the setting for a fixed $z_0 \in \partial\Omega$. Let us consider the local system of coordinates near z_0 and the function $g_{z_0} \in C^{4,\alpha}$ given by Lemma 3.1 (in this case $l = 4$). We will write $g_{z_0} = g$.

Let δ be a positive number such that

$$\delta < \min \left\{ 1, \frac{x_{\rho_\Omega}}{4}, \frac{R}{16} \left(1 - \frac{5}{16} \sqrt{3} \right) \right\} \text{ and } \delta^{1-\gamma} < \varepsilon (C_{10} R^{1+\gamma})^{-1}. \quad (18)$$

Here C_{10} is a positive constant that depends on M . We remark that δ depends on Ω through ρ_Ω^{-1} and M .

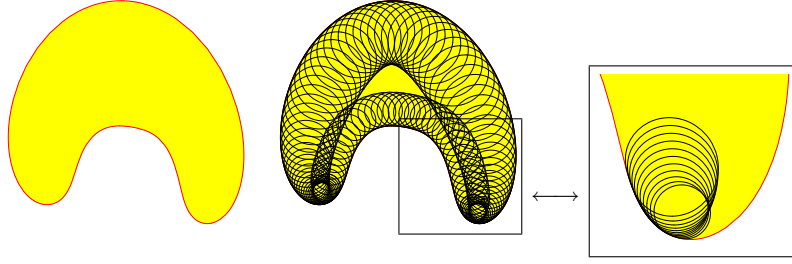


Figure 4: A domain, the finite number of approximated limaçons with their boundaries covering the boundary of the domain, and a zoomed view.

3.2 Approximation by a limaçon in one point

There exists $a \in [\frac{3}{16}, \frac{5}{16}]$ such that $z_0 = (0, 0) \in \bar{\Omega}_{a,R}$ and

$$k''_{a,R}(0) = g''(0),$$

where $k_{a,R} \in C^\infty$ is the map that describes, as in (5), the lower part of the limaçon.

In order to get that $\partial\Omega_{a,R}$ approximates the boundary of Ω in $(0, 0)$ up to the second derivative, we have to impose the condition $g''(0) = k''_{a,R}(0)$. Using (6) this reads as

$$g''(0) = \frac{1}{R} \frac{1-4a}{(1-2a)^2}. \quad (19)$$

Since the map $a \mapsto \frac{1-4a}{(1-2a)^2}$ sends the interval $[\frac{3}{16}, \frac{5}{16}]$ onto $[-\frac{1}{2}, \frac{1}{2}]$ and it holds $|g''(0)|R \leq \frac{1}{2}$ by the definition of R , one finds that $a \in [\frac{3}{16}, \frac{5}{16}]$ exists such that (19) holds.

Note that R is fixed and that it is sufficient to play with the parameter a to fit the limaçon $\Omega_{a,R}$ to the domain Ω around z_0 .

3.3 Construction of the mapping $f_{a,R}$

Again we fix some preliminaries. Let x_a be the number defined in (4) and let us fix $x_a^* := \frac{1}{2}(1 - \sqrt{3}a) \in (\frac{1}{5}x_a, \frac{1}{2}x_a)$. We introduce two cut-off functions:

1. $\varphi_{a,R} \in C^\infty(\mathbb{R})$ such that

$$\begin{aligned} \varphi_{a,R} &\equiv 1 && \text{for } |x| \leq \frac{1}{2}x_a^*R, \\ \varphi_{a,R} &\equiv 0 && \text{for } |x| \geq x_a^*R, \\ \|\varphi_{a,R}\|_{C^{k,\nu}} &\leq \frac{D_{k,\nu}}{R^{k+\nu}} && \text{for } k = 0, \dots, 4 \text{ and } \nu \in (0, 1), \end{aligned}$$

with $D_{k,\nu}$ some positive constants;

2. $\psi_{a,\delta} \in C^\infty(\mathbb{R})$ such that

$$\begin{aligned} \psi_{a,\delta} &\equiv 1 && \text{for } |x| \leq \delta, \\ \psi_{a,\delta} &\equiv 0 && \text{for } |x| \geq 2\delta, \\ \|\psi_{a,\delta}\|_{C^{k,\nu}} &\leq \frac{D'_{k,\nu}}{\delta^{k+\nu}} && \text{for } k = 0, \dots, 4 \text{ and } \nu \in (0, 1), \end{aligned}$$

with $D'_{k,\nu}$ some positive constants.

We define a $C^{4,\alpha}$ -mapping g_δ on $[-Rx_a, Rx_a]$ that follows the boundary of Ω when $|x| \leq \delta$ and the boundary of the limaçon when $Rx_a^* \leq |x| \leq Rx_a$:

$$g_\delta(x) := \begin{cases} g(x) & \text{for } 0 \leq x \leq \delta, \\ k_{a,R}(x) + \sum_{i=0}^2 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (x - \delta)^i + \\ \quad + \psi_{a,\delta}(x) \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (x - \delta)^i & \text{for } \delta < x \leq 2\delta, \\ k_{a,R}(x) + \sum_{i=0}^2 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (x - \delta)^i & \text{for } 2\delta < x \leq \frac{1}{2}Rx_a^*, \\ k_{a,R}(x) + \varphi_{a,R}(x) \sum_{i=0}^2 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (x - \delta)^i & \text{for } \frac{1}{2}Rx_a^* < x \leq Rx_a^*, \\ k_{a,R}(x) & \text{for } Rx_a^* < x \leq Rx_a, \end{cases} \quad (20)$$

and similarly for $x \in [-Rx_a, 0]$.

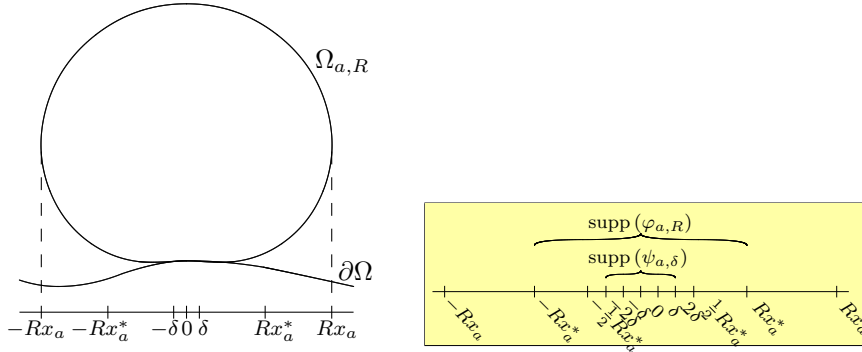


Figure 5: Left: the limaçon that approximates in $(0,0)$ the behavior of $\partial\Omega$ up to the second derivative.

Right: scheme for the support of the cut-off functions $\varphi_{a,R}$ and $\psi_{a,\delta}$.

Remark 3.3.1 In the definition of g_δ we use two cut-off functions. The reason for this construction is that we want g_δ to be close to $k_{a,R}$ in $C^{2,\gamma}$ -sense and also to be a $C^{4,\alpha}$ -mapping. Indeed, considering $\|g_\delta - k_{a,R}\|_{C^{2,\gamma}(-Rx_a, Rx_a)}$ one sees

that the terms $(g - k_{a,R})^{(i)} \Big|_{\delta}$ have a different behavior for $i = 0, 1, 2$ respectively for $i = 3, 4$. One cut-off function can be chosen independent of δ since we will show that for $i = 0, 1, 2$ the term $(g - k_{a,R})^{(i)} \Big|_{\delta} = O(\delta)$. While for $i = 3, 4$ $(g - k_{a,R})^{(i)} \Big|_{\delta}$ will be just bounded, and hence one needs a cut-off function depending on δ in order that the $C^{2,\gamma}$ -norm of $g - k_{a,R}$ is an $O(\delta)$. By the way, close in $C^{2,\gamma}$ -sense is needed for positivity; $C^{4,\alpha}$ is used in the regularity results.

We define the function $f_{a,R} : \bar{\Omega}_{a,R} \rightarrow f_{a,R}(\bar{\Omega}_{a,R})$ by

$$f_{a,R}(x, y) = \left(x, \frac{3R - g_{\delta}(x)}{3R - k_{a,R}(x)} (y - 3R) + 3R \right), \quad (21)$$

which gives $(f_{a,R} - I)(x, y) = \left(0, \frac{k_{a,R}(x) - g_{\delta}(x)}{3R - k_{a,R}(x)} (y - 3R) \right)$. By construction $f_{a,R} \in C^{4,\alpha}(\bar{\Omega}_{a,R})$ and the boundary of $f_{a,R}(\Omega_{a,R})$ coincides with $\partial\Omega$ in a neighborhood of $z_0 = (0, 0)$ of length at least 2δ .

In the next paragraph we show that $f_{a,R}(\Omega_{a,R})$ is ε -close to $\Omega_{a,R}$ in $C^{2,\gamma}$ -sense and that $f_{a,R}(\Omega_{a,R})$ satisfies the uniform $C^{4,\alpha}$ regularity condition.

Remark 3.3.2 Notice that $f_{a,R} \equiv Id$ for $(x, y) \in \bar{\Omega}_{a,R}$ with $|x| \geq Rx_a^*$. While for $|x| < Rx_a^*$ it holds that $f_{a,R} \equiv Id$ for $x = 0$ only. The map $f_{a,R}$ also changes the boundary of $\Omega_{a,R}$ in a neighborhood of the point $(0, 2R)$. That is not a problem since one may notice from the expression of $f_{a,R} - Id$ that in the approximation only the term $\frac{k_{a,R}(x) - g_{\delta}(x)}{3R - k_{a,R}(x)}$ plays a role.

3.4 The mapping is close to the identity in $C^{2,\gamma}$ -sense

In this section we will prove that $f_{a,R}^R$, which is the $f_{a,R}$ from (21) rescaled as in (17), satisfies

$$\|f_{a,R}^R - Id\|_{C^{2,\gamma}(\bar{\Omega}_{a,1})} \leq \varepsilon. \quad (22)$$

By the results of the previous section and our choice of ε , it then follows that the Green function associated to problem (1) on $f_{a,R}(\Omega_{a,R})$ is positive.

We first fix some notation. In the following N_1 and N_2 denote respectively

$$N_1 := \left\| \frac{\partial^4}{\partial x^4} k_{a,R} - \frac{\partial^4}{\partial x^4} g \right\|_{C^0[-\delta, \delta]} + \left| \frac{\partial^3}{\partial x^3} k_{a,R}(0) - \frac{\partial^3}{\partial x^3} g(0) \right|, \quad (23)$$

$$N_2 := \left[\frac{\partial^4}{\partial x^4} k_{a,R} - \frac{\partial^4}{\partial x^4} g \right]_{C^\alpha[-\delta, \delta]} + \left| \frac{\partial^4}{\partial x^4} k_{a,R}(0) - \frac{\partial^4}{\partial x^4} g(0) \right|. \quad (24)$$

Notice that $N_i = N_i(M)$ for $i = 1, 2$. Indeed R depends on M and the dependence of $k_{a,R}$ on a is continuous in $\left[\frac{3}{16}, \frac{5}{16}\right]$ and hence uniform.

We have

$$\begin{aligned} \left\| \frac{\partial^i}{\partial x^i} k_{a,R} - \frac{\partial^i}{\partial x^i} g \right\|_{C^0[-\delta, \delta]} &\leq N_1 \delta^{3-i} \text{ for } i = 0, \dots, 3 \\ \text{and } \left\| \frac{\partial^4}{\partial x^4} k_{a,R} - \frac{\partial^4}{\partial x^4} g \right\|_{C^0[-\delta, \delta]} &\leq N_2. \end{aligned} \quad (25)$$

In order to prove (22) one has first to consider the effect of the scaling.

Proposition 3.4 *Let $\gamma \in (0, 1)$. The function $f_{a,R}^R$ satisfies*

$$\begin{aligned} & \|f_{a,R}^R - Id\|_{C^{2,\gamma}(\bar{\Omega}_{a,1})} \leq 5 \left\| \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \\ & + 5R \left\| \frac{\partial}{\partial x} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} + 9R^2 \left\| \frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \\ & + 4R^{2+\gamma} \left[\frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]}. \end{aligned} \quad (26)$$

We postpone the proof of Proposition 3.4 to Appendix C.1.1.

Proposition 3.5 *Let $\gamma \in (0, 1)$. There is $C_{10} = C_{10}(M) > 0$ such that*

$$\|f_{a,R}^R - Id\|_{C^{2,\gamma}(\bar{\Omega}_{a,1})} \leq C_{10} R^{1+\gamma} \delta^{1-\gamma}. \quad (27)$$

The right-hand side in (27) is less than ε due to our choice of δ in (18).

In order to prove Proposition 3.5 we estimate the terms in the right hand side of (26) separately. The details of the proof are given in Appendix C.1.2.

3.5 Bounded third and fourth derivative of the mapping

In this section we derive the estimate of $\|f_{a,R}^R\|_{C^{4,\alpha}(\bar{\Omega}_{a,1})}$. Again this $f_{a,R}^R$ is the function $f_{a,R}$ from (21) rescaled as in (17). The estimate will imply that $f_{a,R}(\Omega_{a,R})$ satisfies the uniform $C^{4,\alpha}$ regularity condition.

The effect of the scaling is as follows:

Proposition 3.6 *Let $\alpha \in (0, 1)$. There is $C_{11} = C_{11}(M) > 0$ such that:*

$$\begin{aligned} & \|f_{a,R}^R\|_{C^{4,\alpha}(\bar{\Omega}_{a,1})} \leq x_a + 9 + 5C_{11}\delta R + 5R^3 \left\| \frac{\partial^3}{\partial x^3} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} + \\ & + 6R^4 \left\| \frac{\partial^4}{\partial x^4} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} \\ & + R^{3+\alpha} \left[(x, y) \mapsto (y - 3R) \frac{\partial^4}{\partial x^4} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right]_{C^\alpha(\bar{\Omega}_{a,R})}. \end{aligned} \quad (28)$$

We postpone the proof of Proposition 3.6 to Appendix C.2.1.

The estimate we are looking for is then:

Proposition 3.7 *Let $\alpha \in (0, 1)$. There is $C_{19} = C_{19}(M) > 0$ such that:*

$$\|f_{a,R}^R\|_{C^{4,\alpha}(\bar{\Omega}_{a,1})} \leq C_{19} \frac{R^{3+\alpha}}{\delta^{1+\alpha}}.$$

In order to prove Proposition 3.7 it is sufficient to find appropriate estimates for the terms in the right hand side of (28). The details of the proof are in Appendix C.2.2.

3.6 The covering

We are now ready to prove that for any domain Ω with $\partial\Omega \in C^{4,\alpha}$ one may find an appropriate covering by finitely many open domains that are ε -close in $C^{2,\gamma}$ -sense to some limaçon.

Theorem 3.8 *Let Ω , α , γ and ε satisfy the assumptions of Theorem 3.2 and let R defined as in that Theorem. Then there exist finitely many balls B_j , $j \in J_B$ with $\bar{B}_j \subset \Omega$, finitely many open domains $E_j \subset \mathbb{R}^2$, $j \in J_E$, and constants $\bar{M} = \bar{M}(M, \rho_\Omega^{-1}, \varepsilon, \gamma) > 0$ and $\delta > 0$ such that:*

1. $\Omega \subset \bigcup_{j \in J_B} B_j \cup \bigcup_{j \in J_E} E_j$;
2. $(E_j \cap \partial\Omega)^{\circ, \partial\Omega} \neq \emptyset$ for all $j \in J_E$;
3. every E_j with $j \in J_E$ is a domain satisfying the uniform $C^{4,\alpha}$ regularity condition with constant \bar{M} ;
4. each E_j is ε -close in $C^{2,\gamma}$ -sense to a limaçon $\Omega_{a,R}$ with $a \in [\frac{3}{16}, \frac{5}{16}]$.

Furthermore, for $K_j = (\bar{E}_j \cap \partial\Omega)^{\circ, \partial\Omega}$ with $j \in J_E$ it holds:

5. E_j and K_j have the same outward normal for any $x \in K_j$;
6. $\{K_j\}_{j \in J_E}$ is a relatively open covering of $\partial\Omega$;
7. for all $j \in J_E$ the diameter of K_j is larger than δ .

Proof. According to Corollary 3.3 there is a $\delta > 0$ such that for every $z_0 \in \partial\Omega$ there exists a domain E_{z_0} such that the following holds:

- E_{z_0} satisfies the uniform $C^{4,\alpha}$ regularity condition with constant $M_{z_0} = M_{z_0}(M, \rho_\Omega^{-1}, \varepsilon, \gamma)$;
- E_{z_0} is ε -close in $C^{2,\gamma}$ -sense to a limaçon $\Omega_{a,R}$ with $a \in [\frac{3}{16}, \frac{5}{16}]$;
- letting K_{z_0} the connected component of $(\bar{E}_{z_0} \cap \partial\Omega)^{\circ, \partial\Omega}$ that contains z_0 , it holds

$$B_\delta(z_0) \cap \partial\Omega = B_\delta(z_0) \cap K_{z_0}.$$

By compactness of $\partial\Omega$ there exist $z_1, \dots, z_N \in \partial\Omega$ such that $\partial\Omega = \bigcup_{j=1}^N K_{z_j}$. Setting $E_j := E_{z_j}$ and $\bar{M} = \max M_{z_j}$ and K_j accordingly one finds that this family $\{K_j\}_{j=1, \dots, N}$ satisfies the properties of the last three items. A straightforward argument implies that $\Omega \setminus \bigcup_{j=1}^N (E_j \cap \Omega)$ can be covered by finitely open balls B_j with $\bar{B}_j \subset \Omega$. ■

Remark 3.8.1 *In the proof we use that Ω is simply connected. However with a slightly different argument the method would work also for general connected domains.*

4 Proving the estimates

In this section we prove the main results of the paper. First we give pointwise estimates for the solution of (1), and then we prove the splitting of the solution operator between a positive singular part and a sign changing regular part.

4.1 A maximum principle type estimate

The pointwise estimates for the solution of (1) will be obtained using negative Sobolev spaces. We refer to [1, pages 62-65].

Theorem 4.1 *Suppose that the hypotheses of Theorem 3.2 hold true with $0 < \gamma, \alpha < 1$. Then for any $q > 2$ and $\varepsilon \in (0, 4R]$ there exists a constant $C > 0$ with $C = C(\frac{1}{2-q}, M, \rho_\Omega^{-1}, R_\Omega, \varepsilon, \gamma)$ such that for any $f \in L^p(\Omega)$, with $p \in (1, \infty)$, the solution $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ of (1) satisfies*

$$u(x) \leq C \left(\|f^+\|_{L^1(B(x,\varepsilon) \cap \Omega)} + \|u\|_{W^{-1,q}(\Omega)} \right) \text{ for every } x \in \Omega. \quad (29)$$

Proof. Let E_j , with $j \in J$, be the finite covering of Ω of Theorem 3.8 and set $D_j := E_j \cap \Omega$. We first consider the case $\varepsilon = 4R$.

Let $\psi_i, i \in I$, be a partition of unity with boundary associated to the covering $\{D_j\}_{j \in J}$ of Ω (Lemma C.14 in the appendix with $\delta = 2R$) such that for every $i \in I$:

- i. $|D^\alpha \psi_i| \leq c_\alpha R^{-|\alpha|}$ for $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq 4$;
- ii. $\psi_i \neq 0$ at the boundary only if $(\partial E_{j(i)} \cap \partial \Omega)^{\circ, \partial \Omega} \neq \emptyset$.

Here $j(i)$ denotes the $j \in J$ such that $\text{supp}(\psi_i) \subset E_j$. By the choice of ψ_i it also holds that $\psi_i \in C_c^\infty(\Omega \cup (\partial E_{j(i)} \cap \partial \Omega)^{\circ, \partial \Omega})$, $\psi_i \in C_c^\infty(E_{j(i)} \cup (\partial E_{j(i)} \cap \partial \Omega)^{\circ, \partial \Omega})$ and $\psi_i \neq 0$ only on $(E_{j(i)} \cap \Omega) \cup (\partial E_{j(i)} \cap \partial \Omega)^{\circ, \partial \Omega}$. Notice that I is a finite set.

We choose a new family of cut-off functions $\chi_i \in C_c^\infty(\Omega \cup (\partial E_{j(i)} \cap \partial \Omega)^{\circ, \partial \Omega})$, $i \in I$, such that for every $i \in I$:

- i. $\text{supp}(\psi_i) \subset \{x \in \bar{\Omega} : \chi_i(x) = 1\} \subset \text{supp}(\chi_i) \subset (E_{j(i)} \cap \Omega) \cup (E_{j(i)} \cap \partial \Omega)^{\circ, \partial \Omega}$;
- ii. $0 \leq \chi_i(x) \leq 1$;
- iii. $\|\nabla^\alpha \chi_i\|_\infty \leq c_\alpha R^{-|\alpha|}$ for every $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq 4$.

The functions $\chi_i g$ and $\psi_i g$ denote (with abuse of notation) respectively

$$\chi_i g(x) := \begin{cases} \chi_i(x)g(x) & \text{in } \bar{\Omega}, \\ 0 & \text{otherwise,} \end{cases} \quad \psi_i g(x) := \begin{cases} \psi_i(x)g(x) & \text{in } \bar{\Omega}, \\ 0 & \text{otherwise.} \end{cases}$$

In the following, if not explicitly stated, every function will be extended by 0 outside its domain of definition.

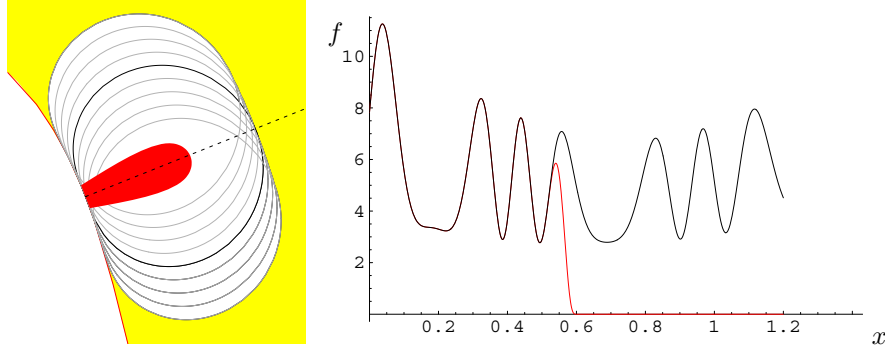


Figure 6: In the picture on the left one finds some E_j 's that cover Ω locally. The dark part shows the support of the cut-off function ψ_i . On the right the effect of the multiplication with the cut-off function considered on the dashed line: in black a function f and in red (lighter) the function $\psi_i f$. The scaling is arbitrary but consistent with the one in the following figures.

Let G_{E_j} be the Green function associated to Δ^2 on E_j with zero Dirichlet boundary condition. Let $v_{g,j}$ the function that satisfies

$$\begin{cases} \Delta^2 v_{g,j} = g & \text{in } E_j, \\ v_{g,j} = \frac{\partial}{\partial \nu} v_{g,j} = 0 & \text{on } \partial E_j. \end{cases}$$

We define

$$\tilde{u}_i(x) := \chi_i(x) v_{\psi_i f, j(i)}(x) \text{ and } \tilde{u}(x) := \sum_{i \in I} \tilde{u}_i(x).$$

Here $j(i)$ denotes the $j \in J$ such that $\text{supp}(\psi_i) \subset E_j$.

Since the Green function $G_{E_j}(x, y)$ is positive and bounded on $E_j \times E_j$ (Theorem 2.4) we have for some $c_1 = c_1(M, \rho_\Omega^{-1})$

$$\begin{aligned} \tilde{u}_i(x) &= \chi_i(x) \int_{E_{j(i)}} G_{E_{j(i)}}(x, y) \psi_i(y) f(y) dy \\ &\leq \chi_i(x) \int_{\text{supp}(\psi_i) \cap \Omega} G_{E_{j(i)}}(x, y) f^+(y) dy \leq c_1 \chi_i(x) \|f^+\|_{L^1(\text{supp}(\psi_i) \cap \Omega)}. \end{aligned}$$

Hence with $\varepsilon_R := 4R$ one gets

$$\begin{aligned} \tilde{u}(x) &\leq c_1 \sum_{i \in I} \chi_i(x) \|f^+\|_{L^1(\text{supp}(\psi_i) \cap \Omega)} \\ &\leq c_2 \|f^+\|_{L^1\left(\bigcup_{\substack{i \in I \\ \chi_i(x) \neq 0}} (\text{supp}(\psi_i) \cap \Omega)\right)} \leq c_2 \|f^+\|_{L^1(B(x, \varepsilon_R) \cap \Omega)}. \end{aligned} \quad (30)$$

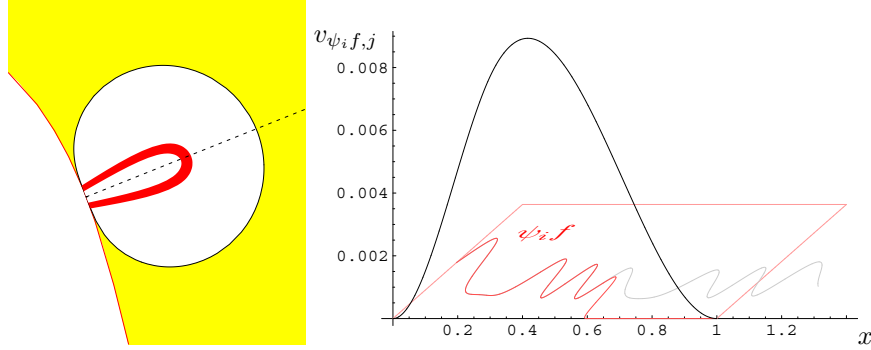


Figure 7: On the left one finds in black the boundary of E_j and in red the set $\{x : \nabla \psi_i(x) \neq 0\}$. In the right one in black the function $v_{\psi_i f, j(i)}$, that is the solution of the clamped plate equation on $E_{j(i)}$ with on the right hand side $\psi_i f$, that is, the truncated f (red in the picture).

We will now estimate the difference $u - \tilde{u}$. For every $i \in I$ one has in $E_{j(i)}$:

$$\Delta^2 v_{\psi_i f, j(i)} = \psi_i \Delta^2 u = \Delta^2 (\psi_i u) - \sum_{\substack{|\alpha+\beta|=4, \\ |\beta| \leq 3}} n_{\alpha, \beta} D^\alpha \psi_i D^\beta u, \quad (31)$$

where $n_{\alpha, \beta}$ are positive coefficients. From (31) we find in E_j that

$$\Delta^2 (v_{\psi_i f, j(i)} - \psi_i u) = - \sum_{\substack{|\alpha+\beta|=4, \\ |\beta| \leq 3}} n_{\alpha, \beta} D^\alpha \psi_i D^\beta u.$$

Furthermore the function $v_{\psi_i f, j(i)} - \psi_i u$ satisfies zero Dirichlet boundary condition on $\partial E_{j(i)}$. Indeed by construction: $u = \frac{\partial}{\partial \nu} u = 0$ on $\partial E_{j(i)} \cap \text{supp}(\psi_i) \subset \partial E_{j(i)} \cap \partial \Omega$ and $\psi_i = \frac{\partial}{\partial \nu} \psi_i = 0$ for $x \in \partial E_{j(i)} \setminus \text{supp}(\psi_i)$.

Hence we may write for $x \in E_{j(i)}$

$$v_{\psi_i f, j(i)}(x) = \psi_i(x)u(x) - R_i(x), \quad (32)$$

where

$$R_i(x) := \int_{E_{j(i)}} G_{E_{j(i)}}(x, y) (\Delta^2(\psi_i(y)u(y)) - \psi_i(y)\Delta^2 u(y)) dy.$$

On the other hand we get from (32)

$$\begin{aligned} \Delta^2 \tilde{u}_i &= \Delta^2 (\chi_i v_{\psi_i f, j(i)}) = \chi_i \Delta^2 v_{\psi_i f, j(i)} + (\Delta^2 (\chi_i v_{\psi_i f, j(i)}) - \chi_i \Delta^2 v_{\psi_i f, j(i)}) \\ &= \chi_i \psi_i f + (\Delta^2 (\chi_i \psi_i u - \chi_i R_i) - \chi_i \Delta^2 (\psi_i u - R_i)). \end{aligned}$$

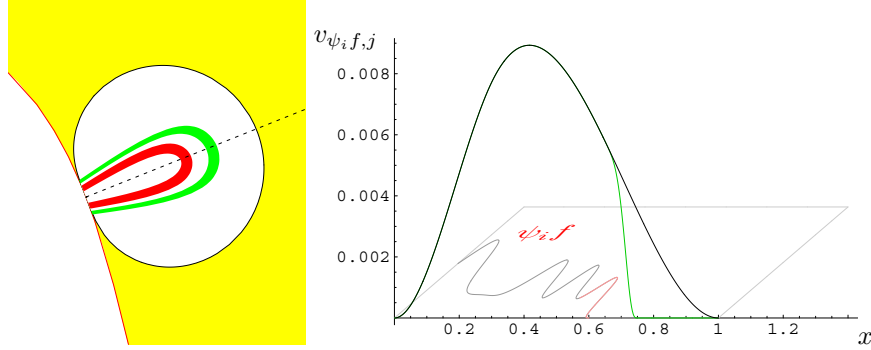


Figure 8: On the left one now also finds in green the set $\{x : \nabla \chi_i(x) \neq 0\}$. On the right in green (lighter) the function $\tilde{u}_i = \chi_i v_{\psi_i f, j(i)}$.

Since $\text{supp}(\psi_i) \subset \{x \in \bar{\Omega} : \chi_i(x) = 1\}$ it holds $\Delta^2(\chi_i \psi_i u) = \chi_i \Delta^2(\psi_i u)$. Hence we get

$$\Delta^2 \tilde{u}_i = \chi_i \psi_i f - \Delta^2(\chi_i R_i) + \chi_i \Delta^2 R_i.$$

Notice that this last relation holds in all of Ω . Hence the function \tilde{u} satisfies in Ω

$$\Delta^2 \tilde{u} = f - \sum_{i \in I} \Delta^2(\chi_i R_i) + \sum_{i \in I} \chi_i \Delta^2 R_i.$$

It follows that $u - \tilde{u}$ satisfies

$$\begin{cases} \Delta^2(u - \tilde{u}) = \sum_{i \in I} \Delta^2(\chi_i R_i) - \sum_{i \in I} \chi_i \Delta^2 R_i & \text{in } \Omega, \\ u - \tilde{u} = 0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial \nu}(u - \tilde{u}) = 0 & \text{on } \partial\Omega. \end{cases} \quad (33)$$

Here we used that $\tilde{u}_i = \frac{\partial}{\partial \nu} \tilde{u}_i = 0$ on $\partial\Omega$ for every $i \in I$.

Writing

$$\begin{aligned} u(x) &= \tilde{u}(x) + \sum_{i \in I} \int_{\Omega} G_{\Omega}(x, y) (\Delta^2(\chi_i R_i) - \chi_i \Delta^2 R_i)(y) dy \\ &= \tilde{u}(x) + \sum_{\substack{i \in I, \\ |\alpha' + \beta'| = 4, \\ |\beta'| \leq 3}} n_{\alpha', \beta'} \int_{\Omega} G_{\Omega}(x, y) D^{\alpha'} \chi_i(y) D^{\beta'} R_i(y) dy \\ &= \tilde{u}(x) + \sum_{\substack{i \in I, |\beta|, |\beta'| \leq 3, \\ |\alpha' + \beta'| = 4, \\ |\alpha + \beta| = 4}} n_{\alpha, \beta, \alpha', \beta'} \int_{\Omega} G_{\Omega}(x, y) D^{\alpha'} \chi_i(y) D^{\beta'} v_{D^{\alpha} \psi_i D^{\beta} u, j(i)}(y) dy, \end{aligned}$$

and using the estimate in (30) we find

$$\begin{aligned}
u(x) &\leq c_2 \|f^+\|_{L^1(B(x, \varepsilon_R) \cap \Omega)} \\
&+ \sum_{\substack{i \in I, |\beta|, |\beta'| \leq 3, \\ |\alpha' + \beta'| = 4, \\ |\alpha + \beta| = 4}} n_{\alpha, \beta, \alpha', \beta'} \left\| \int_{\Omega} G_{\Omega}(\cdot, y) D^{\alpha'} \chi_i(y) D^{\beta'} v_{D^{\alpha} \psi_i D^{\beta} u, j(i)}(y) dy \right\|_{\infty}.
\end{aligned} \tag{34}$$

In the following we will estimate the second term in the right hand side of (34). We fix $i \in I$, $\alpha, \beta, \alpha', \beta' \in \mathbb{N}^2$ with $|\alpha' + \beta'| = |\alpha + \beta| = 4$ and $|\beta'|, |\beta| \leq 3$.

We first notice that it is sufficient to prove (29) for $q > 2$ and near 2. Indeed the result for general $q > 2$ will then follow from the observation that the following inequality holds

$$\|u\|_{W^{-1, q}(\Omega)} \leq |\Omega|^{\frac{1}{q} - \frac{1}{\tilde{q}}} \|u\|_{W^{-1, \tilde{q}}(\Omega)} \text{ for any } \tilde{q} > q > 2.$$

Let fix $q > 2$ with $q - 2$ small. The Sobolev Imbedding Theorem yields that for some $c_3 = c_3(\frac{1}{2-q}, \rho_{\Omega}^{-1}, R_{\Omega})$

$$\begin{aligned}
&\left\| \int_{\Omega} G_{\Omega}(\cdot, y) D^{\alpha'} \chi_i(y) D^{\beta'} v_{D^{\alpha} \psi_i D^{\beta} u, j}(y) dy \right\|_{\infty} \leq \\
&\leq c_3 \left\| \int_{\Omega} G_{\Omega}(\cdot, y) D^{\alpha'} \chi_i(y) D^{\beta'} v_{D^{\alpha} \psi_i D^{\beta} u, j}(y) dy \right\|_{W_0^{1, q}(\Omega)} = \dots
\end{aligned}$$

Here and in the following we write simply j instead of $j(i)$.

We proceed using the regularity result for the ‘‘three-quarter weak solution’’ of problem (1) (see Definition D.12). Indeed by Theorem D.13 the solution operator from $(W^{3, q'}(\Omega) \cap W_0^{2, q'}(\Omega))'$ to the space $W_0^{1, q}(\Omega)$ is an isomorphism. Hence we get for some $c_4 = c_4(\frac{1}{2-q}, M, \rho_{\Omega}^{-1}, R_{\Omega})$

$$\begin{aligned}
\dots &\leq c_4 \left\| D^{\alpha'} \chi_i(\cdot) D^{\beta'} v_{D^{\alpha} \psi_i D^{\beta} u, j}(\cdot) \right\|_{(W^{3, q'}(\Omega) \cap W_0^{2, q'}(\Omega))'} \\
&= c_4 \sup \left\{ \langle D^{\alpha'} \chi_i D^{\beta'} v_{D^{\alpha} \psi_i D^{\beta} u, j}, \varphi \rangle \mid \varphi \in W^{3, q'}(\Omega) \cap W_0^{2, q'}(\Omega) \right. \\
&\quad \left. \text{with } \|\varphi\|_{W^{3, q'}(\Omega)} \leq 1 \right\} = \dots
\end{aligned}$$

Notice that the constant in Theorem D.13 depends on q and q' . However, since we consider q near 2 we can choose a constant that depends only on the distance of q to 2.

Next, we consider a restriction from the space $(W^{3, q'}(\Omega) \cap W_0^{2, q'}(\Omega))'$ to the space $(W^{3, q'}(E_j) \cap W_0^{2, q'}(E_j))'$. One uses that the cut-off function χ_i has support in $(E_j \cap \Omega) \cup (\partial\Omega \cap \partial E_j)^{\circ, \partial\Omega}$. Proceeding formally we take a cut-off function $h_i \in C_c^{\infty}(\Omega \cup (\partial\Omega \cap \partial E_j)^{\circ, \partial\Omega})$ such that:

- i. $\text{supp}(\chi_i) \subset \{x \in \bar{\Omega} : h_i(x) = 1\}$;
- ii. $\text{supp}(h_i) \subset (E_j \cap \Omega) \cup (\partial E_j \cap \partial\Omega)^{\circ, \partial\Omega}$;
- iii. $0 \leq h_i \leq 1$;
- iv. $\|\nabla^\alpha h_i\|_\infty \leq c_\alpha R^{-|\alpha|}$ for every $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq 4$.

Such a cut-off function exists since $\text{supp}(\chi_i) \subset (E_j \cap \Omega) \cup (\partial E_j \cap \partial\Omega)^{\circ, \partial\Omega}$. The function $h_i\varphi$ lies in $W^{3,q'}(E_j) \cap W_0^{2,q'}(E_j)$ for every $\varphi \in W^{3,q'}(\Omega) \cap W_0^{2,q'}(\Omega)$ and moreover it holds

$$\begin{aligned} \langle D^{\alpha'} \chi_i D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}, \varphi \rangle_\Omega &= \langle D^{\alpha'} \chi_i D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}, h_i \varphi \rangle_\Omega \\ &= \langle D^{\alpha'} \chi_i D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}, h_i \varphi \rangle_{E_j}. \end{aligned}$$

Using that there exists a constant c_5 such that $\|h_i \varphi\|_{W^{3,q'}(\Omega)} \leq c_5 R^{-3} \|\varphi\|_{W^{3,q'}(\Omega)}$ we get

$$\begin{aligned} \dots &= c_4 \sup \left\{ \langle D^{\alpha'} \chi_i D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}, h_i \varphi \rangle_{E_j} \mid \varphi \in W^{3,q'}(\Omega) \cap W_0^{2,q'}(\Omega) \right. \\ &\quad \left. \text{with } \|h_i \varphi\|_{W^{3,q'}(\Omega)} \leq c_5 R^{-3} \right\} \\ &\leq c_4 \sup \left\{ \langle D^{\alpha'} \chi_i D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}, \tilde{\varphi} \rangle_{E_j} \mid \tilde{\varphi} \in W^{3,q'}(E_j) \cap W_0^{2,q'}(E_j) \right. \\ &\quad \left. \text{with } \|\tilde{\varphi}\|_{W^{3,q'}(E_j)} \leq c_5 R^{-3} \right\} \\ &\leq c_6 \sup \left\{ \langle D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}, \tilde{\varphi} \rangle_{E_j} \mid \tilde{\varphi} \in W^{3,q'}(E_j) \cap W_0^{2,q'}(E_j) \right. \\ &\quad \left. \text{with } \|\tilde{\varphi}\|_{W^{3,q'}(E_j)} \leq 1 \right\} = \dots \end{aligned}$$

Here $c_6 = c_6(\frac{1}{2-q}, M, \rho_\Omega^{-1}, R_\Omega)$ since R depends on M .

We now proceed by integrating by parts. Since $v_{D^\alpha \psi_i D^\beta u, j}$ and $\tilde{\varphi}$ and their first derivatives are zero on ∂E_j there is no contribution from the boundary. We find

$$\begin{aligned} \dots &= c_6 \sup \left\{ \langle v_{D^\alpha \psi_i D^\beta u, j}, D^{\beta'} \tilde{\varphi} \rangle_{E_j} \mid \tilde{\varphi} \in W^{3,q'}(E_j) \cap W_0^{2,q'}(E_j) \right. \\ &\quad \left. \text{with } \|\tilde{\varphi}\|_{W^{3,q'}(E_j)} \leq 1 \right\} \\ &\leq c_6 \sup \left\{ \langle v_{D^\alpha \psi_i D^\beta u, j}, \varphi \rangle_{E_j} \mid \varphi \in W^{3-|\beta'|, q'}(E_j) \cap W_0^{\min\{2, 3-|\beta'|\}, q'}(E_j), \right. \\ &\quad \left. \|\varphi\|_{W^{3-|\beta'|, q'}(E_j)} \leq 1 \right\} \\ &\leq c_6 \sup \left\{ \langle v_{D^\alpha \psi_i D^\beta u, j}, \varphi \rangle_{E_j} \mid \varphi \in L^{q'}(E_j), \|\varphi\|_{L^{q'}(E_j)} \leq 1 \right\} \\ &= c_6 \|v_{D^\alpha \psi_i D^\beta u, j}\|_{L^q(E_j)} = \dots \end{aligned}$$

Next, we apply the regularity result for weak solution of problem (1) (see Definition D.9). Notice that in order to do that one needs that $\partial E_j \in C^{4,\alpha}$. By the result in Theorem D.10 we get for some $c_7 = c_7(\frac{1}{2-q}, M, \rho_\Omega^{-1}, R_\Omega)$

$$\begin{aligned} \dots &\leq c_7 \|D^\alpha \psi_i D^\beta u\|_{(W^{4,q'}(E_j) \cap W_0^{2,q'}(E_j))'} \\ &= c_7 \sup \left\{ \langle D^\alpha \psi_i D^\beta u, \varphi \rangle_{E_j} \mid \varphi \in W^{4,q'}(E_j) \cap W_0^{2,q'}(E_j), \right. \\ &\quad \left. \|\varphi\|_{W^{4,q'}(E_j)} \leq 1 \right\} \end{aligned}$$

Since we consider q near 2 we can choose the dependance on q of the form $\frac{1}{2-q}$ in the constant that appears in the estimate of Theorem D.10.

We now consider an extension from the space $(W^{4,q'}(E_j) \cap W_0^{2,q'}(E_j))'$ to the space $(W^{4,q'}(\Omega) \cap W_0^{2,q'}(\Omega))'$. Since ψ_i has compact support in $(\Omega \cap E_j) \cup (\partial\Omega \cap \partial E_j)^{\circ, \partial\Omega}$ one has

$$\langle D^\alpha \psi_i D^\beta u, \varphi \rangle_{E_j} = \langle D^\alpha \psi_i D^\beta u, \varphi \rangle_\Omega,$$

which implies

$$\begin{aligned} \dots &= c_7 \sup \left\{ \langle D^\alpha \psi_i D^\beta u, \varphi \rangle_\Omega \mid \varphi \in W^{4,q'}(E_j) \cap W_0^{2,q'}(E_j), \|\varphi\|_{W^{4,q'}(E_j)} \leq 1 \right\} \\ &\leq c_7 \sup \left\{ \langle D^\alpha \psi_i D^\beta u, \varphi \rangle_\Omega \mid \varphi \in W^{4,q'}(\Omega) \cap W_0^{2,q'}(\Omega), \|\varphi\|_{W^{4,q'}(\Omega)} \leq 1 \right\} \\ &\leq c_8 \sup \left\{ \langle D^\beta u, \varphi \rangle_\Omega \mid \varphi \in W^{4,q'}(\Omega) \cap W_0^{2,q'}(\Omega), \|\varphi\|_{W^{4,q'}(\Omega)} \leq 1 \right\} = \dots \end{aligned}$$

Here $c_8 = c_8(\frac{1}{2-q}, M, \rho_\Omega^{-1}, R_\Omega)$.

The last step is an integration by part. We do not have any contribution from the boundary since u and φ and their first derivative are zero on $\partial\Omega$. Hence one finds

$$\begin{aligned} \dots &= c_8 \sup \left\{ \langle u, D^\beta \varphi \rangle_\Omega \mid \varphi \in W^{4,q'}(\Omega) \cap W_0^{2,q'}(\Omega), \|\varphi\|_{W^{4,q'}(\Omega)} \leq 1 \right\} \\ &\leq c_8 \sup \left\{ \langle u, \tilde{\varphi} \rangle_\Omega \mid \tilde{\varphi} \in W^{4-|\beta|,q'}(\Omega) \cap W_0^{\min\{2,4-|\beta|\},q'}(\Omega), \right. \\ &\quad \left. \|\tilde{\varphi}\|_{W^{4-|\beta|,q'}(\Omega)} \leq 1 \right\} \\ &\leq c_8 \sup \left\{ \langle u, \tilde{\varphi} \rangle_\Omega \mid \tilde{\varphi} \in W^{1,q'}(\Omega), \|\tilde{\varphi}\|_{W_0^{1,q'}(\Omega)} \leq 1 \right\} = c_8 \|u\|_{(W_0^{1,q'}(\Omega))'}. \end{aligned}$$

The claim follows for $\varepsilon_R = 4R$. For $\varepsilon \in (0, \varepsilon_R]$ one may repeat the same construction with a refinement of the partition of unity ψ_i , $i \in I$. ■

Remark 4.1.1 *The hypothesis Ω simply connected is required in order to use Theorem D.13. The result can be proved also for general connected domains using a generalization of Theorem D.13.*

4.2 Green function estimates

In this section we prove Theorem 1.1 and we give optimal estimates from below for the Green function of a two-dimensional domain Ω with $\partial\Omega \in C^{16}$. In this section we have to assume more regularity on the boundary of Ω in order to use [8, Th.2.6]. As before, G_Ω denotes the Green function associated to problem (1) on Ω .

We first present some preliminary lemmas.

Lemma 4.2 *Assume that Ω is a bounded domain in \mathbb{R}^2 with $\partial\Omega \in C^{16}$. Then*

$$G_\Omega \in W^{3,p}(\Omega^2) \text{ for any } p \in [1, 2).$$

Proof. In [8] one finds

$$|D^\beta G_\Omega(x, y)| \preceq |x - y|^{-1} \min \left\{ 1, \frac{d(y)}{|x - y|} \right\}^2 \text{ for any } \beta \in \mathbb{N}^2 \text{ with } |\beta| \leq 3. \quad (35)$$

The result follows directly from (35). \blacksquare

Lemma 4.3 *Let Ω be a bounded domain in \mathbb{R}^2 with $\partial\Omega \in C^{16}$. Then for every $\gamma \in (0, 1)$*

$$G_\Omega \in C^{15,\gamma}(\{(x, y) \in \bar{\Omega}^2 : x \neq y\}) \text{ and } G_\Omega \in C^{1,\gamma}(\bar{\Omega}^2) \cap C_0^1(\bar{\Omega}^2).$$

Proof. From general regularity results for elliptic partial differential equations (see [2]) it is well known that $G_\Omega \in C^{15,\gamma}(\{(x, y) \in \bar{\Omega}^2 : x \neq y\})$ for any $\gamma \in (0, 1)$. Indeed, in general, given $l \in \mathbb{N}$, $\beta \in [0, 1)$ and a bounded domain $\mathcal{D} \in C^{l,\beta}$ then the regularity of $G_{\mathcal{D}}$ on $\{(x, y) \in \bar{\mathcal{D}}^2 : x \neq y\}$ is as follows:

$$\begin{aligned} \text{if } \beta = 0 : & \quad G_{\mathcal{D}} \in C^{l-1,\gamma}(\{(x, y) \in \bar{\mathcal{D}}^2 : x \neq y\}) \text{ for any } \gamma \in (0, 1); \\ \text{if } \beta \neq 0 : & \quad G_{\mathcal{D}} \in C^{l,\beta}(\{(x, y) \in \bar{\mathcal{D}}^2 : x \neq y\}). \end{aligned}$$

The result that $G_\Omega \in C^{1,\gamma}(\bar{\Omega}^2)$ follows directly from Lemma 4.2 via the Sobolev imbedding Theorem ([1, Th.4.12 Part 2]). Hence $G_\Omega \in W^{3,p}(\Omega^2) \cap C^{1,\gamma}(\bar{\Omega}^2)$ for $p \in [1, 2)$ and $\gamma \in (0, 1)$. Moreover the function and its first derivatives are zero on $\partial\Omega \times \Omega$ and on $\Omega \times \partial\Omega$. Hence, by continuity and Theorem IX.17 in [4] it follows that $G_\Omega \in C_0^1(\bar{\Omega}^2)$ (and also $G_\Omega \in W_0^{2,p}(\Omega^2)$ for $p \in [1, 2)$). \blacksquare

Proof of Theorem 1.1. Following the construction in Theorem 4.1, see (33), one may write the solution of problem (1) as

$$\begin{aligned} u(x) &= \tilde{u}(x) + \int_{\Omega} G_\Omega(x, z) \sum_{i \in I} (\Delta^2(\chi_i(z)R_i(z)) - \chi_i(z)\Delta^2 R_i(z)) dz \\ &= \sum_{i \in I} \chi_i(x) \int_{E_{j(i)}} G_{E_{j(i)}}(x, z) \psi_i(z) f(z) dz + \\ &\quad + \int_{\Omega} G_\Omega(x, z) \sum_{i \in I} (\Delta^2(\chi_i(z)R_i(z)) - \chi_i(z)\Delta^2 R_i(z)) dz, \end{aligned}$$

where

$$R_i(z) = \int_{E_{j(i)}} G_{E_{j(i)}}(z, z') (\Delta^2(\psi_i(z')u(z')) - \psi_i(z')\Delta^2 u(z')) dz',$$

and $j(i)$ denotes the $j \in J$ such that $\text{supp}(\psi_i) \subset E_j$. Considering formally $f(x) = \delta_y(x)$ we get

$$\begin{aligned} G_\Omega(x, y) &= \sum_{i \in I} \chi_i(x) G_{E_{j(i)}}(x, y) \psi_i(y) + \\ &+ \int_\Omega G_\Omega(x, z) \sum_{i \in I} (\Delta^2(\chi_i(z)R_i(z, y)) - \chi_i(z)\Delta^2 R_i(z, y)) dz, \end{aligned}$$

where

$$R_i(z, y) = \int_{E_{j(i)}} G_{E_{j(i)}}(z, z') (\Delta^2(\psi_i(z')G_\Omega(z', y)) - \psi_i(z')\Delta^2 G_\Omega(z', y)) dz'.$$

We define

$$G_\Omega^{\text{sing}}(x, y) := \sum_{i \in I} \chi_i(x) G_{E_j}(x, y) \psi_i(y), \quad (36)$$

$$G_\Omega^{\text{reg}}(x, y) := G_\Omega(x, y) - G_\Omega^{\text{sing}}(x, y). \quad (37)$$

From the definition it follows that $G_\Omega^{\text{reg}} \in C^{15, \gamma}(\bar{\Omega}^2)$ for any $\gamma \in (0, 1)$. Indeed, writing explicitly R_i and looking at the support of the term inside the integral, we find

$$\begin{aligned} G_\Omega^{\text{reg}}(x, y) &= \\ &= \sum_{i \in I} \sum_{\substack{|\alpha+\beta|=4, \\ |\alpha'+\beta'|=4, \\ |\beta|, |\beta'| \leq 3}} n_{\alpha, \beta, \alpha', \beta'} \int_{B_i} G_\Omega(x, z) D^\alpha \chi_i(z) \cdot \\ &\quad \cdot \left(D^\beta \int_{A_i} G_{E_{j(i)}}(z, z') D^{\alpha'} \psi_i(z') D^{\beta'} G_\Omega(z', y) dz' \right) dz, \end{aligned} \quad (38)$$

with $n_{\alpha, \beta, \alpha', \beta'}$ some positive coefficients and

$$B_i = \{z \in \Omega : \nabla \chi_i(z) \neq 0\} \text{ and } A_i = \{z \in \Omega : \nabla \psi_i(z) \neq 0\}. \quad (39)$$

Since $A_i \cap B_i = \emptyset$ one always has $z \neq z'$ in (38). Hence $G_{E_j}(z, z') \in C^\infty(B_j \times A_j)$. Since the term $D^{\alpha'} \psi_i(z') D^{\beta'} G_\Omega(z', y)$ is integrable it follows that G_Ω^{reg} is as regular as we want in the interior. The regularity up to the boundary is given by the fact that $\partial\Omega \in C^{16}$.

The positivity of G_Ω^{sing} follows from the positivity of G_{E_j} . Furthermore by Lemma 4.3, the definition of G_Ω^{sing} and since $G_\Omega^{\text{reg}} \in C^{15, \gamma}(\bar{\Omega}^2)$ for any $\gamma \in (0, 1)$ holds, it follows that $G_\Omega^{\text{sing}} \in C^{1, \gamma}(\bar{\Omega}^2) \cap C_0^1(\bar{\Omega}^2)$ and moreover that $G_\Omega^{\text{sing}} \in C^{15, \gamma}(\{(x, y) \in \bar{\Omega}^2 : x \neq y\})$ for any $\gamma \in (0, 1)$. Notice that by the boundary condition satisfied by G_Ω and G_Ω^{sing} we also have that $G_\Omega^{\text{reg}} \in C_0^1(\bar{\Omega}^2)$. ■

Remark 4.3.1 The functions G_Ω^{reg} and G_Ω^{sing} defined in the proof of Theorem 1.1 are not yet symmetric. In order to get symmetric functions one may consider $G_{\Omega, \text{new}}^{\text{sym}}(x, y) := \frac{1}{2}G_\Omega^{\text{sym}}(x, y) + \frac{1}{2}G_\Omega^{\text{sym}}(y, x)$.

Optimal estimates from above for the Green function as well as estimates for the absolute value are known. We refer to [18], [13] and [8]. We will next prove optimal estimates from below for G_Ω .

First we prove the following lemma.

Lemma 4.4 Let Ω be a bounded domain in \mathbb{R}^2 with $\partial\Omega \in C^{16}$. Then G_Ω satisfies

$$\|\nabla G_\Omega(\cdot, y)\|_{L^p(\Omega)} \leq c'_{p, \Omega} d(y)^2 \text{ for every } y \in \Omega \text{ and } p \in [1, 2).$$

Proof. Via [8, Th.2.6] one finds

$$\begin{aligned} \|\nabla G_\Omega(\cdot, y)\|_{L^p(\Omega)}^p &\leq c_\Omega \int_\Omega d(y)^p \min\left\{1, \frac{d(y)}{|x-y|}\right\}^p \min\left\{1, \frac{d(x)}{|x-y|}\right\}^p dx \\ &\leq c_\Omega d(y)^{2p} \int_\Omega \frac{1}{|x-y|^p} dx \leq c'_{p, \Omega} d(y)^{2p}, \end{aligned}$$

for $p \in [1, 2)$. ■

Theorem 4.5 Let Ω be a bounded domain in \mathbb{R}^2 with $\partial\Omega \in C^{16}$. Then there exists $c_\Omega > 0$ such that G_Ω satisfies:

$$G_\Omega(x, y) \geq -c_\Omega d(x)^2 d(y)^2 \text{ for every } x, y \in \Omega.$$

Proof. Since $G_\Omega(x, y) = G_\Omega^{\text{sing}}(x, y) + G_\Omega^{\text{reg}}(x, y)$, with G_Ω^{sing} and G_Ω^{reg} defined in (36) and (37) respectively, and G_Ω^{sing} is positive it holds

$$G_\Omega(x, y) \geq -|G_\Omega^{\text{reg}}(x, y)| \text{ for every } x, y \in \Omega.$$

Hence in order to prove the result it is sufficient to get an estimate of the absolute value of G_Ω^{reg} .

We first study the $W^{4,p}$ -norm of $G_\Omega^{\text{reg}}(\cdot, y)$ for $p \in (1, \infty)$.

Let A_i and B_i as defined in (39). From (38) and elliptic regularity theory (see Theorem D.5) it follows that

$$\|G_\Omega^{\text{reg}}(\cdot, y)\|_{W^{4,p}(\Omega)} \leq c \sum_{i \in I} \sum_{\substack{|\alpha+\beta|=4, \\ |\beta| \leq 3}} n_{\alpha, \beta} \|D^\alpha \chi_i(\cdot) D^\beta R_i(\cdot, y)\|_{L^p(B_i)}.$$

We study separately the term $\|D^\alpha \chi_i(\cdot) D^\beta R_i(\cdot, y)\|_{L^p(B_i)}$. One has

$$\begin{aligned} &\|D^\alpha \chi_i(\cdot) D^\beta R_i(\cdot, y)\|_{L^p(B_i)} \\ &\leq c_\Omega \sum_{\substack{|\alpha'+\beta'|=4, \\ |\beta'| \leq 3}} n_{\alpha', \beta'} \left\| D^\beta \int_{A_i} G_{E_j}(\cdot, z') D^{\alpha'} \psi_i(z') D^{\beta'} G_\Omega(z', y) dz' \right\|_{L^p(B_i)}. \end{aligned}$$

We first observe that G_{E_j} is non singular in $B_i \times A_i$. Indeed since $\bar{A}_i \cap \bar{B}_i = \emptyset$, the function $G_{E_j}(z, z')$ is in $C^\infty(B_j \times A_j)$ and all its derivatives are bounded by a constant depending only on Ω .

The next step consists in an integration by part. There are no contribution from the boundary since in $\partial A_i \cap \Omega$ the function ψ_i and its derivatives are zero, while in $\partial A_i \cap \partial \Omega$ both G_{E_j} and G_Ω and their first derivatives are zero.

Let $\beta'' \in \mathbb{N}^2$ denote a multi-index such that $\beta'' < \beta'$, $|\beta''| = |\beta'| - 1$. We obtain

$$\begin{aligned} & \|D^\alpha \chi_i(\cdot) D^\beta R_i(\cdot, y)\|_{L^p(B_i)} \\ & \leq c_\Omega \sum_{\substack{|\alpha' + \beta'| = 4, \\ |\beta'| \leq 3}} n_{\alpha', \beta'} \left\| D^\beta \int_{A_i} D^{\beta''} \left(G_{E_j}(\cdot, z') D^{\alpha'} \psi_i(z') \right) D^{\beta' - \beta''} G_\Omega(z', y) dz' \right\|_{L^p(B_i)} \\ & \leq c_{\Omega, p} \sum_{\substack{|\alpha' + \beta'| = 4, \\ |\beta'| = 2, 3}} n_{\alpha', \beta'} \int_\Omega \left| D^{\beta' - \beta''} G_\Omega(z', y) \right| dz' \leq c'_{\Omega, p} d(y)^2. \end{aligned}$$

In the last step we used Lemma 4.4.

Since $G_\Omega^{\text{reg}}(x, y) \in W^{4, p}(\Omega) \cap W_0^{2, p}(\Omega)$ for any $p \in (1, \infty)$, from [6, Lem.5] it follows that

$$\frac{|G_\Omega^{\text{reg}}(x, y)|}{d(x)^2} \leq c_\Omega \|G_\Omega^{\text{reg}}(\cdot, y)\|_{W^{4, p}(\Omega)}.$$

Hence we obtain

$$|G_\Omega^{\text{reg}}(x, y)| \leq c''_\Omega d(y)^2 d(x)^2.$$

The claim follows. ■

Remark 4.5.1 In [6, Lemma 5] the authors consider a bounded domain Ω with $\partial \Omega$ smooth. One can consider a weaker assumption on the boundary. Indeed, in order to apply the Rellich-Kondrachov Theorem, [1, Th.6.3], it is sufficient that Ω is bounded and satisfies the strong Lipschitz condition, [1, Def.4.9]. Notice that if Ω satisfies the uniform C^l regularity condition with $l \geq 2$ then Ω satisfies also the strong Lipschitz condition.

Appendices

A Improved ε -closeness to the disk

In [12, Prop.2.6] the authors show that $C^{2m, \gamma}$ -closeness to the disk (Definition 2.3) implies the existence of a conformal map that satisfies the C^{2m-1} -closeness condition. This result can be improved. Indeed from $C^{2m, \gamma}$ -closeness to the disk one gets the existence of a conformal map that also satisfies the $C^{2m, \gamma'}$ -closeness condition for $\gamma' \in (0, \gamma)$. We state the result in the following proposition. The proof follows the main steps of the one in [12] except in the last part.

Proposition A.1 *Let $\gamma \in (0, 1)$ and $m \in \mathbb{N}$ be given. Then there exist $\varepsilon_0 = \varepsilon_0(m) > 0$ and $c > 0$ such that for $\varepsilon \in [0, \varepsilon_0]$ and $\gamma' \in (0, 1)$ with $\gamma' < \gamma$ we have the following.*

If Ω^ is ε -close in $C^{2m, \gamma}$ -sense to the disk B , then there exists a biholomorphic mapping $\varphi : \bar{B} \rightarrow \bar{\Omega}^*$, with $\varphi \in C^{2m, \gamma'}(\bar{B})$ and $\varphi^{-1} \in C^{2m, \gamma'}(\bar{\Omega}^*)$, such that*

$$\|\varphi - Id\|_{C^{2m, \gamma'}(\bar{B})} \leq c\varepsilon^{\gamma - \gamma'}.$$

Proof. Let $f : \bar{B} \rightarrow \bar{\Omega}^*$ be a mapping such that $\|f - Id\|_{C^{2m, \gamma}(\bar{B})} \leq \varepsilon$ with $\varepsilon \leq \varepsilon_0$ small enough. According to [5] (see also [22, Sec.4.2]), a holomorphic mapping $\varphi^{-1} : \Omega^* \rightarrow B$, that has the desired qualitative properties, may be constructed in the following way. First set

$$\omega(x) := 2\pi G(x, 0).$$

Here G is the Green function for $-\Delta$ in Ω^* under homogeneous Dirichlet boundary condition. Next define the conjugate harmonic function

$$\omega^*(x) := \int_{1/2}^x \left(-\frac{\partial}{\partial \xi_2} \omega(\xi) d\xi_1 + \frac{\partial}{\partial \xi_1} \omega(\xi) d\xi_2 \right),$$

where the integral is taken with respect to any curve from $\frac{1}{2}$ to x in $\Omega^* \setminus \{0\}$. The function ω^* is well defined up to multiples of 2π . One finds that φ^{-1} is uniquely defined by

$$\varphi^{-1}(x) := \exp(-\omega(x) - i\omega^*(x)) \text{ for } x \in \bar{\Omega}^*,$$

where \mathbb{R}^2 and \mathbb{C} are identified. The function φ^{-1} maps 0 onto 0 and the point $\frac{1}{2}$ somewhere into the positive real half-axis. Moreover, for $x \in \partial\Omega^*$ we find that $|\varphi^{-1}(x)| = |\exp(-i\omega^*(x))| = 1$ and hence $\varphi^{-1}(\partial\Omega^*) \subset \partial B$. For $x \in \Omega^* \setminus \{0\}$ we have $\omega(x) > 0$ and hence $|\varphi^{-1}(x)| < 1$ implying $\varphi^{-1}(\Omega^*) \subset B$.

Setting $r(x) = 2\pi G(x, 0) + \log|x|$ one has that r satisfies

$$\begin{cases} -\Delta r = 0 & \text{in } \Omega^*, \\ r(x) = \theta(x) := \log|x| & \text{on } \partial\Omega^*. \end{cases} \quad (40)$$

In order to have that $\|\varphi^{-1} - Id\|_{C^{2m, \gamma'}(\bar{\Omega}^*)} = \mathcal{O}(\varepsilon^{\gamma - \gamma'})$ (and consequently $\|\varphi - Id\|_{C^{2m, \gamma'}(\bar{B})} = \mathcal{O}(\varepsilon^{\gamma - \gamma'})$) it would be sufficient that

$$\|r\|_{C^{2m, \gamma'}(\bar{\Omega}^*)} = \mathcal{O}(\varepsilon^{\gamma - \gamma'}), \quad (41)$$

since

$$\varphi^{-1}(x) = x \exp(-r(x) - ir^*(x)) \text{ for } x \in \bar{\Omega}^*,$$

again identifying \mathbb{R}^2 and \mathbb{C} . The estimate in (41) follows from the extension of the boundary data $\theta|_{\partial\Omega^*}$ to some $\hat{\theta}$ on $\bar{\Omega}^*$ with

$$\|\hat{\theta}\|_{C^{2m, \gamma'}(\bar{\Omega}^*)} = \mathcal{O}(\varepsilon^{\gamma - \gamma'}). \quad (42)$$

Indeed, the estimate for $\|r\|_{C^0(\bar{\Omega}^*)}$ is immediate by the maximum principle applied to (40). Furthermore, by means of elliptic estimates for second order equations (see [2, Th.7.3] and [11, Chap.6.3-6.4]), we find $\|r\|_{C^{2m,\gamma'}(\bar{\Omega}^*)} = \mathcal{O}(\varepsilon^{\gamma-\gamma'})$. Note that due to the closeness of Ω^* to B in $C^{2m,\gamma}$ -sense, according to Definition 2.3, the constants in these estimates may be chosen independently of Ω^* .

It remains to show the existence of some $\hat{\theta}$ that satisfies (42). This is done as follows. Since Ω^* is ε -close to B in $C^{2m,\gamma}$ -sense one may show that $(\theta \circ f)|_{\partial B}$ can be extended to θ_f on \bar{B} with $\|\theta_f\|_{C^{2m,\gamma'}(\bar{B})} = \mathcal{O}(\varepsilon^{\gamma-\gamma'})$, provided $\|(\theta \circ f)|_{\partial B}\|_{C^{2m,\gamma'}(\partial B)} = \mathcal{O}(\varepsilon^{\gamma-\gamma'})$. That means we only have to estimate the tangential derivatives of $(\theta \circ f)|_{\partial B}$.

Set $\vartheta(t) := \theta(f(\cos(t), \sin(t)))$. We are done, if we have shown that

$$\max_{j=0,\dots,2m} \max_{t \in [0,2\pi]} \left| \left(\frac{d}{dt} \right)^j \vartheta \right| = \mathcal{O}(\varepsilon^\gamma), \quad (43)$$

$$\left[\left(\frac{d}{dt} \right)^{2m} \vartheta \right]_{\gamma'} = \mathcal{O}(\varepsilon^{\gamma-\gamma'}). \quad (44)$$

Notice that (43) was already proved in [12]. The improvement here is that also (44) holds.

We observe that $\vartheta(t) = \mathcal{O}(\varepsilon)$ since it holds that

$$\log |f(\cos(t), \sin(t))| = \log(1 + \mathcal{O}(\varepsilon)) = \mathcal{O}(\varepsilon).$$

Let us denote $\tilde{f}(t) := f(\cos(t), \sin(t))$. Then $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) : \mathbb{R} \rightarrow \mathbb{R}^2$ and

$$\begin{aligned} \left(\frac{d}{dt} \right)^j \vartheta &= \left(\frac{d}{dt} \right)^j (\theta \circ \tilde{f}) \\ &= \sum_{\substack{|\alpha|=1, \\ \alpha \in \mathbb{N}^2}}^j \left((D^\alpha \theta) \circ \tilde{f} \right) \left(\sum_{\substack{p_1 + \dots + p_{|\alpha|} = j \\ 1 \leq p_i}} \sum_{\alpha} d_{j,\alpha,\vec{p}} \prod_{l=1}^{|\alpha|} \tilde{f}_{\beta_l}^{(p_l)} \right), \end{aligned}$$

with some suitable coefficients $d_{j,\alpha,\vec{p}}$ and with $\beta_l = 1$ if $1 \leq l \leq \alpha_1$ and $\beta_l = 2$ otherwise.

We want to compare this with the corresponding expression with f replaced by Id . Writing $\tilde{f}_0(t) = Id \circ (\cos(t), \sin(t))$ we find

$$\begin{aligned} \left(\frac{d}{dt} \right)^j \vartheta &= \sum_{\substack{|\alpha|=1, \\ \alpha \in \mathbb{N}^2}}^j \left(((D^\alpha \theta) \circ \tilde{f} - (D^\alpha \theta) \circ \tilde{f}_0) + (D^\alpha \theta) \circ \tilde{f}_0 \right) \\ &\quad \times \left(\sum_{\substack{p_1 + \dots + p_{|\alpha|} = j \\ 1 \leq p_i}} \sum_{\alpha} d_{j,\alpha,\vec{p}} \prod_{l=1}^{|\alpha|} \left((\tilde{f}_{\beta_l}^{(p_l)} - \tilde{f}_{0,\beta_l}^{(p_l)}) + \tilde{f}_{0,\beta_l}^{(p_l)} \right) \right). \end{aligned}$$

Since $\theta(\tilde{f}_0(t)) = \log |(\cos(t), \sin(t))| \equiv 0$, all expressions containing \tilde{f}_0 only (and not a difference), sum up to zero. In the remaining sum, every term contains at least one factor of the form

$$(D^\alpha \theta) \circ \tilde{f} - (D^\alpha \theta) \circ \tilde{f}_0 \quad \text{or} \quad \tilde{f}_{\beta_l}^{(p_l)} - \tilde{f}_{0, \beta_l}^{(p_l)},$$

with $|\alpha|, p_l \in \{1, \dots, 2m\}$. For ε small, each of this factors is at most $\mathcal{O}(\varepsilon^\gamma)$. Choosing ε_0 sufficiently small, the other factors remain uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0)$. This shows (43). In order to verify (44) we remark that $\left[\tilde{f}_{\beta_l}^{(p_l)} - \tilde{f}_{0, \beta_l}^{(p_l)} \right]_\gamma = \mathcal{O}(\varepsilon)$ for $p_l \in \{1, \dots, 2m\}$ by the definition of ε -closeness. It remains to study the term $\left[(D^\alpha \theta) \circ \tilde{f} - (D^\alpha \theta) \circ \tilde{f}_0 \right]_{\gamma'}$ for $|\alpha| \in \{1, \dots, 2m\}$.

One finds

$$\begin{aligned} & \left[(D^\alpha \theta) \circ \tilde{f} - (D^\alpha \theta) \circ \tilde{f}_0 \right]_{\gamma'} \\ = & \sup_{\substack{t, s \in [0, 2\pi], \\ |t-s| > \varepsilon}} \frac{\left| (D^\alpha \theta) \circ \tilde{f}(t) - (D^\alpha \theta) \circ \tilde{f}_0(t) - (D^\alpha \theta) \circ \tilde{f}(s) + (D^\alpha \theta) \circ \tilde{f}_0(s) \right|}{|t-s|^{\gamma'}} \\ + & \sup_{\substack{t, s \in [0, 2\pi], \\ |t-s| < \varepsilon}} \frac{\left| (D^\alpha \theta) \circ \tilde{f}(t) - (D^\alpha \theta) \circ \tilde{f}_0(t) - (D^\alpha \theta) \circ \tilde{f}(s) + (D^\alpha \theta) \circ \tilde{f}_0(s) \right|}{|t-s|^{\gamma'}} = \dots \end{aligned}$$

Since $D^\alpha \theta \in C^\gamma$ for $|\alpha| \leq 2m$, $\left\| \tilde{f} - \tilde{f}_0 \right\|_{C^1(0, 2\pi)} = \mathcal{O}(\varepsilon)$ and $\tilde{f}, \tilde{f}_0 \in C^1[0, 2\pi]$, we get

$$\begin{aligned} \dots & \leq c \sup_{\substack{t, s \in [0, 2\pi], \\ |t-s| > \varepsilon}} \frac{\left| \tilde{f}(t) - \tilde{f}_0(t) \right|^\gamma + \left| \tilde{f}(s) - \tilde{f}_0(s) \right|^\gamma}{|t-s|^{\gamma'}} \\ & + c \sup_{\substack{t, s \in [0, 2\pi], \\ |t-s| < \varepsilon}} \frac{\left| \tilde{f}(t) - \tilde{f}(s) \right|^\gamma + \left| \tilde{f}_0(t) - \tilde{f}_0(s) \right|^\gamma}{|t-s|^{\gamma'}} \\ & \leq 2c \sup_{\substack{t, s \in [0, 2\pi], \\ |t-s| > \varepsilon}} \frac{\varepsilon^\gamma}{|t-s|^{\gamma'}} + 2c' \sup_{\substack{t, s \in [0, 2\pi], \\ |t-s| < \varepsilon}} |t-s|^{\gamma-\gamma'} \leq \mathcal{O}(\varepsilon^{\gamma-\gamma'}). \end{aligned}$$

■

B Previously known estimates for G_{Ω_a}

For easy reference we recall here some results from [7] and [8].

Proposition B.1 *For every limaçon Ω_a with $a \in [0, \bar{a}]$ the following two-sided estimate holds for $(x, y) \in \Omega_a^2$:*

$$G_{\Omega_a}(x, y) \sim d_{\Omega_a}(x)d_{\Omega_a}(y) \min \left\{ 1, \frac{d_{\Omega_a}(x)d_{\Omega_a}(y)}{|x-y|^2} \right\}.$$

Proposition B.2 *Let $k \in \mathbb{N}^2$ with $1 \leq |k| \leq 3$. For every limaçon Ω_a with $a \in [0, \bar{a}]$ the following estimates hold for $x, y \in \Omega_a$:*

$$\begin{aligned} \text{when } |k| = 3 & : |D_x^k G_{\Omega_a}(x, y)| \preceq |x-y|^{-1} \min \left\{ 1, \frac{d_{\Omega_a}(y)}{|x-y|} \right\}^2, \\ \text{when } |k| = 2 & : |D_x^k G_{\Omega_a}(x, y)| \preceq \log \left(1 + \frac{d_{\Omega_a}(y)^2}{|x-y|^2} \right) \\ & \sim \log \left(2 + \frac{d_{\Omega_a}(y)}{|x-y|} \right) \min \left\{ 1, \frac{d_{\Omega_a}(y)}{|x-y|} \right\}^2, \\ \text{when } |k| = 1 & : |D_x^k G_{\Omega_a}(x, y)| \preceq d_{\Omega_a}(y) \min \left\{ 1, \frac{d_{\Omega_a}(x)d_{\Omega_a}(y)}{|x-y|^2} \right\}. \end{aligned}$$

Remark B.2.1 *We refer to [8, Th.2.6] for estimates of the derivatives of the Green function associated to polyharmonic Dirichlet boundary value problems in domains $\Omega \subset \mathbb{R}^n$ with $n \geq 2$. There it is assumed that Ω is bounded and that $\partial\Omega \in C^r$ with $r > 4m + 2$.*

C Technical lemmas

In this section we give the proof of some results needed in the proof of Theorem 3.2. We assume the same hypothesis and we use the same notation as in the proof of this theorem. In particular, we recall that the domain Ω satisfies the uniform $C^{4,\alpha}$ regularity condition with constant M .

C.1 The mapping is close to identity

C.1.1 The effect of the scaling: proof of Proposition 3.4

In the following lemma we give the effect on the norms of the scaling defined in (17).

Lemma C.1 *Let Ω be a subset of \mathbb{R}^n and let $f : \bar{\Omega} \rightarrow \bar{\Omega}'$ be a $C^{2,\gamma}$ -function. Let f^R be the f scaled as in (17). Then it holds*

$$\begin{aligned} \|f^R - Id\|_{C^{2,\gamma}(R^{-1}\bar{\Omega})} &= \frac{1}{R} \|f - Id\|_{C^0(\bar{\Omega})} + \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} (f - Id) \right\|_{C^0(\bar{\Omega})} \\ &+ R \sum_{i,j=1}^n \left\| \frac{\partial^2}{\partial x_i \partial x_j} f \right\|_{C^0(\bar{\Omega})} + R^{1+\gamma} \sum_{i,j=1}^n \left[\frac{\partial^2}{\partial x_i \partial x_j} f \right]_{C^\gamma(\bar{\Omega})} \end{aligned} \quad (45)$$

The proof is obvious and will be skipped.

Proof of Proposition 3.4. We estimate separately the terms in the right-hand side of (45) for $f = f_{a,R}$ and $\Omega = \Omega_{a,R}$.

1. Since $-y \leq R$ and $k_{a,R} - g_\delta \equiv 0$ for $|x| \in [Rx_a^*, Rx_a]$ we find

$$\|f_{a,R} - Id\|_{C^0(\bar{\Omega}_{a,R})} \leq 4R \left\| \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]}.$$

2. We also have

$$\begin{aligned} & \sum_{i=1}^2 \left\| \frac{\partial}{\partial x_i} (f_{a,R} - Id) \right\|_{C^0(\bar{\Omega}_{a,R})} \\ & \leq \left\| \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} + 4R \left\| \frac{\partial}{\partial x} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]}. \end{aligned}$$

3. From the definition of the function $f_{a,R}$ in (21) we get

$$\begin{aligned} & \sum_{i,j=1}^2 \left\| \frac{\partial^2}{\partial x_i \partial x_j} f_{a,R} \right\|_{C^0(\bar{\Omega}_{a,R})} \\ & \leq \left\| \frac{\partial}{\partial x} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} + 4R \left\| \frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]}. \end{aligned}$$

4. One finds

$$\begin{aligned} & \sum_{i,j=1}^2 \left[\frac{\partial^2}{\partial x_i \partial x_j} f_{a,R} \right]_{C^\gamma(\bar{\Omega}_{a,R})} \\ & = \left[(x, y) \mapsto (y - 3R) \frac{\partial^2}{\partial x^2} \frac{k_{a,R}(x) - g_\delta(x)}{3R - k_{a,R}(x)} \right]_{C^\gamma(\bar{\Omega}_{a,R})} + \\ & \quad + \left[\frac{\partial}{\partial x} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} = \dots \end{aligned} \tag{46}$$

Since it holds that

$$[(x, y) \mapsto f(x)g(y)]_{C^\alpha[a,b]^2} \leq \|f\|_{C^0[a,b]} \|g\|_{C^\alpha[a,b]} + \|g\|_{C^0[a,b]} \|f\|_{C^\alpha[a,b]}$$

one gets from (46) that

$$\begin{aligned} \dots & \leq 3R^{1-\gamma} \left\| \frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \\ & \quad + 4R \left[\frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} + 2R^{1-\gamma} \left\| \frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]}, \end{aligned}$$

and the claim follows. \blacksquare

C.1.2 Proof of Proposition 3.5

We divide the rather technical proof of Proposition 3.5 in several lemmas. Using the result of Proposition 3.4, to bound $\|f_{a,R}^R - Id\|_{C^{2,\gamma}(\bar{\Omega}_{a,1})}$ it is sufficient to get the estimates of the terms in the right hand side of (26) separately. We will do so in the next lemmas.

In the following $C_i = C_i(M) > 0$, for $i = 1, \dots, 9$. The constants N_i , $i = 1, 2$ are defined in (23) and (24).

Lemma C.2 For $k_{a,R}$ and g_δ respectively as in (5) and (20) it holds that

$$\left\| \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq C_1 \delta R.$$

Proof. By the definition of g_δ in (20), and (25) one has

$$\begin{aligned} & \|k_{a,R} - g_\delta\|_{C^0[-Rx_a^*, Rx_a^*]} \leq \|k_{a,R} - g\|_{C^0[-\delta, \delta]} + \\ & + \sum_{\sigma=\pm} \left\| \varphi_{a,R}(\cdot) \sum_{i=0}^2 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_{\sigma\delta} (\cdot - \sigma\delta)^i \right\|_{C^0[\sigma\delta, \sigma Rx_a^*]} + \\ & + \sum_{\sigma=\pm} \left\| \psi_{a,\delta}(\cdot) \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_{\sigma\delta} (\cdot - \sigma\delta)^i \right\|_{C^0[\sigma\delta, \sigma 2\delta]} \\ & \leq N_1 \delta^3 + 2 \sum_{i=0}^2 \frac{1}{i!} N_1 \delta^{3-i} R^i + \frac{2}{3!} N_1 \delta^3 + \frac{2}{4!} N_2 \delta^4 \leq C_1 \delta R^2. \end{aligned}$$

The claim follows since $|k_{a,R}| \leq 2R$. ■

Lemma C.3 Let $k_{a,R}$ and g_δ be given respectively as in (5) and (20). Then it holds

$$\left\| \frac{\partial}{\partial x} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq C_2 \delta.$$

Proof. Using Lemma C.2 and (7) one finds directly

$$\left\| \frac{\partial}{\partial x} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq \frac{1}{R} \left\| \frac{\partial}{\partial x} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} + C_1 \delta b_1 \leq \dots$$

By the definition of g_δ and the choice of the cut-off functions $\varphi_{a,R}$ and $\psi_{a,\delta}$ we get

$$\begin{aligned} \dots & \leq \frac{N_1}{R} \delta^2 + \frac{2}{R} \sum_{i=1}^2 \frac{1}{(i-1)!} N_1 \delta^{3-i} R^{i-1} + \frac{2}{R} \frac{D_{1,0}}{R} \sum_{i=0}^2 \frac{1}{i!} N_1 \delta^{3-i} R^i \\ & + \frac{1}{2!} \frac{2}{R} N_1 \delta^2 + \frac{1}{3!} \frac{2}{R} \frac{D'_{1,0}}{\delta} N_1 \delta^3 + \frac{1}{3!} \frac{2}{R} N_2 \delta^3 + \frac{1}{4!} \frac{2}{R} \frac{D'_{1,0}}{\delta} N_2 \delta^4 + C_1 b_1 \delta \leq C_2 \delta. \end{aligned}$$

Here we used (25) and that $\delta < R$ and $\delta < 1$. ■

Lemma C.4 For $k_{a,R}$ and g_δ respectively as in (5) and (20) it holds that

$$\left\| \frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq C_3 \frac{\delta}{R}.$$

Proof. Since $\left(\frac{\alpha}{\beta}\right)'' = \frac{1}{\beta}\alpha'' - 2\frac{\beta'}{\beta}\left(\frac{\alpha}{\beta}\right)' - \frac{\beta''}{\beta}\frac{\alpha}{\beta}$, using Lemmas C.2 and C.3 and (7) one finds

$$\begin{aligned} \left\| \frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} &\leq \frac{1}{R} \left\| \frac{\partial^2}{\partial x^2} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} \\ &\quad + \frac{2}{R} b_1 C_2 \delta + \frac{1}{R} \frac{b_2}{R} C_1 \delta R \leq \dots \end{aligned}$$

By the definition of g_δ in (20) one gets

$$\begin{aligned} \dots &\leq \frac{1}{R} N_1 \delta + \frac{2}{R} N_1 \delta + \frac{4}{R} \frac{D_{1,0}}{R} \sum_{i=1}^2 \frac{1}{(i-1)!} N_1 \delta^{3-i} R^{i-1} + \frac{2}{R} \frac{D_{2,0}}{R^2} \sum_{i=0}^2 \frac{1}{i!} N_1 \delta^{3-i} R^i \\ &\quad + \frac{2}{R} N_1 \delta + \frac{2}{R} \frac{1}{2} 2 \frac{D'_{1,0}}{\delta} N_1 \delta^2 + \frac{2}{R} \frac{1}{3!} N_1 \delta^3 \frac{D'_{2,0}}{\delta^2} + \frac{2}{R} \frac{1}{2} N_2 \delta^2 \\ &\quad + \frac{2}{R} \frac{2}{3!} \frac{D'_{1,0}}{\delta} N_2 \delta^3 + \frac{2}{R} \frac{1}{4!} \frac{D'_{2,0}}{\delta^2} N_2 \delta^4 + \frac{1}{R} (2b_1 C_2 + b_2 C_1) \delta \leq C_3 \frac{\delta}{R}. \end{aligned}$$

The constant C_3 depends on Ω through N_1 and N_2 . ■

Remark C.4.1 Notice that the proof also implies that

$$\left\| \frac{\partial^2}{\partial x^2} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq C_4 \delta.$$

Lemma C.5 For $k_{a,R}$ and g_δ respectively as in (5) and (20) it holds that

$$\left[\frac{\partial^2}{\partial x^2} (k_{a,R} - g_\delta) \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \leq C_5 \delta^{1-\gamma}.$$

Proof. Writing explicitly the function g_δ yields

$$\begin{aligned} \left[\frac{\partial^2}{\partial x^2} (k_{a,R} - g_\delta) \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} &\leq 2N_1 \delta^{1-\gamma} \\ &\quad + 2 \left[\frac{\partial^2}{\partial x^2} \left(\varphi_{a,R}(\cdot) \sum_{i=0}^2 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (\cdot - \delta)^i \right) \right]_{C^\gamma[\delta, Rx_a^*]} \\ &\quad + 2 \left[\frac{\partial^2}{\partial x^2} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (\cdot - \delta)^i \right) \right]_{C^\gamma[\delta, 2\delta]} \end{aligned} \quad (47)$$

It is convenient to study separately the terms on the right-hand side of (47). In the following $\tilde{C}_i = \tilde{C}_i(M) > 0$, $i = 1, 2$.

1. By (25) one has

$$\begin{aligned} & \left[\frac{\partial^2}{\partial x^2} \left(\varphi_{a,R}(\cdot) \sum_{i=0}^2 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_{\delta} (\cdot - \delta)^i \right) \right]_{C^\gamma[\delta, Rx_a^*]} \\ & \leq \sum_{i=0}^2 \frac{1}{i!} N_1 \delta^{3-i} \left[(\cdot - \delta)^i \frac{\partial^2}{\partial x^2} \varphi_{a,R} \right]_{C^\gamma[\delta, Rx_a^*]} \\ & \quad + 2 \sum_{i=1}^2 \frac{1}{(i-1)!} N_1 \delta^{3-i} \left[(\cdot - \delta)^{i-1} \frac{\partial}{\partial x} \varphi_{a,R} \right]_{C^\gamma[\delta, Rx_a^*]} + N_1 \delta [\varphi_{a,R}]_{C^\gamma[\delta, Rx_a^*]} \leq \dots \end{aligned}$$

Via the definition of the cut-off function $\varphi_{a,R}$ we get

$$\begin{aligned} \dots & \leq \sum_{i=1}^2 \frac{1}{(i-1)!} N_1 \delta^{3-i} R^{i-1} R^{1-\gamma} \frac{D_{2,0}}{R^2} + \sum_{i=0}^2 \frac{1}{i!} N_1 \delta^{3-i} \frac{D_{2,\gamma}}{R^{2+\gamma}} R^i + 2N_1 \delta R^{1-\gamma} \frac{D_{1,0}}{R} \\ & \quad + 2 \sum_{i=1}^2 \frac{1}{(i-1)!} N_1 \delta^{3-i} \frac{D_{1,\gamma}}{R^{1+\gamma}} R^{i-1} + N_1 \delta \frac{D_{0,\gamma}}{R^\gamma} \leq \tilde{C}_1 \delta^{1-\gamma}. \end{aligned}$$

2. Since

$$\begin{aligned} & \left[\frac{\partial^2}{\partial x^2} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_{\delta} (\cdot - \delta)^i \right) \right]_{C^\gamma[\delta, 2\delta]} \\ & \leq \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_{\delta} \left[(\cdot - \delta)^i \frac{\partial^2}{\partial x^2} \psi_{a,\delta}(\cdot) \right]_{C^\gamma[\delta, 2\delta]} \\ & \quad + \sum_{i=3}^4 \frac{2}{(i-1)!} (g - k_{a,R})^{(i)} \Big|_{\delta} \left[(\cdot - \delta)^{i-1} \frac{\partial}{\partial x} \psi_{a,\delta}(\cdot) \right]_{C^\gamma[\delta, 2\delta]} \\ & \quad + \sum_{i=3}^4 \frac{1}{(i-2)!} (g - k_{a,R})^{(i)} \Big|_{\delta} \left[(\cdot - \delta)^{i-2} \psi_{a,\delta}(\cdot) \right]_{C^\gamma[\delta, 2\delta]} \leq \dots, \end{aligned}$$

from (25) and the choice of $\psi_{a,\delta}$ one obtains

$$\begin{aligned} \dots & \leq \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_{\delta} \left(\delta^i \frac{D'_{2,\gamma}}{\delta^{2+\gamma}} + i \delta^{i-1} \delta^{1-\gamma} \frac{D'_{2,0}}{\delta^2} \right) \\ & \quad + \sum_{i=3}^4 \frac{2}{(i-1)!} (g - k_{a,R})^{(i)} \Big|_{\delta} \left(\delta^{i-1} \frac{D'_{1,\gamma}}{\delta^{1+\gamma}} + (i-1) \delta^{i-2} \delta^{1-\gamma} \frac{D'_{1,0}}{\delta} \right) \\ & \quad + \sum_{i=3}^4 \frac{1}{(i-2)!} (g - k_{a,R})^{(i)} \Big|_{\delta} \left(\delta^{i-2} \frac{D'_{0,\gamma}}{\delta^\gamma} + (i-2) \delta^{i-3} \delta^{1-\gamma} \right) \leq \tilde{C}_2 \delta^{1-\gamma}. \end{aligned}$$

The claim follows. \blacksquare

Lemma C.6 *Let $k_{a,R}$ and g_δ be given respectively in (5) and (20). Then it holds*

$$\left[\frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \leq \frac{C_9}{R} \delta^{1-\gamma}.$$

Proof. We have

$$\begin{aligned} & \left[\frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \leq \left[\frac{1}{3R - k_{a,R}} \frac{\partial^2}{\partial x^2} (k_{a,R} - g_\delta) \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} + \\ & + 2 \left[\frac{1}{3R - k_{a,R}} \frac{\partial}{\partial x} k_{a,R} \frac{\partial}{\partial x} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \\ & + \left[\frac{k_{a,R} - g_\delta}{(3R - k_{a,R})^2} \frac{\partial^2}{\partial x^2} k_{a,R} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]}. \end{aligned} \quad (48)$$

We study the terms in the right-hand side of (48) separately.

1. From (7), Remark C.4.1 and Lemma C.5 it follows that

$$\left[\frac{1}{3R - k_{a,R}} \frac{\partial^2}{\partial x^2} (k_{a,R} - g_\delta) \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \leq \frac{1}{R^2} b_1 2R^{1-\gamma} C_4 \delta + \frac{C_5}{R} \delta^{1-\gamma} \leq \frac{C_6}{R} \delta^{1-\gamma}.$$

2. Using (7) and Lemmas C.3 and C.4 one obtains

$$\begin{aligned} & \left[\frac{1}{3R - k_{a,R}} \frac{\partial}{\partial x} k_{a,R} \frac{\partial}{\partial x} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \\ & \leq b_1 C_2 \delta \left[\frac{1}{3R - k_{a,R}} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} + \frac{1}{R} C_2 \delta \left[\frac{\partial}{\partial x} k_{a,R} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \\ & \quad + b_1 \frac{1}{R} \left[\frac{\partial}{\partial x} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \\ & \leq \frac{1}{R^2} b_1 2R^{1-\gamma} C_2 b_1 \delta + \frac{b_2}{R} 2R^{1-\gamma} \frac{C_2}{R} \delta + C_3 \frac{\delta}{R} 2R^{1-\gamma} \frac{b_1}{R} \leq \frac{C_7}{R} \delta^{1-\gamma}. \end{aligned}$$

3. Since

$$\begin{aligned} & \left[\frac{k_{a,R} - g_\delta}{(3R - k_{a,R})^2} \frac{\partial^2}{\partial x^2} k_{a,R} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \\ & \leq \left[\frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \frac{1}{R} \frac{b_2}{R} + \left[\frac{1}{3R - k_{a,R}} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \frac{b_2}{R} C_1 \delta R + \\ & \quad + \left[\frac{\partial^2}{\partial x^2} k_{a,R} \right]_{C^\gamma[-Rx_a^*, Rx_a^*]} \frac{1}{R} C_1 \delta R \leq \dots, \end{aligned}$$

applying (7) and Lemmas C.2 and C.3 one finds

$$\dots \leq C_2 \delta 2R^{1-\gamma} \frac{b_2}{R^2} + \frac{1}{R^2} 2b_1 R^{1-\gamma} b_2 C_1 \delta + \frac{b_3}{R^2} 2R^{1-\gamma} C_1 \delta \leq \frac{C_8}{R} \delta^{1-\gamma}.$$

The claim follows directly from (48) using the results of the previous points 1, 2 and 3. \blacksquare

The proof of Proposition 3.5 follows from Lemmas C.2, C.3, C.4 and C.6.

C.2 Bounded third and fourth derivative of the mapping

C.2.1 The effect of the scaling: proof of Proposition 3.6

Proof of Proposition 3.6. Let $f_{a,R,1}$ and $f_{a,R,2}$ be respectively the first and the second component of $f_{a,R}$. From the definition of $f_{a,R}$ we find: $f_{a,R,1}(x, y) = x$ and $f_{a,R,2}(x, y) = \frac{3R-g\delta(x)}{3R-k_{a,R}(x)}(y-3R)+3R$. Hence $\|f_{a,R,1}\|_{C^{4,\alpha}(\bar{\Omega}_{a,1})} \leq x_a+1$ and Lemma C.1 yields

$$\begin{aligned} \|f_{a,R,2}^R\|_{C^{4,\alpha}(\bar{\Omega}_{a,1})} &\leq \sum_{\substack{|\beta|=0, \\ \beta \in \mathbb{N}^2}}^4 R^{|\beta|-1} \|D^\beta f_{a,R,2}\|_{C^0(\bar{\Omega}_{a,R})} \\ &\quad + R^{3+\alpha} \sum_{\substack{|\beta|=4, \\ \beta \in \mathbb{N}^2}} [D^\beta f_{a,R,2}]_{C^\alpha(\bar{\Omega}_{a,R})} \leq \dots \end{aligned}$$

By observing that

$$\frac{1}{R} \|f_{a,R,2}\|_{C^0(\bar{\Omega}_{a,R})} \leq 3 + 4 \left\| \frac{3R-g\delta}{3R-k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)},$$

using that $f_{a,R,2}$ is linear in y one finds

$$\begin{aligned} \dots &\leq 3 + 5 \sum_{i=0}^3 R^i \left\| \frac{\partial^i}{\partial x^i} \frac{3R-g\delta}{3R-k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} + \\ &\quad + 6R^4 \left\| \frac{\partial^4}{\partial x^4} \frac{3R-g\delta}{3R-k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} \\ &\quad + R^{3+\alpha} \left[(x, y) \mapsto (y-3R) \frac{\partial^4}{\partial x^4} \frac{3R-g\delta}{3R-k_{a,R}} \right]_{C^\alpha(\bar{\Omega}_{a,R})}. \end{aligned}$$

The claim follows from Lemmas C.2, C.3 and C.4 since $\frac{3R-g\delta}{3R-k_{a,R}} = 1 + \frac{k_{a,R}-g\delta}{3R-k_{a,R}}$. \blacksquare

C.2.2 Proof of Proposition 3.7

We also divide the technical proof of Proposition 3.7 in several lemmas.

In the following $C_i = C_i(M) > 0$, for $i = 12, \dots, 18$. The constants N_i , $i = 1, 2$ are defined in (23) and (24).

Lemma C.7 For $k_{a,R}$ and g_δ respectively as in (5) and (20) it holds that

$$\left\| \frac{\partial^3}{\partial x^3} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq C_{12}.$$

Proof. By the definition of g_δ we have

$$\begin{aligned} & \left\| \frac{\partial^3}{\partial x^3} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq \left\| \frac{\partial^3}{\partial x^3} (k_{a,R} - g) \right\|_{C^0[-\delta, \delta]} + \\ & + 2 \left\| \frac{\partial^3}{\partial x^3} \left(\varphi_{a,R}(\cdot) \sum_{i=0}^2 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta, Rx_a^*]} \\ & + 2 \left\| \frac{\partial^3}{\partial x^3} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta, 2\delta]}. \end{aligned} \quad (49)$$

It is convenient to study the terms on the right-hand side of (49) separately. In the following $\bar{C}_i = \bar{C}_i(M) > 0$ for $i = 1, 2$.

1. It follows directly from (25) that $\left\| \frac{\partial^3}{\partial x^3} (k_{a,R} - g) \right\|_{C^0[-\delta, \delta]} \leq N_1$.

2. Via (25) and the definition of the cut-off function $\varphi_{a,R}$ we get

$$\begin{aligned} & \left\| \frac{\partial^3}{\partial x^3} \left(\varphi_{a,R}(\cdot) \sum_{i=0}^2 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta, Rx_a^*]} \\ & \leq \sum_{i=0}^2 \frac{1}{i!} N_1 \delta^{3-i} \frac{D_{3,0}}{R^3} R^i + 3 \sum_{i=1}^2 \frac{1}{(i-1)!} N_1 \delta^{3-i} \frac{D_{2,0}}{R^2} R^{i-1} + 3N_1 \delta \frac{D_{1,0}}{R} \leq \bar{C}_1 \delta < \bar{C}_1. \end{aligned}$$

3. One finds

$$\begin{aligned} & \left\| \frac{\partial^3}{\partial x^3} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta, 2\delta]} \leq \\ & \leq \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta \frac{D'_{3,0}}{\delta^3} \delta^i + 3 \sum_{i=3}^4 \frac{1}{(i-1)!} (g - k_{a,R})^{(i)} \Big|_\delta \frac{D'_{2,0}}{\delta^2} \delta^{i-1} + \\ & + 3 \sum_{i=3}^4 \frac{1}{(i-2)!} (g - k_{a,R})^{(i)} \Big|_\delta \frac{D'_{1,0}}{\delta} \delta^{i-2} + \sum_{i=3}^4 (g - k_{a,R})^{(i)} \Big|_\delta \delta^{i-3} \leq \bar{C}_2. \end{aligned}$$

The claim follows. ■

Lemma C.8 For $k_{a,R}$ and g_δ respectively as in (5) and (20) it holds that

$$\left\| \frac{\partial^3}{\partial x^3} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} \leq \frac{C_{13}}{R}.$$

Proof. Since

$$\left(\frac{\alpha}{\beta}\right)''' = \frac{\alpha'''}{\beta} - 3\frac{\beta'}{\beta}\left(\frac{\alpha}{\beta}\right)'' - 3\frac{\beta''}{\beta}\left(\frac{\alpha}{\beta}\right)' - \frac{\beta'''}{\beta}\frac{\alpha}{\beta},$$

using Lemma C.7, (7) and Lemmas C.4, C.3, C.2 we get

$$\left\|\frac{\partial^3}{\partial x^3}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right\|_{C^0(-Rx_a^*,Rx_a^*)} \leq \frac{1}{R}C_{12} + 3\frac{b_1}{R}\frac{C_3\delta}{R} + 3\frac{b_2}{R^2}C_2\delta + \frac{b_3}{R^3}C_1\delta R \leq \frac{C_{13}}{R}.$$

■

Lemma C.9 For $k_{a,R}$ and g_δ respectively as in (5) and (20) it holds that

$$\left\|\frac{\partial^4}{\partial x^4}(k_{a,R}-g_\delta)\right\|_{C^0[-Rx_a^*,Rx_a^*]} \leq \frac{C_{14}}{\delta}.$$

Proof. From the definition of g_δ it follows

$$\begin{aligned} & \left\|\frac{\partial^4}{\partial x^4}(k_{a,R}-g_\delta)\right\|_{C^0[-Rx_a^*,Rx_a^*]} \leq \left\|\frac{\partial^4}{\partial x^4}(k_{a,R}-g)\right\|_{C^0[-\delta,\delta]} + \\ & + 2\left\|\frac{\partial^4}{\partial x^4}\left(\varphi_{a,R}(\cdot)\sum_{i=0}^2\frac{1}{i!}(g-k_{a,R})^{(i)}\Big|_\delta(\cdot-\delta)^i\right)\right\|_{C^0[\delta,Rx_a^*]} + \\ & + 2\left\|\frac{\partial^4}{\partial x^4}\left(\psi_{a,\delta}(\cdot)\sum_{i=3}^4\frac{1}{i!}(g-k_{a,R})^{(i)}\Big|_\delta(\cdot-\delta)^i\right)\right\|_{C^0[\delta,2\delta]}. \end{aligned} \quad (50)$$

It is convenient to study the terms on the right-hand side of (50) separately. Here $\tilde{C}_i = \tilde{C}_i(M) > 0$ for $i = 1, 2$.

1. From (25) it follows directly that $\left\|\frac{\partial^4}{\partial x^4}(k_{a,R}-g)\right\|_{C^0[-\delta,\delta]} \leq N_2$.
2. By (25) and the definition of the cut-off function $\varphi_{a,R}$ we get that

$$\begin{aligned} & \left\|\frac{\partial^4}{\partial x^4}\left(\varphi_{a,R}(\cdot)\sum_{i=0}^2\frac{1}{i!}(g-k_{a,R})^{(i)}\Big|_\delta(\cdot-\delta)^i\right)\right\|_{C^0[\delta,Rx_a^*]} \\ & \leq \sum_{i=0}^2\frac{1}{i!}N_1\delta^{3-i}\frac{D_{4,0}}{R^4}R^i + 4\sum_{i=1}^2\frac{1}{(i-1)!}N_1\delta^{3-i}\frac{D_{3,0}}{R^3}R^{i-1} + 6N_1\delta\frac{D_{2,0}}{R^2} \leq \frac{\tilde{C}_1}{R^2}\delta < \frac{\tilde{C}_1}{R}. \end{aligned}$$

3. From (25) and the choice of $\psi_{a,\delta}$ one obtains

$$\begin{aligned}
& \left\| \frac{\partial^4}{\partial x^4} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_{\delta} (\cdot - \delta)^i \right) \right\|_{C^0[\delta, 2\delta]} \\
& \leq \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_{\delta} \frac{D'_{4,0}}{\delta^4} \delta^i + 4 \sum_{i=3}^4 \frac{1}{(i-1)!} (g - k_{a,R})^{(i)} \Big|_{\delta} \frac{D'_{3,0}}{\delta^3} \delta^{i-1} \\
& \quad + 6 \sum_{i=3}^4 \frac{1}{(i-2)!} (g - k_{a,R})^{(i)} \Big|_{\delta} \frac{D'_{2,0}}{\delta^2} \delta^{i-2} + 4 \sum_{i=3}^4 (g - k_{a,R})^{(i)} \Big|_{\delta} \frac{D'_{1,0}}{\delta} \delta^{i-3} \\
& \quad + (g - k_{a,R})^{(4)} \Big|_{\delta} \leq \frac{\tilde{C}_2}{\delta}.
\end{aligned}$$

The claim follows. \blacksquare

Lemma C.10 *Let $k_{a,R}$ and g_{δ} be given respectively in (5) and (20). Then it holds that*

$$\left\| \frac{\partial^4}{\partial x^4} \frac{k_{a,R} - g_{\delta}}{3R - k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} \leq \frac{C_{15}}{\delta R}.$$

Proof. From

$$\left(\frac{\alpha}{\beta} \right)^{(iv)} = \frac{\alpha^{(iv)}}{\beta} - 4 \frac{\beta'}{\beta} \left(\frac{\alpha}{\beta} \right)''' - 6 \frac{\beta''}{\beta} \left(\frac{\alpha}{\beta} \right)'' - 4 \frac{\beta'''}{\beta} \left(\frac{\alpha}{\beta} \right)' - \frac{\beta^{(iv)}}{\beta} \frac{\alpha}{\beta}, \quad (51)$$

using Lemma C.9, (7) and Lemmas C.8, C.4, C.3, C.2 we get

$$\begin{aligned}
\left\| \frac{\partial^4}{\partial x^4} \frac{k_{a,R} - g_{\delta}}{3R - k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} & \leq \frac{1}{R} \frac{C_{14}}{\delta} + 4 \frac{b_1}{R} \frac{C_{13}}{R} + 6 \frac{b_2}{R^2} C_3 \frac{\delta}{R} + 4 \frac{b_3}{R^3} C_2 \delta \\
& \quad + \frac{b_4}{R^4} C_1 \delta R \leq \frac{C_{15}}{\delta R}.
\end{aligned}$$

\blacksquare

Lemma C.11 *For $k_{a,R}$ and g_{δ} respectively as in (5) and (20) it holds that*

$$\left[\frac{\partial^4}{\partial x^4} (k_{a,R} - g_{\delta}) \right]_{C^{\alpha}[-Rx_a^*, Rx_a^*]} \leq \frac{C_{16}}{\delta^{1+\alpha}}.$$

Proof. From the definition of g_{δ} one finds

$$\begin{aligned}
& \left[\frac{\partial^4}{\partial x^4} (k_{a,R} - g_{\delta}) \right]_{C^{\alpha}[-Rx_a^*, Rx_a^*]} \leq \left[\frac{\partial^4}{\partial x^4} (k_{a,R} - g) \right]_{C^{\alpha}[-\delta, \delta]} + \\
& \quad + 2 \left[\frac{\partial^4}{\partial x^4} \left(\varphi_{a,R}(\cdot) \sum_{i=0}^2 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_{\delta} (\cdot - \delta)^i \right) \right]_{C^{\alpha}[\delta, Rx_a^*]} \\
& \quad + 2 \left[\frac{\partial^4}{\partial x^4} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_{\delta} (\cdot - \delta)^i \right) \right]_{C^{\alpha}[\delta, 2\delta]}. \quad (52)
\end{aligned}$$

It is convenient to study the terms on the right-hand side of (52) separately. In the following $\tilde{C}_i = \tilde{C}_i(M) > 0$ for $i = 1, 2, 3$.

1. Since Ω is a $C^{4,\alpha}$ domain with constant M we have

$$\left[\frac{\partial^4}{\partial x^4} (k_{a,R} - g) \right]_{C^\alpha[-\delta,\delta]} \leq \left[\frac{\partial^4}{\partial x^4} k_{a,R} \right]_{C^\alpha[-\delta,\delta]} + M \leq \tilde{C}_1.$$

Notice that the constant \tilde{C}_1 depends only on M .

2. One has

$$\begin{aligned} & \left[\frac{\partial^4}{\partial x^4} \left(\varphi_{a,R}(\cdot) \sum_{i=0}^2 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (\cdot - \delta)^i \right) \right]_{C^\alpha[\delta, Rx_a^*]} \\ & \leq \sum_{i=0}^2 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta \left[(\cdot - \delta)^i \frac{\partial^4}{\partial x^4} \varphi_{a,R}(\cdot) \right]_{C^\alpha[\delta, Rx_a^*]} \\ & \quad + 4 \sum_{i=1}^2 \frac{1}{(i-1)!} (g - k_{a,R})^{(i)} \Big|_\delta \left[(\cdot - \delta)^{i-1} \frac{\partial^3}{\partial x^3} \varphi_{a,R}(\cdot) \right]_{C^\alpha[\delta, Rx_a^*]} \\ & \quad + 6 (g - k_{a,R})^{(2)} \Big|_\delta \left[\frac{\partial^2}{\partial x^2} \varphi_{a,R} \right]_{C^\alpha[\delta, Rx_a^*]} \leq \dots \end{aligned}$$

Via (25) and the definition of the cut-off function $\varphi_{a,R}$ we get

$$\begin{aligned} \dots & \leq \sum_{i=0}^2 \frac{1}{i!} N_1 \delta^{3-i} \frac{D_{4,\alpha}}{R^{4+\alpha}} R^i + \sum_{i=1}^2 \frac{1}{(i-1)!} N_1 \delta^{3-i} \frac{D_{4,0}}{R^4} R^{i-1} R^{1-\alpha} \\ & \quad + 4 \sum_{i=1}^2 \frac{1}{(i-1)!} N_1 \delta^{3-i} \frac{D_{3,\alpha}}{R^{3+\alpha}} R^{i-1} + 4N_1 \delta \frac{D_{3,0}}{R^3} R^{1-\alpha} + 6N_1 \delta \frac{D_{2,\alpha}}{R^{2+\alpha}} \leq \frac{\tilde{C}_2}{R^{2+\alpha}} \delta. \end{aligned}$$

3. Since

$$\begin{aligned} & \left[\frac{\partial^4}{\partial x^4} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta (\cdot - \delta)^i \right) \right]_{C^\alpha[\delta, 2\delta]} \\ & \leq \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_\delta \left[(\cdot - \delta)^i \frac{\partial^4}{\partial x^4} \psi_{a,\delta}(\cdot) \right]_{C^\alpha[\delta, 2\delta]} \\ & \quad + \sum_{i=3}^4 \frac{4}{(i-1)!} (g - k_{a,R})^{(i)} \Big|_\delta \left[(\cdot - \delta)^{i-1} \frac{\partial^3}{\partial x^3} \psi_{a,\delta}(\cdot) \right]_{C^\alpha[\delta, 2\delta]} \\ & \quad + \sum_{i=3}^4 \frac{6}{(i-2)!} (g - k_{a,R})^{(i)} \Big|_\delta \left[(\cdot - \delta)^{i-2} \frac{\partial^2}{\partial x^2} \psi_{a,\delta}(\cdot) \right]_{C^\alpha[\delta, 2\delta]} \\ & \quad + 4 \sum_{i=3}^4 (g - k_{a,R})^{(i)} \Big|_\delta \left[(\cdot - \delta)^{i-3} \frac{\partial}{\partial x} \psi_{a,\delta}(\cdot) \right]_{C^\alpha[\delta, 2\delta]} \\ & \quad + (g - k_{a,R})^{(4)} \Big|_\delta [\psi_{a,\delta}]_{C^\alpha[\delta, 2\delta]} \leq \dots, \end{aligned}$$

from (25) and the choice of $\psi_{a,\delta}$ one obtains

$$\begin{aligned}
\cdots &\leq \sum_{i=3}^4 \frac{1}{i!} (g - k_{a,R})^{(i)} \Big|_{\delta} \left(\delta^i \frac{D'_{4,\alpha}}{\delta^{4+\alpha}} + i\delta^{i-1} \delta^{1-\alpha} \frac{D'_{4,0}}{\delta^4} \right) \\
&\quad + \sum_{i=3}^4 \frac{4}{(i-1)!} (g - k_{a,R})^{(i)} \Big|_{\delta} \left(\delta^{i-1} \frac{D'_{3,\alpha}}{\delta^{3+\alpha}} + (i-1) \delta^{i-2} \delta^{1-\alpha} \frac{D'_{3,0}}{\delta^3} \right) + \\
&\quad + \sum_{i=3}^4 \frac{6}{(i-2)!} (g - k_{a,R})^{(i)} \Big|_{\delta} \left(\delta^{i-2} \frac{D'_{2,\alpha}}{\delta^{2+\alpha}} + (i-2) \delta^{i-3} \delta^{1-\alpha} \frac{D'_{2,0}}{\delta^2} \right) \\
&\quad + 4 \sum_{i=3}^4 (g - k_{a,R})^{(i)} \Big|_{\delta} \left(\delta^{i-3} \frac{D'_{1,\alpha}}{\delta^{1+\alpha}} + (i-3) \delta^{i-4} \delta^{1-\alpha} \frac{D'_{1,0}}{\delta} \right) \\
&\quad + (g - k_{a,R})^{(4)} \Big|_{\delta} \frac{D'_{0,\alpha}}{\delta^{\alpha}} \leq \frac{\tilde{C}_3}{\delta^{1+\alpha}}.
\end{aligned}$$

The claim follows. ■

Lemma C.12 For $k_{a,R}$ and g_{δ} respectively as in (5) and (20) it holds that

$$\left[\frac{\partial^4}{\partial x^4} \frac{k_{a,R} - g_{\delta}}{3R - k_{a,R}} \right]_{C^{\alpha}[-Rx_a^*, Rx_a^*]} \leq \frac{C_{17}}{R\delta^{1+\alpha}}.$$

Proof. From (51) by Lemma C.9, (7) and Lemmas C.11, C.8, C.4, C.3, C.2 one obtains

$$\begin{aligned}
&\left[\frac{\partial^4}{\partial x^4} \frac{k_{a,R} - g_{\delta}}{3R - k_{a,R}} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \leq \left[\frac{1}{3R - k_{a,R}} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{C_{14}}{\delta} + \frac{1}{R} \frac{C_{16}}{\delta^{1+\alpha}} + \\
&\quad + 4 \left[\frac{1}{3R - k_{a,R}} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} b_1 \frac{C_{13}}{R} + 4 \left[\frac{\partial}{\partial x} k_{a,R} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{1}{R} \frac{C_{13}}{R} \\
&\quad + 4 \left[\frac{\partial^3}{\partial x^3} \frac{k_{a,R} - g_{\delta}}{3R - k_{a,R}} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{b_1}{R} + 6 \left[\frac{1}{3R - k_{a,R}} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{b_2}{R} C_3 \frac{\delta}{R} \\
&\quad + 6 \left[\frac{\partial^2}{\partial x^2} k_{a,R} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{1}{R} C_3 \frac{\delta}{R} + 6 \left[\frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_{\delta}}{3R - k_{a,R}} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{b_2}{R^2} \\
&\quad + 4 \left[\frac{1}{3R - k_{a,R}} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{b_3}{R^2} C_2 \delta + 4 \left[\frac{\partial^3}{\partial x^3} k_{a,R} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{1}{R} C_2 \delta \\
&\quad + 4 \left[\frac{\partial}{\partial x} \frac{k_{a,R} - g_{\delta}}{3R - k_{a,R}} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{b_3}{R^3} + \left[\frac{1}{3R - k_{a,R}} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{b_4}{R^3} C_1 \delta R \\
&\quad + \left[\frac{\partial^4}{\partial x^4} k_{a,R} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{1}{R} C_1 \delta R + \left[\frac{k_{a,R} - g_{\delta}}{3R - k_{a,R}} \right]_{C^{\alpha}(-Rx_a^*, Rx_a^*)} \frac{b_4}{R^4}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{b_1}{R^2} 2R^{1-\alpha} \frac{C_{14}}{\delta} + \frac{1}{R} \frac{C_{16}}{\delta^{1+\alpha}} + 4 \frac{b_1}{R^2} 2R^{1-\alpha} b_1 \frac{C_{13}}{R} + 4 \frac{b_2}{R^2} 2R^{1-\alpha} \frac{C_{13}}{R} \\
&+ 4 \frac{C_{15}}{\delta R} 2R^{1-\alpha} \frac{b_1}{R} + 6 \frac{b_1}{R^2} 2R^{1-\alpha} \frac{b_2}{R} C_3 \frac{\delta}{R} + 6 \frac{b_3}{R^2} 2R^{1-\alpha} \frac{1}{R} C_3 \frac{\delta}{R} \\
&+ 6 \frac{C_{13}}{R} 2R^{1-\alpha} \frac{b_2}{R^2} + 4 \frac{b_1}{R^2} 2R^{1-\alpha} \frac{b_3}{R^2} C_2 \delta + 4 \frac{b_4}{R^3} 2R^{1-\alpha} \frac{1}{R} C_2 \delta \\
&+ \frac{b_5}{R^4} 2R^{1-\alpha} \frac{1}{R} C_1 \delta R + 4 C_3 \frac{\delta}{R} 2R^{1-\alpha} \frac{b_3}{R^3} + \frac{b_1}{R^2} 2R^{1-\alpha} \frac{b_4}{R^3} C_1 \delta R \\
&+ C_2 \delta 2R^{1-\alpha} \frac{b_4}{R^4} \leq C_{16} \frac{1}{R \delta^{1+\alpha}}.
\end{aligned}$$

The claim follows. \blacksquare

Lemma C.13 For $k_{a,R}$ and g_δ respectively as in (5) and (20) it holds that

$$\left[(x, y) \mapsto (y - 3R) \frac{\partial^4 k_{a,R} - g_\delta}{\partial x^4 3R - k_{a,R}} \right]_{C^\alpha(\bar{\Omega}_{a,R})} \leq \frac{C_{18}}{\delta^{1+\alpha}}.$$

Proof. Since

$$[(x, y) \mapsto f(x)g(y)]_{C^\alpha[a,b]^2} \leq [f]_{C^\alpha[a,b]} \|g\|_{C^0[a,b]} + \|f\|_{C^0[a,b]} [g]_{C^\alpha[a,b]},$$

one finds

$$\begin{aligned}
&\left[(x, y) \mapsto (y - 3R) \frac{\partial^4 k_{a,R} - g_\delta}{\partial x^4 3R - k_{a,R}} \right]_{C^\alpha(\bar{\Omega}_{a,R})} \\
&\leq 3R^{1-\alpha} \left\| \frac{\partial^4 k_{a,R} - g_\delta}{\partial x^4 3R - k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} + 4R \left[\frac{\partial^4 k_{a,R} - g_\delta}{\partial x^4 3R - k_{a,R}} \right]_{C^\alpha(-Rx_a^*, Rx_a^*)} \leq \dots
\end{aligned}$$

By Lemma C.10 and Lemma C.12 we get

$$\dots \leq 3R^{1-\alpha} \frac{C_{15}}{\delta R} + 4R \frac{C_{17}}{R \delta^{1+\alpha}} < \frac{C_{18}}{\delta^{1+\alpha}}.$$

\blacksquare

The boundedness of $f_{a,R}$ in $C^{4,\alpha}$ -norm follows directly from Proposition 3.6 and Lemmas C.8, C.10 and C.13.

C.3 Partition of unity

In this section we present a general result about partition of unity that we will use in the proof of Theorem 4.1.

Lemma C.14 (Partition of unity with boundary) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\{D_j\}_{j \in J} \subset \Omega$ be a finite open covering of Ω such that $\partial\Omega \subset \bigcup_{j \in J} (\partial D_j \cap \partial\Omega)^{\circ, \partial\Omega}$. For every $\delta > 0$ there exist finitely many smooth functions $\psi_i \in C^\infty(\bar{\Omega})$, $i \in I$, such that:

1. $\psi_i \geq 0$ for all $i \in I$ and $\sum_{i \in I} \psi_i(x) = 1$ for all $x \in \bar{\Omega}$;
2. for every $i \in I$ there exists $j = j(i)$ such that $\text{supp}(\psi_i) \subset D_j \cup (\partial D_j \cup \partial\Omega)^{\circ, \partial\Omega}$,
3. $\text{diam}(\text{supp}(\psi_j)) \leq \delta$ for all $i \in I$.

D Elliptic regularity and interpolation

Elliptic regularity results for linear equations can be found in numerous places. However, if one goes beyond second order and if one needs to know how the constants depend on the domain there is no easy reference. For that reason we will collect such type of results in the present section. For the explicit dependence of these constants we will go back to the original source of Agmon, Douglis and Nirenberg ([2]).

This section is organized as follows. First we recall some classical results and the Calderon-Zygmund inequality for $n = 2$. Then we consider a strong and a weak formulation of problem (1). Finally we study three intermediate versions (between strong and weak) of problem (1).

Throughout this section the following condition will appear..

Condition D.1 *The number α lies in $(0, 1)$ and Ω is a bounded simply connected domain (open subset) in \mathbb{R}^2 satisfying the uniform $C^{4,\alpha}$ regularity condition with constant M .*

D.1 Classical results

In this section we recall some results from [11]. For sake of brevity we do not give the most general statements.

Theorem D.2 [11, Th.9.13] *Let Ω be a bounded domain in \mathbb{R}^n satisfying the uniform $C^{1,1}$ regularity condition with constant M . Then it holds*

$$\|u\|_{W^{2,2}(\Omega)} \leq C \left(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)} \right) \text{ for every } u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

with $C = C(n, M, \rho_\Omega^{-1}, R_\Omega)$.

Remark D.2.1 *The dependence of the constant can be deduced from the proof in [11, Th.9.13].*

We will use the Calderon-Zygmund inequality for $n = 2$. This inequality is usually proved by contradiction. Since we are interested in the dependence of the constant on the domain, we give here a direct proof.

Lemma D.3 *Let Ω be a bounded domain in \mathbb{R}^2 . Then there is $C = C(R_\Omega)$ such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)} \text{ for every } u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Proof. Let $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. For $n = 2$ Aleksandrov's maximum principle ([11, Th.9.1]) implies that $\sup_\Omega |u| \leq C \|\Delta u\|_{L^2(\Omega)}$ for some $C = C(R_\Omega)$. Hence we find $\|u\|_{L^2(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)} |\Omega|^{\frac{1}{2}}$. ■

Corollary D.4 *Suppose Ω is a bounded domain in \mathbb{R}^2 satisfying the uniform $C^{1,1}$ regularity condition with constant M . Then there is $C = C(M, \rho_\Omega^{-1}, R_\Omega)$ such that*

$$\|u\|_{W^{2,2}(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)} \text{ for every } u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Proof. The claim follows directly from Theorem D.2 and Lemma D.3. ■

D.2 Regularity for strong solutions

The classical regularity result that we like to recall in an explicit statement is the following.

Theorem D.5 *Assume Condition D.1. For every $f \in L^p(\Omega)$ with $p \in (1, \infty)$ there exists a unique solution $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ of (1).*

Moreover the solution satisfies

$$\frac{1}{2} \|f\|_{L^p(\Omega)} \leq \|u\|_{W^{4,p}(\Omega)} \leq C_s \|f\|_{L^p(\Omega)}, \quad (53)$$

with $C_s = C_s(p, p', M, \rho_\Omega^{-1}, R_\Omega)$ where C_s satisfies the convention of Notation 1.4.

Before proving Theorem D.5 we present some estimates.

Lemma D.6 *Let Ω be a bounded domain in \mathbb{R}^2 satisfying the uniform $C^{1,1}$ regularity condition with constant M and let $p \in (1, \infty)$. Then there is $C = C(p, p', M, \rho_\Omega^{-1}, R_\Omega)$ such that:*

$$\|u\|_{L^p(\Omega)} \leq C \|\Delta^2 u\|_{L^p(\Omega)} \text{ for every } u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega).$$

Proof. Since $n = 2$ we find by Sobolev inequalities that

$$\|u\|_{L^p(\Omega)} \leq C_1 \|u\|_{W^{2,2}(\Omega)} \text{ and } \|u\|_{L^{p'}(\Omega)} \leq C_2 \|u\|_{W^{2,2}(\Omega)}.$$

Notice that $C_1 = C_1(p, M, \rho_\Omega^{-1}, R_\Omega)$ and $C_2 = C_2(p', M, \rho_\Omega^{-1}, R_\Omega)$. Hence we have by Corollary D.4, by integrating by parts and by Hölder that

$$\begin{aligned} \|u\|_{L^{p'}(\Omega)} \|u\|_{L^p(\Omega)} &\leq C_1 C_2 \|u\|_{W^{2,2}(\Omega)}^2 \leq C_3 \int_\Omega |\Delta u|^2 dx = \\ &= C_3 \int_\Omega u \Delta^2 u dx \leq C_3 \|\Delta^2 u\|_{L^p(\Omega)} \|u\|_{L^{p'}(\Omega)}, \end{aligned}$$

with $C_3 = C_3(p, p', M, \rho_\Omega^{-1}, R_\Omega)$. The claim follows. ■

Lemma D.7 *Assume Condition D.1. Then for $1 < p < \infty$ there exists $C = C(p, p', M, \rho_\Omega^{-1}, R_\Omega)$ such that*

$$\|u\|_{W^{4,p}(\Omega)} \leq C \|\Delta^2 u\|_{L^p(\Omega)} \text{ for every } u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega).$$

Remark D.7.1 Usually Lemma D.7 is proved by contradiction and this does not explain what the constant depends on. However by using Lemma D.6 we find the explicit quantities.

Proof. The result follows from [2, Th.15.2] and Lemma D.6. The proof of [2, Th.15.2] shows that the dependence of the constant is as given in the statement. ■

Proof of Theorem D.5.

- Uniqueness follows by a standard integration by parts. Indeed, if $\Delta^2 u = 0$ then

$$\int_{\Omega} |\Delta u|^2 dx = \int_{\Omega} u \Delta^2 u dx = 0,$$

and with the boundary condition one finds $u \equiv 0$.

- Estimate: By definition of the norm in $W^{4,p}(\Omega)$ one finds $\frac{1}{2} \|\Delta^2 u\|_{L^p(\Omega)} \leq \|u\|_{W^{4,p}(\Omega)}$. The other side of inequality (53) follows from Lemma D.7.
- Existence: For $f \in C^\alpha(\Omega)$ the existence of a solution $u \in C^{4,\alpha}(\Omega) \cap C_0^1(\bar{\Omega})$ is given by [2, Th.12.7]. Such a solution satisfies (53), ([2, Th.9.3]). By an approximation argument the existence in $W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ follows. ■

Remark D.7.2 The hypothesis $\partial\Omega \in C^{4,\alpha}$ is needed in order to use Theorem 12.7 in [2]. For the rest of the paper it would be sufficient to assume $\partial\Omega \in C^4$.

For $1 < p < \infty$ we formally fix the operator $T_{4,p}$ by

$$\begin{aligned} D(T_{4,p}) &:= W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega), \\ T_{4,p}u &:= \Delta^2 u \text{ for } u \in D(T_{4,p}). \end{aligned} \tag{54}$$

The operator $T_{4,p}$ is the inverse of the solution operator. The following result is a consequence of Theorem D.5.

Corollary D.8 Let $1 < p < \infty$. Assuming Condition D.1 the operator $T_{4,p}$ defined in (54) gives an isomorphism from $W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ onto $L^p(\Omega)$. Moreover one has

$$\frac{1}{C_s} \leq \|T_{4,p}\|_{(W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)) \rightarrow L^p(\Omega)} \leq 2,$$

where C_s is the constant appearing in Theorem D.5.

D.3 Regularity for weak solutions

In the following section we give the explicit definition of what we will call a weak solution for problem (1) and we recall the classical regularity result in this setting.

Definition D.9 Let $p \in (1, \infty)$ and $F \in \left(W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'$. We call $u \in L^p(\Omega)$ a weak solution of problem (1) with right hand side F if the following holds

$$\int_{\Omega} u(x) \Delta^2 v(x) dx = F(v) \text{ for every } v \in W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega).$$

Theorem D.10 Assume Condition D.1 and let $p \in (1, \infty)$. Then for every $F \in \left(W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'$ there exists a unique u weak solution of problem (1) with right hand side F .

Moreover u satisfies

$$\frac{1}{2} \|F\|_{\left(W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'} \leq \|u\|_{L^p(\Omega)} \leq C_w \|F\|_{\left(W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'},$$

with $C_w = C_w(p, p', M, \rho_{\Omega}^{-1}, R_{\Omega})$.

Proof. Let i_p be the canonical isometry $L^p(\Omega) \rightarrow \left(L^{p'}(\Omega)\right)'$, that is, $i_p(u)(v) = \int_{\Omega} u(x) v(x) dx$ for every $v \in L^{p'}(\Omega)$.

Existence of u follows by a duality argument. Indeed, by Corollary D.8 we may define

$$U(f) := F(T_{4,p'}^{-1}(f)) \text{ for every } f \in L^{p'}(\Omega).$$

The solution u is given by $u := i_p^{-1}(U)$. Uniqueness and the estimate follow from Corollary D.8. \blacksquare

For $1 < p < \infty$ let us formally fix the operator $T_{0,p}$ by

$$\begin{aligned} D(T_{0,p}) &:= L^p(\Omega), \\ (T_{0,p}(u))(v) &:= i_p(u)(T_{4,p'}(v)) \text{ for every } v \in W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega). \end{aligned} \tag{55}$$

From Theorem D.10 it follows:

Corollary D.11 Let $1 < p < \infty$ and assume Condition D.1. The operator $T_{0,p}$ defined in (55) gives an isomorphism from $L^p(\Omega)$ onto $\left(W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'$. Moreover one has

$$\frac{1}{C_w} \leq \|T_{0,p}\|_{\left(L^p(\Omega) \rightarrow \left(W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'\right)} \leq 2,$$

where C_w is the constant appearing in Theorem D.10.

D.4 Regularity between weak and strong

In the following section we consider via interpolation solutions between the ‘strong’ and the ‘weak’ ones defined in the previous sections.

We first give the three intermediate notions of solution.

Definition D.12 *Let $p \in (1, \infty)$.*

1. *Let $F \in \left(W_0^{1,p'}(\Omega)\right)'$. We say that $u \in W^{3,p}(\Omega) \cap W_0^{2,p}(\Omega)$ is a “one-quarter weak solution” of problem (1) with right hand side F if it satisfies*

$$- \int_{\Omega} (\nabla \Delta u(x)) \cdot (\nabla v(x)) \, dx = F(v) \text{ for every } v \in W_0^{1,p'}(\Omega).$$

2. *Let $F \in \left(W_0^{2,p'}(\Omega)\right)'$. We say that $u \in W_0^{2,p}(\Omega)$ is a “one-half weak solution” of problem (1) with right hand side F if it satisfies*

$$\int_{\Omega} (\Delta u(x)) (\Delta v(x)) \, dx = F(v) \text{ for every } v \in W_0^{2,p'}(\Omega).$$

3. *Let $F \in \left(W^{3,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'$. We say that $u \in W_0^{1,p}(\Omega)$ is a “three-quarter weak solution” of problem (1) with right hand side F if it satisfies*

$$- \int_{\Omega} (\nabla u(x)) \cdot (\nabla \Delta v(x)) \, dx = F(v) \text{ for every } v \in W^{3,p'}(\Omega) \cap W_0^{2,p'}(\Omega).$$

Theorem D.13 *Assume Condition D.1 and let $1 < p < \infty$.*

1. *Then for every $F \in \left(W_0^{1,p'}(\Omega)\right)'$ there exists a unique u “one-quarter weak solution” of problem (1) with right hand side F .*

Moreover u satisfies

$$\frac{1}{C_1} \|F\|_{\left(W_0^{1,p'}(\Omega)\right)'} \leq \|u\|_{W^{3,p}(\Omega)} \leq C_1 \|F\|_{\left(W_0^{1,p'}(\Omega)\right)'},$$

with $C_1 = C_1(p, p', M, \rho_{\Omega}^{-1}, R_{\Omega})$.

2. *Then for every $F \in \left(W_0^{2,p'}(\Omega)\right)'$ there exists a unique u “one-half weak solution” of problem (1) with right hand side F .*

Moreover u satisfies

$$\frac{1}{C_2} \|F\|_{\left(W_0^{2,p'}(\Omega)\right)'} \leq \|u\|_{W^{2,p}(\Omega)} \leq C_2 \|F\|_{\left(W_0^{2,p'}(\Omega)\right)'},$$

with $C_2 = C_2(p, p', M, \rho_{\Omega}^{-1}, R_{\Omega})$.

3. Then for every $F \in \left(W^{3,p'}(\Omega) \cap W_0^{2,p'}(\Omega) \right)'$ there exists a unique u “three-quarter weak solution” of problem (1) with right hand side F .

Moreover u satisfies

$$\frac{1}{C_3} \|F\|_{\left(W^{3,p'}(\Omega) \cap W_0^{2,p'}(\Omega) \right)'} \leq \|u\|_{W^{1,p}(\Omega)} \leq C_3 \|F\|_{\left(W^{3,p'}(\Omega) \cap W_0^{2,p'}(\Omega) \right)'},$$

with $C_3 = C_3(p, p', M, \rho_\Omega^{-1}, R_\Omega)$.

Remark D.13.1 *Theorem D.13 part 2 has been studied in [21, Chap.7].*

Our aim in giving the proof of Theorem D.13 is to show how the constants in the estimates depend on the domain. We proceed through interpolation: $[\cdot, \cdot]_\theta$ denotes the complex interpolation with parameter $\theta \in (0, 1)$.

For sake of conciseness we fix the following notation:

$$\begin{aligned} A_{0,p} &:= L^p(\Omega) & A_{4,p} &:= W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega), \\ B_{0,p} &:= (A_{4,p'})' & B_{4,p} &:= L^p(\Omega) (\cong (A_{0,p'})'), \end{aligned}$$

and for $\theta \in (0, 1)$

$$A_{4\theta,p} := [A_{0,p}, A_{4,p}]_\theta \quad \text{and} \quad B_{4\theta,p} := [B_{0,p}, B_{4,p}]_\theta.$$

With this notation we have $T_{0,p} : A_{0,p} \rightarrow B_{0,p}$ and $T_{4,p} : A_{4,p} \rightarrow B_{4,p}$, where $T_{0,p}$ is defined in (55) and $T_{4,p}$ is defined in (54).

Lemma D.14 *Assume Condition D.1 and let $1 < p < \infty$. The operator $T_{4,p}$ is a restriction of $T_{0,p}$ to $A_{4,p}$ in the sense that for every $u \in A_{4,p}$*

$$T_{0,p}(u) \in (B_{4,p'})' \quad \text{and} \quad T_{0,p}(u) = i_p(T_{4,p}(u)).$$

Proof. Let $u \in A_{4,p}$. For every $v \in A_{4,p'}$ we have

$$(T_{0,p}(u))(v) = \int_{\Omega} u \Delta^2 v \, dx = \int_{\Omega} v \Delta^2 u \, dx = \int_{\Omega} v T_{4,p}(u) \, dx.$$

The claim follows. ■

As a consequence of Lemma D.14 in the following lemma we find via interpolation a family of isomorphisms which are extensions of $T_{4,p}$ and restrictions of $T_{0,p}$.

Lemma D.15 *Assume Condition D.1 and fix $\theta \in (0, 1)$. Consider the operator $T_{4\theta,p}$ such that $D(T_{4\theta,p}) := A_{4\theta,p}$ and $T_{4\theta,p}(u) := T_{0,p}(u)$ for $u \in D(T_{4\theta,p})$ and $1 < p < \infty$.*

Then $T_{4\theta,p}$ is an isomorphism from $A_{4\theta,p}$ onto $B_{4\theta,p}$ and moreover

$$\frac{1}{\max\{C_s, C_w\}} \leq \|T_{4\theta,p}\|_{(A_{4\theta,p} \rightarrow B_{4\theta,p})} \leq 2, \quad (56)$$

where C_s and C_w are the constants appearing in Theorems D.5 and D.10 respectively.

Proof. The claim follows from Corollaries D.8 and D.11 since the complex interpolation functor is exact and of type θ ([23, Th.1.9.3a]). ■

Remark D.15.1 Notice that (56) implies that for every $u \in A_{4\theta,p}$ it holds

$$\frac{1}{2} \|T_{4\theta,p}(u)\|_{B_{4\theta,p}} \leq \|u\|_{A_{4\theta,p}} \leq \max\{C_s, C_w\} \|T_{4\theta,p}(u)\|_{B_{4\theta,p}}.$$

In the following we consider the operators $T_{1,p}$, $T_{2,p}$ and $T_{3,p}$; i.e. the operators $T_{4\theta_i,p}$ defined in Lemma D.15 with $\theta_i = \frac{1}{4}i$ and $i = 1, 2, 3$. Notice that the solution operator for the “three-quarter weak solution” of problem (1) is the inverse of $T_{1,p}$. Analogously the solution operator for the “one-half weak solution” of problem (1) is the inverse of $T_{2,p}$ and the solution operator for the “one-quarter weak solution” of problem (1) is the inverse of $T_{3,p}$.

For these operators we have that

$$A_{i,p} = W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega) \text{ with equivalent norms,} \quad (57)$$

where $A_{i,p} = D(T_{i,p})$. Identity (57) can be found in Triebel for C^∞ -domains. We first show that in order (57) to hold it is sufficient that $\partial\Omega \in C^{4,\alpha}$. Furthermore we give the dependence on the domain of the constants $D_{1,p,i}$ and $D_{2,p,i}$ that appear in

$$D_{1,p,i} \|u\|_{W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)} \leq \|u\|_{A_{i,p}} \leq D_{2,p,i} \|u\|_{W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)},$$

for $u \in W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)$.

We first recall a classical result from [23].

Proposition D.16 [23, Th.4.3.3] Let B denote the unit ball in \mathbb{R}^n . Then for $i = 1, 2, 3$ and $1 < p < \infty$ one has

$$\left[L^p(B), W^{4,p}(B) \cap W_0^{2,p}(B) \right]_{\frac{1}{4}i} = W^{i,p}(B) \cap W_0^{\min\{i,2\},p}(B)$$

as Banach spaces (same elements, equivalent norms). Hence there exist constants $C_{1,p,i}$ and $C_{2,p,i}$ such that for every $u \in W^{i,p}(B) \cap W_0^{\min\{i,2\},p}(B)$ one has

$$C_{1,p,i} \|u\|_{W^{i,p}(B)} \leq \|u\|_{[L^p(B), W^{4,p}(B) \cap W_0^{2,p}(B)]_{\frac{1}{4}i}} \leq C_{2,p,i} \|u\|_{W^{i,p}(B)}.$$

Theorem D.17 Let assume Condition D.1. Then for $1 < p < \infty$ and $i = 1, 2, 3$ it holds

$$[A_{0,p}, A_{4,p}]_{\frac{1}{4}i} = W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega).$$

Hence there exist constants $D_{1,p,i}$ and $D_{2,p,i}$ such that for every $u \in W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)$ one has

$$D_{1,p,i} \|u\|_{W^{i,p}(\Omega)} \leq \|u\|_{[A_{0,p}, A_{4,p}]_{\frac{1}{4}i}} \leq D_{2,p,i} \|u\|_{W^{i,p}(\Omega)},$$

with $D_{j,p,i} = D_{j,p,i}(p, M, \rho_\Omega^{-1}, R_\Omega)$ for $j = 1, 2$.

Proof. Let S denote a $C^{4,\alpha}$ transformation from $\bar{\Omega}$ onto \bar{B} .

Considering the operator

$$E_p : L^p(\Omega) \rightarrow L^p(B) \text{ such that } E_p(f) := f \circ S^{-1},$$

one finds that the following properties hold:

- E_p is an isomorphism;
- for $i = 1, \dots, 4$ the restriction of E_p to $W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)$ is an isomorphism onto $W^{i,p}(B) \cap W_0^{\min\{i,2\},p}(B)$;
- there are constants $\bar{C}_{1,p}$ and $\bar{C}_{2,p}$ such that

$$\bar{C}_{1,p} \|E_p(u)\|_{W^{i,p}(B)} \leq \|u\|_{W^{i,p}(\Omega)} \leq \bar{C}_{2,p} \|E_p(u)\|_{W^{i,p}(B)}, \quad (58)$$

for every $i = 0, 1, \dots, 4$ and $u \in W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)$. Furthermore the constants $\bar{C}_{1,p}$ and $\bar{C}_{2,p}$ depend only on p , R_Ω , ρ_Ω^{-1} and the M of Condition D.1.

For $\theta \in (0, 1)$ the operator E_p induces isomorphisms

$$E_{p,\theta} : A_{4\theta,p} \rightarrow \left[L^p(B), W^{4,p}(B) \cap W_0^{2,p}(B) \right]_\theta,$$

and since the complex interpolation functor is exact ([23, Th.1.9.3a]) one has

$$\begin{aligned} & \bar{C}_{1,p} \|E_{p,\theta}(u)\|_{[L^p(B), W^{4,p}(B) \cap W_0^{2,p}(B)]_\theta} \\ & \leq \|u\|_{A_{4\theta,p}} \leq \bar{C}_{2,p} \|E_{p,\theta}(u)\|_{[L^p(B), W^{4,p}(B) \cap W_0^{2,p}(B)]_\theta}, \end{aligned} \quad (59)$$

(See Theorem 1.2.4 in [23]).

Hence, by (59) and Proposition D.16, we have that

$$\begin{aligned} A_{i,p} &= \left[L^p(\Omega), W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \right]_{\frac{1}{4}i} \\ &= \left(E_{p,\frac{1}{4}i} \right)^{-1} \left(\left[L^p(B), W^{4,p}(B) \cap W_0^{2,p}(B) \right]_{\frac{1}{4}i} \right) \\ &= \left(E_{p,\frac{1}{4}i} \right)^{-1} \left(W^{i,p}(B) \cap W_0^{\min\{i,2\},p}(B) \right) = W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega). \end{aligned}$$

Furthermore we explicitly find the constants that give the equivalence of the norms. Indeed from (58), (59) and Proposition D.16 it follows

$$\begin{aligned} \|u\|_{A_{i,p}} &\leq \bar{C}_{2,p} \|E_{p,\frac{1}{4}i}(u)\|_{[L^p(B), W^{4,p}(B) \cap W_0^{2,p}(B)]_{\frac{1}{4}i}} \\ &\leq \bar{C}_{2,p} C_{2,p,i} \|E_{p,\frac{1}{4}i}(u)\|_{W^{i,p}(B)} \leq \frac{\bar{C}_{2,p}}{\bar{C}_{1,p}} C_{2,p,i} \|u\|_{W^{i,p}(\Omega)}, \end{aligned}$$

and

$$\begin{aligned} \|u\|_{W^{i,p}(\Omega)} &\leq \bar{C}_{2,p} \|E_{p,\frac{1}{4}i}(u)\|_{W^{i,p}(B)} \\ &\leq \frac{\bar{C}_{2,p}}{C_{1,p,i}} \|E_{p,\frac{1}{4}i}(u)\|_{[L^p(B), W^{4,p}(B) \cap W_0^{2,p}(B)]_{\frac{1}{4}i}} \leq \frac{\bar{C}_{2,p}}{\bar{C}_{1,p}} \frac{1}{C_{1,p,i}} \|u\|_{A_{i,p}}. \end{aligned}$$

■

Remark D.17.1 *The existence of the $C^{4,\alpha}$ transformation from $\bar{\Omega}$ onto \bar{B} depends upon the regularity of Ω and the fact that Ω is simply connected. This technical assumption can be removed.*

Corollary D.18 *Let assume Condition D.1. Then for $1 < p < \infty$ and $i = 1, 2, 3$ it holds*

$$[B_{0,p}, B_{4,p}]_{\frac{1}{4}i} = \left(W^{4-i,p'}(\Omega) \cap W_0^{\min\{4-i,2\},p'}(\Omega) \right)'.$$

Moreover there exist constants $\bar{D}_{j,p,i} = \bar{D}_{j,p,i}(p', M, \rho_\Omega^{-1}, R_\Omega)$ for $j = 1, 2$ such that

$$\begin{aligned} &\bar{D}_{1,p,i} \|u\|_{\left(W^{4-i,p'}(\Omega) \cap W_0^{\min\{4-i,2\},p'}(\Omega) \right)'} \\ &\leq \|u\|_{[B_{0,p}, B_{4,p}]_{\frac{1}{4}i}} \leq \bar{D}_{2,p,i} \|u\|_{\left(W^{4-i,p'}(\Omega) \cap W_0^{\min\{4-i,2\},p'}(\Omega) \right)'}, \end{aligned}$$

holds for every $u \in \left(W^{4-i,p'}(\Omega) \cap W_0^{\min\{4-i,2\},p'}(\Omega) \right)'$.

Proof. The result follows from Theorem D.17 through duality results for complex interpolation spaces ([23, Th.1.11.3]).

■

Corollary D.19 *Assume Condition D.1 and let $1 < p < \infty$.*

Then for $i = 1, 2, 3$ there exist isomorphisms

$$T_{i,p} : W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega) \rightarrow \left(W^{4-i,p'}(\Omega) \cap W_0^{\min\{4-i,2\},p'}(\Omega) \right)',$$

which are restrictions of $T_{0,p}$ and extensions of $T_{4,p}$.

Moreover there exists constants $C_i = C_i(p, p', M, \rho_\Omega^{-1}, R_\Omega)$ such that for every $u \in W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)$ it holds

$$\begin{aligned} &\frac{1}{C_i} \|T_{i,p}(u)\|_{\left(W^{4-i,p'}(\Omega) \cap W_0^{\min\{4-i,2\},p'}(\Omega) \right)'} \\ &\leq \|u\|_{W^{i,p}(\Omega)} \leq C_i \|T_{i,p}(u)\|_{\left(W^{4-i,p'}(\Omega) \cap W_0^{\min\{4-i,2\},p'}(\Omega) \right)'}. \end{aligned}$$

Proof. The result follows from Lemma D.15, Theorem D.17 and Corollary D.18.

■

Theorem D.13 follows directly from the previous corollary.

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