UNIQUENESS FOR THE HOMOGENEOUS DIRICHLET WILLMORE BOUNDARY VALUE PROBLEM

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ABSTRACT. We give a sufficient condition for curves on a plane or on a sphere such that if these give the boundary of a Willmore surface touching tangentially along the boundary the plane or the sphere respectively, the surface is necessarily a piece of the plane or a piece of the sphere. The condition we require is that the curves bound a strictly star-shaped domain with respect to the Euclidean geometry in the plane and with respect to the spherical geometry in the sphere, respectively.

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1. Introduction

A Willmore surface is a critical point for the Willmore functional, that for an immersed surface $f : \Sigma \hookrightarrow \mathbb{R}^n$ is given by

$$\mathcal{W}(f) = \int_\Sigma |\vec{H}|^2 \, d\mu_g \, ,$$

with $\vec{H}$ the mean-curvature vector and $d\mu_g$ the area form induced by the canonical metric in $\mathbb{R}^n$. The Willmore functional appeared already at the beginning of the 19-th century as a measure for the elastic energy of thin plates. Nowadays it has applications also in the modeling of biological membranes and in image processing. An important feature is that, for closed surfaces without boundary, the Willmore functional is invariant under conformal transformations (see Willmore, Ch.7.3 [23] or Weiner [22]). The Euler-Lagrange equation (called Willmore equation) is

$$\Delta^\perp \vec{H} + g^{ik} g^{jl} A^\circ_{ij} \langle A^\circ_{kl}, \vec{H} \rangle = 0 \tag{1.1}$$

with $\Delta^\perp$ the Laplace in the normal bundle and A° the trace-free part of the second fundamental form (see the end of this section for the

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notation). In codimension one, one may write $\vec{H} = -H\vec{n}$ with $\vec{n}$ a normal vector field and $H = \frac{1}{2}(\kappa_1+\kappa_2)$ where κ_1 and κ_2 are the principal curvatures of the surface. Then equation (1.1) can be rewritten as

$$\Delta_g H + 2H(H^2 - K) = 0\,, \tag{1.2}$$

with Δ_g the Laplace-Beltrami operator and K the Gauss curvature. (For surfaces with boundary we consider only interior variations.) The Willmore equation (1.1) is a system of quasilinear elliptic equations of fourth order. Moreover, the ellipticity is not uniform.

Existence of closed (without boundary) Willmore surfaces of prescribed genus has been proved in [20] and [1]. Concerning regularity in [18] it is proven that any Willmore surface is real analytic. In all these works the conformal invariance of the Willmore functional plays a key role.

We are interested in studying Willmore surfaces with boundary satisfying prescribed boundary conditions. The first to study boundary value problems for Willmore surfaces was Nitsche in [15]. He describes several choices of boundary value problems for the Willmore equation and established existence results for small data. Most of the works in the literature concerns Dirichlet boundary data. By this we mean that the boundary of the surface is fixed and also that the tangent spaces of the surface along the boundary are fixed. (See [3], [4] and [8] for results on natural boundary conditions.)

The studies on Willmore surfaces with boundary in the literature follow two streams. On one side, there are existence results under special symmetries. In [6] and [7] existence of Willmore surfaces of revolution generated by graphs satisfying arbitrary symmetric Dirichlet boundary conditions has been proved. In this case, the boundary consists of two circles with the same radius and center on the axis with respect to which we rotate. The second boundary condition prescribes the derivative of the function at the boundary. One has existence of Willmore surfaces for all choices of the radius and for all values of the derivative at the boundary. This is in great contrast with the corresponding results for minimal surfaces, where there is a critical value of the radius under which there do not exist minimal surfaces having the two circles as boundary. A more general approach is in [19]. Schätzle in [19] proves existence of Willmore immersions in $\mathbb{S}^n$ satisfying Dirichlet boundary conditions. Under certain smallness assumptions on the energy, he can then project these surfaces into $\mathbb{R}^n$ to get embedded Willmore surfaces. This is the only result known at the moment which is valid in any codimension.

Palmer in [17] proves (among other results) the following uniqueness result. A Willmore surface of disk type which has its boundary on a circle and which intersects the plane of the circle in a constant angle is a spherical cap or a flat disk. In this work we wish to extend this result, allowing the curve to be more general than a circle. The main result of the present paper is the following.

Theorem 1.1. *Let P be a plane in $\mathbb{R}^3$ and $\Omega \subset P$ be a smooth strictly star-shaped bounded domain. Let D denote the unit disk in $\mathbb{R}^2$ and let $f : D \hookrightarrow \mathbb{R}^3$ be a smooth Willmore immersion so that:*

1. *the boundary of the surface is given by $\partial\Omega$;*
2. *the surface touches tangentially the plane P along the boundary.*

Then, $f(D) = \Omega \subset P$.

Remark 1.2. If the surface admits a parametrisation as the graph of a smooth function $u : \Omega \to \mathbb{R}$, the boundary conditions require that $u|_{\partial\Omega} = 0$ and $\nabla u|_{\partial\Omega} = 0$. The result states that u has to be identically zero.

Via an inversion we get the corresponding result for curves on the sphere. Before stating the result we need to explain what we mean by saying that a curve is *strictly star-shaped with respect to the spherical geometry*. It is convenient first to recall the definition of strictly star-shaped domain in the plane. A smooth domain Ω in the plane is called strictly star-shaped if there exists a point $p \in \Omega$ (called 'star-center') such that $(p-x)\cdot\nu > 0$ for every $x \in \partial\Omega$ with ν the interior normal (to $\partial\Omega$ in x). In other words, at each point x of the boundary the tangent vector to the boundary curve does not have the same direction as $p-x$ which is the line segment (a geodesic in the euclidean geometry) from x to p. Here we use the smoothness of Ω. There is a transversality between the boundary curve and the geodesics going through the star-center. In this spirit, we give the following definition.

Definition 1.3. We say that a closed smooth curve on $\mathbb{S}^2$ is *strictly star-shaped with respect to the spherical geometry* if the following holds. There exists a point $p \in \mathbb{S}^2$ such that if a geodesic in $\mathbb{S}^2$ going through p meets the curve, the tangent vector to the curve in the point of intersection does not have the same direction as the tangent vector to the geodesic in the same point.

Remark 1.4. Notice that if a curve on $\mathbb{S}^2$ satisfies the definition of strictly star-shaped with respect to the spherical geometry with a point p, so it does also with respect to $-p$. Further, via inversions boundary curves of strictly star-shaped domains in the plane are sent to curves in

$\mathbb{S}^2$ that are strictly star-shaped with respect to the spherical geometry and vice versa. See Figure 1.

The result corresponding to Theorem 1.1 for Willmore immersions in the unit sphere is now as follows.

Corollary 1.5. *Let D denote the unit disk in $\mathbb{R}^2$ and let $f : D \hookrightarrow \mathbb{R}^3$ be a smooth Willmore immersion so that:*

1. *the boundary of the surface is a closed smooth curve on $\mathbb{S}^2$ which is strictly star-shaped with respect to the spherical geometry;*
2. *the surface touches tangentially $\mathbb{S}^2$ along the boundary.*

Then, $f(D)$ is a spherical cap.

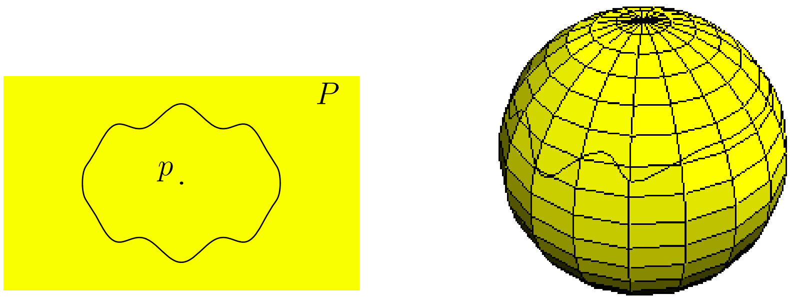

FIGURE 1. On the left: strictly star-shaped domain on the plane. On the right: Curve on $\mathbb{S}^2$ which is strictly star-shaped with respect to the spherical geometry and with respect to the North Pole.

In general we do not expect uniqueness for the Willmore Dirichlet boundary value problems, not even in the presence of some extra symmetries. Indeed, in the case of surfaces of revolution generated by symmetric graphs with symmetric boundary data, for certain values of the parameters, one can numerically find two different minimisers. So there is numerical evidence not only of two solutions of the Euler-Lagrange equation but also of two different surfaces with the same Willmore energy.

The proof of Theorem 1.1 consists of two steps. In the first we prove that, under the assumptions, f being a Willmore immersion implies that the mean curvature is zero at the boundary and also that $A(X, X) = 0$ at the boundary for any tangential vector field X. Here A denotes the second fundamental form. We achieve this by choosing in the first variation of the Willmore functional as a test function the Willmore immersion itself. This uses the scaling invariance of the problem and is inspired by the proof of the Pohozaev identity. This first result is valid in any codimension and is presented in Section 2. The second step is as in the work of Palmer [17]. By a result of Bryant [2] we may associate to the Willmore surface a holomorphic function. By the first step of the proof this function is zero at the boundary and therefore identically zero. Then a classification theorem of Bryant yields the result. For sake of completeness we present the argument in Section 3 together with the proof of Corollary 1.5. In Section 4 we present some generalisations of the argument used in Section 2 based on the Pohozaev identity. More precisely, we derive other integral identities containing only boundary integrals using the other invariances of the problem and we comment on a slightly more general functional that could be considered.

1.1. **Notation.** In local coordinates (x^1, x^2) on D the first fundamental form and the area element are given by

$$g_{ij} = \langle \partial_i f, \partial_j f \rangle \quad \text{and} \quad d\mu_g = \sqrt{\det(g)} dx^1 dx^2 ,$$

with $\det(g) := \det(g_{ij})$. Here and in the following $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$. In order to introduce the mean curvature vector we introduce first the projections onto the tangent and normal space. For a vector field V along f ($V : D \to \mathbb{R}^n$) we denote by $V^\top$ and $V^\perp$ the tangential and normal component of V along f, respectively. Then the mean curvature vector is given by

$$\vec{H} = \frac{1}{2} g^{ij} A_{ij} ,$$

with (g^{ij}) the inverse of the matrix (g_{ij}) and A_{ij} the second fundamental form defined by $A_{ij} = (\partial_i \partial_j f)^\perp$. Notice the factor $\frac{1}{2}$ in our definition of the mean curvature vector. Here and in the following we use the convention of summing over repeated indices. The trace free part of the second fundamental form is given by $A^\circ_{ij} = A_{ij} - g_{ij}\vec{H}$. For a normal vector $\vec{\phi}$ the Laplace in the normal bundle is defined as

$$\Delta^\perp \vec{\phi} = g^{ij} \nabla_i \nabla_j \vec{\phi} ,$$

with ∇_i the connection in the normal bundle.

In codimension one, the shape operator or Weingarten map is given by $-g^{ik}\langle \partial_k \partial_j f, \vec{n} \rangle$ where $\vec{n}$ is a locally defined normal vector field. The mean curvature and Gauss curvature are defined as $\frac{1}{2}$ times the trace of the shape operator and the determinant of the shape operator respectively. Further, the mean curvature vector and Laplace in the normal bundle have the simpler form

$$\vec{H} = -H\vec{n} \text{ and } \Delta^\perp \vec{H} = -(\Delta H)\vec{n}$$

with Δ the Laplace-Beltrami operator given by

$$\Delta f = \frac{1}{\sqrt{\det(g)}}\partial_i\left(\sqrt{\det(g)}g^{ij}\partial_j f\right).$$

Moreover,

$$g^{ik}g^{jl}A^\circ_{ij}\langle A^\circ_{kl},\vec{H}\rangle = 2\vec{H}(|\vec{H}|^2 - K).$$

2. A third condition at the boundary

Many proofs of uniqueness for elliptic equations of second order are based on comparison principles that do hot hold in general for higher order elliptic equations as the Willmore equation. Another approach to uniqueness was used by Pohozaev in the proof of the so-called Pohozaev-identity. The main idea is to use the invariances of the problem. This leads to equations involving only boundary integrals from which, under suitable assumptions, new informations at the boundary can be gained. This is the approach we shall follow in this section to prove the following result.

Proposition 2.1. *Let P be a plane in $\mathbb{R}^n$ and $\Omega \subset P$ be a smooth strictly star-shaped bounded domain. Let D be the unit disk in $\mathbb{R}^2$ and let $f : D \hookrightarrow \mathbb{R}^n$, $n \geq 3$, be a smooth Willmore immersion so that:*

1. *the boundary of the surface is given by $\partial\Omega$;*
2. *the surface touches tangentially the plane P along the boundary.*

Then, the second fundamental form of f satisfies $A(X, Y) = 0$ at the boundary for any X, Y smooth tangential vector fields.

Thanks to this result we have a third boundary condition. Being the Willmore problem of fourth order, this is not enough to obtain uniqueness via a unique continuation kind of result. For that we would need a fourth condition at the boundary. In the next section we will present how, in codimension one, Theorem 1.1 follows from Proposition 2.1 and the classification theorem of Bryant following an argument of Palmer [17]. Notice that Proposition 2.1, which we now prove, holds in any codimension.

We first observe that the boundary conditions immediately give the result in Proposition 2.1 under the additional hypothesis that one of the two vector fields is tangential to the boundary curve.

Lemma 2.2. *Assume the hypothesis of Proposition 2.1 and let τ be a smooth tangential vector field that is also tangential to the boundary curve $\partial\Omega$. Then for any smooth tangential vector field X we have*

$$A(\tau, X) = 0.$$

Proof. By definition,

$$A(\tau, X) = (D_\tau D_X f)^\perp = (D_\tau(Df \cdot X))^\perp = 0\,,$$

since $D_\tau(Df \cdot X)$ is a tangent vector, due to the boundary conditions. □

Remark 2.3. When the surface admits a parametrisation as the graph of a function $u : \Omega \to \mathbb{R}$ (see Remark 1.2), this lemma states the well known fact that the zero Dirichlet boundary conditions imply that $u_{\tau\tau}$ and $u_{\tau\nu}$ are zero on $\partial\Omega$. Here τ denotes the tangent to the boundary (planar) curve and ν the normal to it.

We prove Proposition 2.1 using the first variation of the Willmore functional and by choosing, inspired by the invariances of the problem, the right test-function. This is the point where we need the assumption that the domain Ω is strictly star-shaped. Let Σ be a compact surface with boundary $\partial\Sigma$. For a smooth family of immersions $h : \Sigma \times I \hookrightarrow \mathbb{R}^n$, $h(\cdot, t)$ with $t \in I \subset \mathbb{R}$ and $\phi = \frac{\partial}{\partial t}h$, the first variation of the Willmore functional is given by

$$\begin{aligned}\frac{d}{dt}\mathcal{W}(h) = &\int_\Sigma \langle \Delta^\perp \vec{H} + g^{ik}g^{jl}A^\circ_{ij}\langle A^\circ_{kl}, \vec{H}\rangle, \phi\rangle \, d\mu_g \\ &+ \int_{\partial\Sigma} \Big(2\langle \phi, (D_\eta \vec{H})^\perp\rangle - d\left(\langle \phi, \vec{H}\rangle\right)(\eta) - |\vec{H}|^2\langle \phi, Dh\cdot\eta\rangle\Big)\, ds_g\,,\end{aligned} \tag{2.1}$$

with $Dh \cdot \eta$ the interior conormal to the boundary and ds_g the length element of the boundary curve. The formula can be found in [14, Thm. 2.1]. In codimension one, the first integral can be rewritten as

$$-\int_\Sigma (\Delta H + 2H(H^2 - K))\phi^\perp \, d\mu_g\,.$$

Proof of Proposition 2.1. Without loss of generality we assume that $P = \{x \in \mathbb{R}^n : x^i = 0 \text{ for } i \geq 3\}$ and that Ω is strictly star-shaped with respect to zero. Since f is tangential along the boundary we may write $f = Df \cdot \xi$ on ∂D and for the interior conormal to the boundary $Df \cdot \eta$. The fact that Ω is star-shaped with respect to zero gives that

$$\langle f, Df\cdot\eta\rangle = \langle Df\cdot\xi, Df\cdot\eta\rangle = g(\xi,\eta) < 0 \text{ on } \partial D\,.$$

Due to Lemma 2.2 it is sufficient to show that $A(\eta,\eta) = 0$. In the formula for the first variation (2.1) we choose $\Sigma = D$ and $h(\cdot,t) = tf(\cdot)$. Then, $h(\cdot,1) = f$ and $\frac{\partial}{\partial t}h = f$. By this choice, the scaling invariance of the Willmore functional and the fact that f is a Willmore immersion we find

$$\begin{aligned} 0 &= \frac{d}{dt}\mathcal{W}(h)\Big|_{t=1} \qquad (2.2)\\ &= \int_{\partial D} \Big(2\langle f, (D_\eta \vec{H})^\perp\rangle - d\left(\langle f, \vec{H}\rangle\right)(\eta) - |\vec{H}|^2 \langle f, Df\cdot \eta\rangle \Big)\, ds_g\,. \end{aligned}$$

Making use of the boundary conditions we simplify further these boundary integrals. First of all, since f is tangential along the boundary

$$\langle f, (D_\eta \vec{H})^\perp\rangle = 0\,. \qquad (2.3)$$

To study the second term it is convenient to write $f^\top = Df\cdot\xi$. Notice that $f = f^\top$ at the boundary. Then,

$$\begin{aligned} d\langle f, \vec{H}\rangle(\eta) &= d\langle f^\perp, \vec{H}\rangle(\eta)\\ &= \langle D_\eta(f^\perp), \vec{H}\rangle + \langle f^\perp, D_\eta \vec{H}\rangle\,. \end{aligned}$$

Since $f^\perp = 0$ on ∂D, we find

$$\begin{aligned} d\langle f, \vec{H}\rangle(\eta) &= \langle D_\eta(f - Df\cdot\xi), \vec{H}\rangle\\ &= -\langle D_\eta(D_\xi f), \vec{H}\rangle = -\langle (D_\eta D_\xi f)^\perp, \vec{H}\rangle\\ &= -\langle A(\eta,\xi), \vec{H}\rangle\,. \qquad (2.4) \end{aligned}$$

Writing $\xi = g(\xi,\tau)\tau + g(\xi,\eta)\eta$ with $Df\cdot\tau$ the tangential vector to the boundary curve, we find

$$\begin{aligned} A(\xi,\eta) &= g(\xi,\tau)A(\tau,\eta) + g(\xi,\eta)A(\eta,\eta)\\ &= g(\xi,\eta)A(\eta,\eta) = 2g(\xi,\eta)\vec{H}\,, \qquad (2.5) \end{aligned}$$

where we have used Lemma 2.2. Then combining (2.4) and (2.5) we obtain

$$d\langle f, \vec{H}\rangle(\eta) = -2g(\xi,\eta)|\vec{H}|^2\,.$$

From this equation, (2.2) and (2.3) it follows

$$0 = \int_{\partial D} |\vec{H}|^2 g(\xi,\eta)\, ds_g\,.$$

Since Ω is strictly star-shaped, $g(\xi,\eta) < 0$ on ∂D and hence necessarily $H \equiv 0$ and $A(\eta,\eta)\equiv 0$ on the boundary. □

Remark 2.4. In the proof we have used the invariance with respect to rescaling. Using the other invariances we would get other boundary integrals from which it is possible to get other informations by making some extra assumptions. We comment on that and on other generalisations in Section 4.

Remark 2.5. Proposition 2.1 is true also on arbitrary compact surfaces with boundary whenever the term $g(\xi, \eta)$ is strictly positive (or strictly negative) at the boundary. This is for instance the case in an annulus when the interior conormal at the boundary points in the same direction at points on the same ray.

3. Proof of the main result

Here we have to restrict to the case of codimension one, that is to $n = 3$. In the previous section we have proved that the boundary of the surface consists of umbilic points. Theorem 1.1 follows from this observation using the classification result of Bryant (Theorem C.3 below; see [2] and [9]) as done by Palmer in [17, page 1587]. We present the proof for the sake of completeness. The restriction to the case of codimension one is due to the need of the classification theorem of Bryant to close the argument. An important point is that, Bryant in [2, Thm. B] shows that to Willmore surfaces one can associate a holomorphic function q (see (3.3) below). This result is analogous to the characterisation of surfaces with constant mean curvature via the observation that their Hopf function (see (3.1) below) is holomorphic (see [13, Chap.4]). To get his result Bryant uses the conformal Gauss map (see [9, Sect. 3]) through which a Willmore surface in $\mathbb{R}^3$ corresponds to a minimal surface in $\mathbb{S}^4_1$ (that is the unit sphere in $\mathbb{R}^4_1$). This construction goes back to Thomsen [21]. We present in the appendix a brief survey of the concepts and results we need.

Proof of Theorem 1.1. We follow [17, page 1587].

By a reparametrization we may assume that $f : D \hookrightarrow \mathbb{R}^3$ is conformal. Let (x^1, x^2) denote the coordinates in D and $z = x^1 + ix^2$ be the associated complex structure. Let $\vec{n} : D \to \mathbb{R}$ denote a unit normal vector field to the surface. Let φ denote the Hopf function associated to f, i.e.

$$\varphi = \frac{1}{2}((f_{x^1x^1}, \vec{n}) - (f_{x^2x^2}, \vec{n}) - 2i(f_{x^1x^2}, \vec{n})) = 2(f_{zz}, \vec{n}) \,, \tag{3.1}$$

with $\partial_z = \frac{1}{2}(\partial_{x^1} - i\partial_{x^2})$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_{x^1} + i\partial_{x^2})$.

By Proposition 2.1 we know that the Hopf function and the mean curvature are zero at the boundary. Then, differentiating both functions along the boundary ∂D in the tangential direction one finds $0 = \partial_\theta \varphi$ and $0 = \partial_\theta H$. Since $\partial_\theta = i(z\partial_z - \bar{z}\partial_{\bar{z}})$ we get

$$z\varphi_z = \bar{z}\varphi_{\bar{z}} \text{ and } zH_z = \bar{z}H_{\bar{z}} \text{ on } \partial D. \tag{3.2}$$

Bryant in [2, Thm. B] (see Lemma B.4 and Proposition B.5 in the appendix and also [9, Sect.5, Prop.1]) proves that the function q defined by

$$q = \begin{cases} \frac{1}{4}\varphi^2(H^2 + \Delta \log \varphi) & \text{if } \varphi \neq 0\,, \\ -\varphi_z H_z & \text{if } \varphi = 0\,, \end{cases} \tag{3.3}$$

is a holomorphic function if f is a Willmore immersion. Here, $\Delta = 4e^{-\mu}\partial_z\partial_{\bar{z}}$ and e^{μ} denotes the conformal factor (i.e. $e^{\mu} = 2f_z \cdot f_{\bar{z}} = f_u \cdot f_u = f_v \cdot f_v$).

Since $\varphi = 0$ on ∂D using (3.2) we have

$$-q = \varphi_z H_z = z\bar{z}\varphi_z H_z = (\bar{z})^2 \varphi_{\bar{z}} H_z \text{ on } \partial D\,.$$

By the Codazzi equation $\varphi_{\bar{z}} = e^{\mu} H_z$ (see (B3) in the appendix) and using (3.2) we get

$$-q = (\bar{z})^2 e^{\mu} H_z H_z = (\bar{z})^3 e^{\mu} z H_z H_z = (\bar{z})^4 e^{\mu} H_{\bar{z}} H_z \text{ on } \partial D\,.$$

Hence, $z^4 q$ is a holomorphic function that is real valued on ∂D. By the maximum principle, $z^4 q = a \in \mathbb{R}$ in D. Since q is holomorphic, we get that necessarily $a = 0$ and hence $q \equiv 0$ in D.

The classification theorem of Bryant [2, Thm. D and E] (see Theorem C.3 in the appendix and also [9, Sect. 6, Prop.]) yields then that f is a piece of a sphere or, after a Möbius transformation, a piece of a minimal surface.

Due to the boundary conditions, the surface cannot be a piece of a proper sphere. Then, there exists a conformal transformation h in $\mathbb{R}^3$ such that $h \circ f$ is a minimal surface with a boundary component made of umbilic points. Here we use that the set of umbilic points is a conformal invariant, see [2, page 32] or [12, Lemma P6.7]. Then, the Hopf function of $h \circ f$ is holomorphic. Being also zero at the boundary, it is necessarily identically zero. Here we use that the Hopf function of surfaces with constant mean curvature is holomorphic (see (B3)). The claim follows. □

Proof of Corollary 1.5. Let $p \in \mathbb{S}^2$ be a point with respect to which the boundary curve is star-shaped in the spherical geometry. Notice that the curve is also star-shaped with respect to $-p$. We may than assume that $f^{-1}(\{-p\}) = \emptyset$. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the inversion with respect to the sphere of radius $\sqrt{2}$ and center $-p$. By the assumptions, $\Phi \circ f : D \hookrightarrow \mathbb{R}^3$ satisfies the hypotheses of Theorem 1.1. The claim follows. □

4. Generalisations

In Section 2, in the proof of Proposition 2.1 we use the scaling invariance of the Willmore functional. One could use also the invariance with respect to translations and rotations. We give here the boundary integrals one would get. Here $f : \Sigma \hookrightarrow \mathbb{R}^3$ is a smooth Willmore immersion, Σ is compact and has boundary $\partial\Sigma$. For completeness we recall the formula one gets using the scaling invariance. Choosing in (2.1) $h(\cdot,t) = tf(\cdot)$ and computing the derivative at $t = 1$ we find

$$0 = \int_{\partial\Sigma} \left(2\langle f, (D_\eta \vec{H})^\perp \rangle - d\left(\langle f, \vec{H} \rangle \right)(\eta) - |\vec{H}|^2 \langle f, Dh \cdot \eta \rangle \right) \, ds_g \, .$$

To use the invariance with respect to translation we choose in (2.1) $h(\cdot,t) = f(\cdot) + t\vec{V}$ with $\vec{V}$ a fixed vector in $\mathbb{R}^3$. Then, for $t = 0$

$$0 = \int_{\partial\Sigma} \left(2\langle \vec{V}, (D_\eta \vec{H})^\perp \rangle - d\left(\langle \vec{V}, \vec{H} \rangle \right)(\eta) - |\vec{H}|^2 \langle \vec{V}, Df \cdot \eta \rangle \right) \, ds_g \tag{4.1}$$

We consider rotations choosing $h(\cdot,t) = R(t)f(\cdot)$ in (2.1) with $R(t) \in SO(n)$ smooth and such that $R(0) = Id$. Then, for $t = 0$

$$\begin{aligned} 0 \;=\; & \int_{\partial\Sigma} \Big(2\langle R'(0)f, (D_\eta \vec{H})^\perp \rangle - d\left(\langle R'(0)f, \vec{H} \rangle \right)(\eta) \\ & \qquad\qquad - |\vec{H}|^2 \langle R'(0)f, Df \cdot \eta \rangle \Big) \, ds_g \, . \end{aligned}$$

Not only in the case of Dirichlet boundary conditions, but also for other choices of boundary conditions these equations together with some extra assumptions yield new informations at the boundary. For instance, with the result of Proposition 2.1, under the same assumptions and taking $n = 3$, (4.1) gives

$$0 = \int_{\partial\Sigma} \langle \vec{V}, \vec{n} \rangle D_\eta H \, ds_g \, ,$$

from which, under additional hypotheses (as a fixed sign for $\langle \vec{V}, \vec{n} \rangle$ and also assuming that H has a fixed sign in Ω), $DH = 0$ on $\partial\Sigma$ can be derived. That for fourth order problems one needs a sign condition in order to derive via the Pohozaev identity uniqueness is well established in the literature, see for instance [16] and [10, Sec. 7.5.1].

Our uniqueness result can be directly generalised to critical points of the functional

$$\widetilde{\mathcal{W}}(f) = \int_\Sigma (|\vec{H}|^2 - cK) \, d\mu_g \, ,$$

defined on smooth immersions $f : \Sigma \hookrightarrow \mathbb{R}^n$. Here K denotes the Gauss curvature and c is some real constant. Notice that one needs $0 \le c \le 1$ so that $\widetilde{\mathcal{W}}(f)$ is bounded below. Our result applies immediately since the functional $\widetilde{\mathcal{W}}$ has the same critical points as the Willmore functional. Similarly the extensions discussed above apply also to this functional. Notice that, the difference between $\mathcal{W}(f)$ and $\widetilde{\mathcal{W}}(f)$ is a fixed constant for surfaces satisfying the same Dirichlet boundary conditions and of the same topological type.

Appendix A. The conformal Gauss map

The conformal Gauss map associates to each point of a two-dimensional surface its central sphere which can be considered as a point in the unit sphere $\mathbb{S}^4_1$ in the five dimensional Minkowski space. This map is important in the study of Willmore surfaces since by this transformation, a Willmore surface corresponds (away from its umbilic points) to a minimal surface in $\mathbb{S}^4_1$. This observation goes back to Thomsen, [21]. In this section we describe the geometric constructions that lead to the definition of the conformal Gauss map. We follow the presentation in [9].

Let $f : \Sigma \to \mathbb{R}^3$ be a smooth immersion of a two-dimensional orientable surface Σ. Let $n : \Sigma \to \mathbb{S}^2$ be a normal vectorfield. We consider the central sphere of f at $f(s)$, $s \in \Sigma$. This is the 2-dimensional sphere in $\mathbb{R}^3$ going through $f(s) \in \mathbb{R}^3$ and with mean curvature equal to the mean curvature of f in $f(s)$. We denote the central sphere by $S_r(p)$ with $r = r(s) \in \mathbb{R} \cup \{\pm\infty\}$ the 'radius' and $p = p(s) \in \mathbb{R}^3$ the center. If $H(s)$ denotes the mean curvature of f in $f(s)$, then $r = 1/H(s)$ and $p = f(s) + rn(s)$.

Let Φ denote the inverse of the stereographic projection into $\mathbb{R}^3$ given by $\Phi : \mathbb{R}^3 \to \mathbb{S}^3 \setminus \{(0,0,0,1)^t\}$ with

$$\Phi((y^1, y^2, y^3)^t) = \frac{1}{1 + \|y\|^2}(2y^1, 2y^2, 2y^3, \|y\|^2 - 1)^t .$$

Since Φ is conformal, $\Phi(S_r(p)) \subset \mathbb{S}^3$ is a two-dimensional sphere. There exists a unique three-dimensional sphere that intersects $\mathbb{S}^3$ orthogonally along $\Phi(S_r(p))$. We denote its center by $Z(\Phi(S_r(p)))$. In this way we get a mapping

$$\begin{aligned} f_1 : \Sigma &\to \mathbb{R}^4 \cup \{\infty\} , \\ s &\mapsto Z(\Phi(S_r(p))) = (2p, \|p\|^2 - r^2 - 1)\frac{1}{\|p\|^2 - r^2 + 1} , \end{aligned} \tag{A1}$$

where, as before, $r = r(s) = 1/H(s)$ and $p = p(s) = f(s) + rn(s) \in \mathbb{R}^3$. Here $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$. (For this formula it is convenient to see Φ as the restriction to $\mathbb{R}^3$ of $G : \mathbb{R}^4 \to \mathbb{R}^4$ with G the inversion with respect to the 3-sphere of radius $\sqrt{2}$ and center $(0,0,0,1)^t$. If $\Phi(S_r(p))$ is an equatorial sphere in $\mathbb{S}^3$, then $f_1(s) = \infty$. This is the case if $\|p\|^2 + 1 = r^2$.)

Notice that in (A1) we write a vector in $\mathbb{R}^4$ via two components. The first is a vector in $\mathbb{R}^3$, while the second is a real number. Similarly, in the following we write elements in $\mathbb{R}^5$ via three components. The first is a vector in $\mathbb{R}^3$, while the other two components are real numbers. The formulas become nicer with this convention.

Now, to take care of the points sent to ∞, we look at $\mathbb{R}^4$ as the subset $\{[y,1] : y \in \mathbb{R}^4\}$ of $\mathbb{RP}^4$. We get then the map

$$\begin{aligned} f_2 : \Sigma &\to \mathbb{RP}^4 , \\ s &\mapsto \big[p, \frac{1}{2}(\|p\|^2 - r^2 - 1), \frac{1}{2}(\|p\|^2 - r^2 + 1)\big] . \end{aligned}$$

As a final step we consider as a target $\mathbb{R}^5$ with the Lorentzian metric

$$g(X,Y) = \sum_{i=1}^{4} X^i Y^i - X^5 Y^5 , \quad X, Y \in \mathbb{R}^5, \tag{A2}$$

signature $(+,+,+,+,-)$, and the map

$$\begin{aligned} Y : \Sigma &\to (\mathbb{R}^5, g) , \\ s &\mapsto \frac{1}{r}\big(p, \frac{1}{2}(\|p\|^2 - r^2 - 1), \frac{1}{2}(\|p\|^2 - r^2 + 1)\big) . \end{aligned}$$

This is the conformal Gauss map associated to $f : \Sigma \to \mathbb{R}^3$. The normalisation factor $1/r$ is chosen in such a way that

$$Y(\Sigma) \subset \mathbb{S}^4_1 := \{Y \in (\mathbb{R}^5, g) : g(Y,Y) = 1\} ,$$

i.e. the image of Y is a subset of the unit sphere in $(\mathbb{R}^5, g)$. Since $r = r(s) = 1/H(s)$ and $p = p(s) = f(s) + rn(s)$ we have

$$\begin{aligned} Y(s) &= H(s) \left(f(s), \frac{1}{2}(\|f(s)\|^2 - 1), \frac{1}{2}(\|f(s)\|^2 + 1) \right) \\ &\quad + (n(s), (f(s), n(s)), (f(s), n(s))) , \end{aligned}$$

with $(\cdot,\cdot)$ the Euclidean scalar product in $\mathbb{R}^3$. It is convenient to write the conformal Gauss map as

$$\begin{aligned} s &\mapsto Y(s) = H(s)X(s) + T(s) \quad \text{with} \\ X(s) &= \left(f(s), \frac{1}{2}(\|f(s)\|^2 - 1), \frac{1}{2}(\|f(s)\|^2 + 1) \right) \qquad \text{(A3)} \\ \text{and } T(s) &= (n(s), (f(s), n(s)), (f(s), n(s))) . \end{aligned}$$

Notice that $g(X(s), X(s)) = 0$ while $g(T(s), T(s)) = 1$.

We will see in the next section that the conformal Gauss map is (indeed) conformal with degeneracies at the umbilic points of Σ. Further, we study the properties of the conformal Gauss map associated to a Willmore surface.

Remark A.1. Notice that if $f : \Sigma \to \mathbb{R}^3$ is a sphere than the image of the associated conformal Gauss map is a fixed point in $\mathbb{S}^4_1$.

Remark A.2. The name conformal Gauss map has been used by Bryant. Thomsen in [21] used instead the concept of sphere congruence. A *sphere congruence* is a smooth mapping $S : \Sigma \to \{\text{spheres in } \mathbb{R}^3\}$ with Σ a two-dimensional manifold. This mapping induces a new mapping $Y : \Sigma \to \mathbb{S}^4_1$ which assigns to each sphere in $\mathbb{R}^3$ a point in $\mathbb{S}^4_1$ with the same construction as above. That is, to a sphere in $\mathbb{R}^3$ with center p and radius r we associate the vector

$$\frac{1}{r}(p, \frac{1}{2}(\|p\|^2 - r^2 - 1), \frac{1}{2}(\|p\|^2 - r^2 + 1)) \in \mathbb{R}^5 . \tag{A4}$$

In the same way we may associate to points in $\mathbb{R}^3$ a vector in $\mathbb{R}^5$. Renormalizing (A4) by multiplying it by r and taking $r \to 0$ we see that

$$\mathbb{R}^3 \ni x \mapsto X = (x, \frac{1}{2}(\|x\|^2 - 1), \frac{1}{2}(\|x\|^2 + 1)) \in \mathbb{R}^5 . \tag{A5}$$

Notice that the images of points in $\mathbb{R}^3$ are $X \in L$ with $L := \{X \in \mathbb{R}^5 \text{ such that } g(X, X) = 0 \text{ and } X^5 - X^4 = 1\}$. The mapping given in (A5) is an isometry from $\mathbb{R}^3$ to L.

We need also the concept of *enveloping surface* of a sphere congruence S. This is a map $f : \Sigma \to \mathbb{R}^3$ such that for all $s \in \Sigma$ it holds

$$f(s) \in S(s) \ \text{ and } \ df(s)(T_s\Sigma) \subset T_{f(s)}S(s) . \tag{A6}$$

Equivalent relations may be stated in $(\mathbb{R}^5, g)$. Indeed, denoting by $X(s)$ the representative in $\mathbb{R}^5$ of $f(s)$ according to (A5), the formulas in (A6) are equivalent to

$$g(X(s), Y(s)) = 0 \ \text{ and } \ g(X(s), dY(s)) = 0 . \tag{A7}$$

In the proof of the classification theorem of Bryant in Appendix C we will see that if f is a Willmore surface and a certain holomorphic differential is identically zero, then f is an enveloping surface for its own conformal Gauss map.

For more informations on sphere congruence and conditions on the existence of a enveloping surface we refer to [12] and the references therein.

Appendix B. A holomorphic differential for Willmore surfaces

The results we collect here are due to Thomsen [21] and Bryant [2]. We follow the presentation in [9].

Let $f : \Sigma \to \mathbb{R}^3$ (with the standard scalar product) be a smooth immersion of an orientable surface. We have thus fixed a conformal structure on Σ. Locally there exists conformal coordinates. Let denote this conformal coordinates by u and v. We associate a complex coordinate to this conformal structure by considering $z = u + iv$. The first fundamental form of f is given by $I = e^\mu dz d\bar{z}$ with e^μ the conformal factor. Let n be a unit normal field along the surface. For the second fundamental form we have the representation

$$II = \mathrm{Re}\{\varphi dz^2 + He^\mu dz d\bar{z}\}$$

with H the mean curvature of f, e^μ the conformal factor and φ the Hopf differential given by

$$\varphi = \frac{1}{2}((f_{uu}, n) - (f_{vv}, n) - 2i(f_{uv}, n)) = 2(f_{zz}, n)\,. \tag{B1}$$

Notice that $\partial_z = \frac{1}{2}(\partial_u - i\partial_v)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v)$.

We also have

$$\begin{aligned} f_{zz} = \mu_z f_z + \tfrac{1}{2}\varphi n\,, \quad f_{z\bar{z}} = \tfrac{1}{2} He^\mu n\,, \\ \text{and} \quad n_z = -Hf_z - \varphi e^{-\mu} f_{\bar{z}}\,, \end{aligned} \tag{B2}$$

and the integrability conditions

$$\begin{aligned} \varphi_{\bar{z}} &= e^\mu H_z \quad \text{(Equation of Codazzi)}, \\ |\varphi|^2 e^{-2\mu} &= H^2 - K \quad \text{(Equation of Gauss)}, \end{aligned} \tag{B3}$$

with K the Gaussian curvature of f.

With the same choice of complex coordinate, we find for the conformal Gauss map associated to f as given in (A3) that

$$\begin{aligned} &Y_z = H_z X - \varphi e^{-\mu} X_{\bar{z}}, \\ &g(Y_z, Y_z) = 0 \text{ and } g(Y_z, Y_{\bar{z}}) = (H^2 - K)(f_z, f_{\bar{z}})\,, \end{aligned} \tag{B4}$$

with g the metric given in (A2). Here we have used the formula for n_z in (B2) and (B3). The induced metric is given by $ds_Y^2 = (H^2 - K) ds_f^2$. Thus Y is a conformal map (with degeneracies at the umbilic points of f) with respect to the conformal structure induced on Σ by f and an immersion away from the umbilic points of f.

For Willmore immersions (i.e. solutions to (1.2)) the formulas in (B4) imply immediately the following.

Proposition B.1. *Let $f : \Sigma \to \mathbb{R}^3$ be a Willmore surface and $Y : \Sigma \to \mathbb{S}^4_1$ be the conformal Gauss map associated to f. Then,*

$$\mathit{Area}(Y) = \int_\Sigma ds_Y^2 = \int_\Sigma (H^2 - K)\, ds_f^2\,.$$

Moreover, if Y is a minimal surface, then $f : \Sigma \to \mathbb{R}^3$ is a Willmore immersion.

Remark B.2. We call Willmore surface a solution to equation (1.2). Notice that these are also critical points for the functional $\int (H^2 - K)\, dS$ with respect to interior variations. In our setting, $\int K\, dS$ is equal to a constant. See the discussion in the introduction.

Further, the converse of Proposition B.1 is also true.

Proposition B.3. *An immersion $f : \Sigma \to \mathbb{R}^3$ is a Willmore immersion if and only if the associated conformal Gauss map is an harmonic map.*

Idea of the proof. One first shows that

$$\Delta Y + 2(H^2 - K)Y = (\Delta H + 2(H^2 - K)H)X\,. \tag{B5}$$

with X as defined in (A3). Since $Y \in \mathbb{S}^4_1$, Y is normal to $\mathbb{S}^4_1$ and therefore taking the tangential component in (B5) we find

$$(\Delta Y)^{TQ} = [(\Delta H + 2(H^2 - K)H)X]^{TQ}\,.$$

If f is Willmore, we directly get $(\Delta Y)^{TQ} = 0$ and so Y is a harmonic map. On the other hand if Y is an harmonic map, Y is a critical point for the Dirichlet energy. Being Y conformal, it is also a critical point for the area functional. Proposition B.1 yields that f is a Willmore immersion. ☐

We consider now the quartic differential

$$Q : \quad g(Y_{zz}, Y_{zz})dz^4\,.$$

Lemma B.4. *The quartic differential $g(Y_{zz}, Y_{zz})dz^4$ can be written as qdz^4 with*

$$q = \begin{cases} \frac{1}{4}\varphi^2(H^2 + \Delta_f \log \varphi) & \text{where } \varphi \neq 0\,, \\ -\varphi_z H_z & \text{where } \varphi = 0, \end{cases}$$

with $\Delta_f = 4e^{-\mu}\partial_z\partial_{\bar z}$ and φ the Hopf differential given in (B1).

Proof. Starting from the formula for Y_z given in (B4) and differentiating it once again we find

$$Y_{zz} = H_{zz}X + H_z X_z - (\varphi e^{-\mu})_z X_{\bar z} - \varphi e^{-\mu} X_{z\bar z}\,. \tag{B6}$$

For the last term starting from the formula for X given in (A3) and using (B2) one gets

$$X_{z\bar z} = \frac{1}{2} H e^{\mu} T + \frac{e^{\mu}}{2}(0, 1, 1)\,. \tag{B7}$$

Using that $g(X,X) = 0$, $g(X,X_z) = 0$, $g(X_z,X_z) = 0$, $g(X_z,X_{\bar z}) = \frac{1}{2}e^\mu$, $g(X,T) = 0$ and $g(X_z,T) = 0$, formulas (B6) and (B7) yield

$$g(Y_{zz},Y_{zz}) = \varphi H_{zz} - H_z(\varphi e^{-\mu})_z e^\mu + \frac{1}{4}H^2\varphi^2\,.$$

The claim follows using the equation of Codazzi (B3). □

Proposition B.5. *If $f:\Sigma\to\mathbb{R}^3$ is a Willmore surface, then Q is a holomorphic quartic differential.*

Proof. Let $Y : M \to \mathbb{S}^4_1$ be the conformal Gauss map associated to f. Recalling (B4) we have

$$g(Y_z,Y_z) = 0 \;\text{ and }\; g(Y_z,Y_{\bar z}) = \frac{1}{2}(H^2-K)e^\mu\,.$$

Let α denote the second fundamental form of Y, i.e. $\alpha_{ij} = (Y_{ij})^\perp$ and η denote the mean curvature vector, i.e.

$$\eta = \frac{\alpha_{11}+\alpha_{22}}{2E} \;\text{ with }\; E = (H^2-K)e^\mu\,.$$

Being f a Willmore immersion, Proposition B.3 gives us that Y is an harmonic map. On the other hand,

$$Y_{z\bar z} = \frac{1}{2}E\eta$$

and so we get that the mean curvature vector η is normal to $\mathbb{S}^4_1\subset\mathbb{R}^5$. Since $\mathbb{S}^4_1$ is the unit sphere in $(\mathbb{R}^5,g)$, $\eta=\beta Y$ for some $\beta\in\mathbb{R}$ and, by a direct computation, one finds $\eta = -Y$. Therefore, we have

$$Y_{z\bar z} = -\frac{1}{2}EY\,.$$

This is the crucial information for showing that Q is holomorphic. Indeed,

$$\begin{aligned}\partial_{\bar z} g(Y_{zz},Y_{zz}) &= 2g((Y_{z\bar z})_z,Y_{zz})\\ &= -E_z g(Y,Y_{zz}) - Eg(Y_z,Y_{zz}) = 0\,,\end{aligned}$$

since $g(Y,Y_z)=0$ and $g(Y_z,Y_z)=0$. □

Appendix C. The classification theorem of Bryant

We follow once again the presentation in [9].

Let $f:\Sigma\to\mathbb{R}^3$ be a Willmore surface that is not totally umbilic. By Theorem C in [2] the set of umbilic points of f is closed and it has no interior. Let $\Sigma\setminus\Sigma'$ be the preimages of the umbilic points of f. Eschenburg, Tribuzy [11] prove that one can smoothly define on each point of $Y(\Sigma)$ the tangent space. More precisely, the map

$\Sigma' \ni s \mapsto dY_s(T_s\Sigma)$ can be smoothly extended to all of Σ. Therefore the normal bundle is defined everywhere. The induced metric on the normal bundle has signature $(+,-)$ and so we may find two real normal vectors N_1 and N_2 such that

$$g(N_i, N_i) = 0\,,\ i = 1,2, \text{ and } g(N_1, N_2) = 1\,. \tag{C1}$$

Lemma C.1. *One has*

$$\begin{aligned} g(Y_{zz}, Y_{zz}) &= 2g(Y_{zz}, N_1)g(Y_{zz}, N_2)\,, \\ \text{and } \partial_{\bar z} g(Y_{zz}, N_i) &= (-1)^{i-1} g(N_{1,\bar z}, N_2) g(Y_{zz}, N_i)\,, \end{aligned} \tag{C2}$$

for $i = 1, 2$.

Proof. Since $g(Y_z, Y) = 0$ and $g(Y_z, Y_z) = 0$ one sees that Y_{zz} lies in the span of Y_z, N_1 and N_2. The first claim follows from (B4) and (C1).

For the second equality one first notices that

$$\partial_{\bar z} g(Y_{zz}, N_i) = g(Y_{zz}, N_{i,\bar z})\,,$$

and that $N_{i,\bar z}$ lies in the span of Y_z and N_i. □

Notice that this lemma gives another proof of Proposition B.5. The next result gives a crucial observation for the proof of Bryant's classification theorem.

Proposition C.2. *If* $g(Y_{zz}, Y_{zz}) \equiv 0$*, then* $g(Y_{zz}, N_j) \equiv 0$ *and* $N_{jz} = \lambda(z) N_j$ *for* $j = 1$ *or* 2 *and some scalar function* λ*.*

Proof. Since the functions $g(Y_{zz}, N_j)$ satisfy the differential equation given in (C2) and $g(N_{1,\bar z}, N_2)$ is bounded on compact subsets of Σ, it follows from Carleman's theorem [5] that each $g(Y_{zz}, N_j)$ has isolated zeroes or it is identically zero. Therefore from the first equality in Lemma C.1 and $g(Y_{zz}, Y_{zz}) \equiv 0$, we infer that $g(Y_{zz}, N_j) \equiv 0$ for $j = 1$ or 2. This implies that $g(Y_z, N_{j,z}) = 0$. Further, $g(Y, N_{j,z}) = 0$, $g(N_{j,z}, Y_{\bar z}) = 0$ and $g(N_j, N_{j,z}) = 0$. The claim follows. □

We are now ready to state the classification theorem of Bryant.

Theorem C.3 (Bryant classification theorem)**.** *Let* $f : \Sigma \to \mathbb{R}^3$ *be a Willmore immersion and* $Y : \Sigma \to \mathbb{S}^4_1$ *the associated conformal Gauss map. Assume further that* $g(Y_{zz}, Y_{zz}) \equiv 0$*. Then* f *is either totally umbilic or* f *is the Möbius transform of a minimal immersion.*

Proof. If f is not totally umbilic, by the discussion at the beginning of the section we have two real normal vectors N_1 and N_2 such that $g(N_i, N_i) = 0$, $g(N_i, Y) = 0$, $g(N_i, Y_z) = 0 = g(N_i, Y_{\bar z})$, for $i = 1, 2$, and $g(N_1, N_2) = 1$. According to the definition given in Remark A.2 and the characterisation in (A7) $[N_1]$ and $[N_2]$ ($[N_i] \in \mathbb{RP}^4$) are enveloping surfaces for the conformal Gauss map associated to f. Even more, we may choose

$$N_1(s) = X(s) = (f(s), \frac{1}{2}(\|f(s)\|^2 - 1), \frac{1}{2}(\|f(s)\|^2 + 1))\,.$$

That is, one of the enveloping surfaces is the Willmore immersion itself. The other normal direction $N_2 = \hat{X}$ is called the conformal transform of X.

Since $g(Y_{zz}, Y_{zz}) \equiv 0$, $N_{2,z} = \lambda(z) N_2$ by Proposition C.2. This differential equation and the fact that N_2 is a real vector imply that $[N_2]$ is a well defined fixed vector. Therefore $[N_2] = [\hat{X}]$ can be identified with a point in $\mathbb{S}^3$ and as such it is the image of a fixed point $\hat{x}$ in $\mathbb{R}^3 \cup \{\infty\}$. Via an inversion h in $\mathbb{R}^3$ we can send $\hat{x}$ to infinity. Accordingly all the spheres that passes through $\hat{x}$ are sent to planes. These planes are, by construction, the central spheres of the immersion $h \circ f : \Sigma \to \mathbb{R}^3$. Therefore $h \circ f$ is a minimal immersion. □

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