Expected lifetime of h-conditioned Brownian motion

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Abstract

Let τ_Ω denote the lifetime of Brownian motion in a domain $\Omega \subset \mathbb{R}^m$. We obtain the asymptotic behaviour of its expected lifetime $E_x^y[\tau_\Omega]$ as $y \to x$, where the Brownian motion is conditioned to start at x and to exit at y.

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1 Introduction

Let Ω be an open and connected set in Euclidean space $\mathbb{R}^m$ with $m \geq 2$ and let Δ be the Dirichlet Laplacian for Ω acting in $L^2(\Omega)$. Let $p_\Omega(x, y; t)$ denote the Dirichlet heat kernel on $\Omega \times \Omega \times (0, \infty)$ associated to the parabolic operator $-\Delta + \frac{\partial}{\partial t}$. It is well known that the resolvent of $-\Delta$ has an integral kernel $G_\Omega(x, y)$ on $\Omega \times \Omega$ given by

$$G_\Omega(x, y) = \int_0^\infty p_\Omega(x, y; t)dt, \tag{1}$$

whenever the integral in the right hand side of (1) converges. This is always the case if $m \geq 3$ since by monotonicity of the Dirichlet heat kernel

$$0 < p_\Omega(x, y; t) \leq p(x, y; t), \tag{2}$$

where

$$p(x, y; t) = p_{\mathbb{R}^m}(x, y; t) = \frac{1}{(4\pi t)^{m/2}}\, e^{-|x-y|^2/(4t)}. \tag{3}$$

Hence we have that

$$0 < G_\Omega(x, y) \leq G(x, y) = \frac{\Gamma(\frac{1}{2}m - 1)}{4\pi^{m/2}}\, |x - y|^{2-m}. \tag{4}$$

In Proposition 6 below we will see that the integral in (1) converges for planar sets under very mild conditions.

The main subject of this paper is the function $H_\Omega : \Omega \times \Omega \to [0, \infty]$ defined by

$$H_\Omega(x, y) = \int_\Omega \frac{G_\Omega(x, z)G_\Omega(z, y)}{G_\Omega(x, y)} dz. \tag{5}$$

For Ω in $\mathbb{R}^2$ we assume that $\mathbb{R}^2 \backslash \Omega$ contains a compact set with strictly positive logarithmic capacity. See also Proposition 6. For sufficiently smooth domains the function $H_\Omega(x, y)$ can be extended to $\overline{\Omega} \times \Omega$. Estimates for $H_\Omega(x, y)$ with $x \in \partial\Omega$ are discussed in Theorem 4.

The function H_Ω shows up in analysis when studying positivity preserving properties of systems of second order elliptic partial differential equations [13, 15, 12]. In probability, see [9, 10], $H_\Omega(x, y)$ for $\Omega \subset \mathbb{R}^m$ with $m \geq 2$ and $x \in \Omega$ is the expectation of the lifetime τ_Ω of Brownian motion $\{X(t)\}_{t\geq 0}$ killed on exiting Ω, starting in x, and conditioned to exit at y:

$$H_\Omega(x, y) = \mathbb{E}_x^y \left[\tau_\Omega\right]. \tag{6}$$

We may extend this formula for $y \in \Omega \subset \mathbb{R}^m$ with $m \geq 2$ to find

$$H_\Omega(x, y) = \lim_{\varepsilon \downarrow 0} \mathbb{E}_x \left[\tau_{\Omega\backslash B_\varepsilon(y)} | X(\tau_{\Omega\backslash B_\varepsilon(y)}) \in \partial B_\varepsilon(y)\right], \tag{7}$$

where $B_\varepsilon(y) = \{x \in \mathbb{R}^m; |x-y| < \varepsilon\}$. Many authors have investigated the behaviour of H_Ω in terms of the geometry of Ω. See [5–8,11]. It is well known that H_Ω is continuous on $\Omega \times \Omega$ whenever H_Ω is finite and $H_\Omega(x,y)$ is positive for $x \neq y$. Moreover, if $m \geq 2$ then $\lim_{y\to x} H_\Omega(x,y) = 0$ for any $x \in \Omega$.

The main results of this paper concern the behaviour of H_Ω near this minimum and are stated in Section 2. The proofs of the Theorem 1 and Proposition 6 are deferred to Sections 3 and 5 respectively. In Section 4 we give sketch of the proof of Theorem 4 .

2 Main results

The function H_Ω contains G_Ω and its iterate $G_\Omega \circ G_\Omega$. Note that the condition

$$\lambda := \inf \operatorname{spec}(-\Delta) > 0 \tag{8}$$

guarantees that both these terms are well defined. Indeed, G_Ω is defined by the now convergent integral in (1). Its iterate is also well defined since by (1), Fubini's theorem and the semigroup property for heat kernels

$$\begin{aligned} &\int_\Omega G_\Omega(x,z)G_\Omega(z,y)\ dz \\ &\quad = \int_0^\infty dt_1 \int_0^\infty dt_2 \int_\Omega p_\Omega(x,z;t_1)\ p_\Omega(z,y;t_2)\ dz \\ &\quad = \int_0^\infty dt_1 \int_0^\infty dt_2\ p_\Omega(x,y;t_1+t_2) = \int_0^\infty t\, p_\Omega(x,y;t)\, dt. \end{aligned} \tag{9}$$

Our first result states the precise asymptotic behaviour of $H_\Omega(x,y)$ for y near an interior point x.

Theorem 1. *Let Ω be an open and connected set in $\mathbb{R}^m$, and let $x \in \Omega$.*

(i) If $m \geq 5$ then for $y \to x$

$$H_\Omega(x,y) = \tfrac{1}{2(m-4)}\, |y-x|^2 + O(|y-x|^{m-2}). \tag{10}$$

(ii) If $m = 4$ and $\lambda > 0$ then for $y \to x$

$$H_\Omega(x,y) = \tfrac{1}{2}|y-x|^2 \log \frac{1}{|y-x|} + O(|y-x|^2). \tag{11}$$

(iii) If $m = 3$ and $\lambda > 0$ then

$$\int_0^\infty t\, p_\Omega(x,x;t)dt < \infty, \tag{12}$$

and for $y \to x$

$$H_\Omega(x,y) = 4\pi \left(\int_0^\infty t\, p_\Omega(x,x;t)dt \right) |y-x| + o(|y-x|). \tag{13}$$

(iv) If $m = 2$ and $\lambda > 0$ then (12) holds, and for $y \to x$

$$H_\Omega(x,y) = 2\pi \left(\int_0^\infty t\, p_\Omega(x,x;t)dt \right) \frac{-1}{\log|y-x|} + o(\frac{-1}{\log|y-x|}). \quad (14)$$

Remark 2. *It follows from the proof of Theorem 1 that the remainder estimates in (10) and (11) are uniform on compact subsets of Ω.*

Remark 3. *Note that, by (9), (12) is equivalent with*

$$\int_\Omega G_\Omega(x,z)^2 dz < \infty. \quad (15)$$

Note that $\int_\Omega G_\Omega(x,z)^2 dz = \infty$ for any $\Omega \subseteq \mathbb{R}^m$ with $m \geq 4$.

The next result concerns the asymptotic behaviour of H_Ω when one of the points lies on the boundary. It is well known that if $\partial\Omega$ is sufficiently smooth and if $x_0 \in \partial\Omega$ then

$$\widetilde{H}_\Omega(x_0,y) := \lim_{x \to x_0} H_\Omega(x,y) \quad (16)$$

exists and is non-trivial. Indeed, one has $G_\Omega(x_0,y) = 0$ and $\frac{\partial}{\partial\nu_x} G_\Omega(x_0,y) = K_\Omega(x_0,y)$, where $K_\Omega(x_0,y)$ is the Poisson kernel at $x_0 \in \partial\Omega$. It follows that

$$\widetilde{H}_\Omega(x_0,y) = \int_\Omega \frac{G_\Omega(x_0,z)K_\Omega(z,y)}{K_\Omega(x_0,y)} dz. \quad (17)$$

For general domains the asymptotic behaviour of $\lim_{x\to x_0} H_\Omega(x,y)$ as $y \to x_0$ will depend on the particular subsequence. We have the following in the case where $y - x_0$ is perpendicular to the tangent plane at x_0.

Theorem 4. *Let Ω be an open and connected set in $\mathbb{R}^m$ with $\partial\Omega$ of class C^2. Suppose that $\lambda_0 > 0$ for $m = 2,3,4$. Then $\widetilde{H}_\Omega(x_0,y)$ in (16) exists for any $y_0 \in \partial\Omega$. For $m \geq 3$ and $\eta \to 0$*

$$\widetilde{H}_\Omega\left(x_0, x_0 + n(x_0)\eta\right) = \tfrac{1}{2m-4}\eta^2(1+o(1)), \quad (18)$$

and for $m = 2$ and $\eta \to 0$

$$\widetilde{H}_\Omega\left(x_0, x_0 + n(x_0)\eta\right) = \tfrac{1}{2}\eta^2\left(\log \eta^{-1}\right)(1+o(1)). \quad (19)$$

Here $n(x_0)$ is the inward pointing unit normal vector at x_0.

Remark 5. *It is possible to weaken the hypothesis $\lambda_0 > 0$ in Theorems 1 and 4 for the cases $m = 2,3,4$. However, this will not change the asymptotic formulae in (11) and in (13)-(19) respectively.*

Proposition 6. *Let Ω be an open and connected set in $\mathbb{R}^2$. The integral in* (1) *converges if and only if $\mathbb{R}^2 \backslash \Omega$ contains a compact set with strictly positive logarithmic capacity.*

In the above we have always assumed that $m > 1$. However, $H(x, y)$ is well defined for an open interval in $\mathbb{R}$. A direct computation yields for $\Omega = (0, 1)$ that

$$G_\Omega(x, y) = (x \wedge y) - xy, \tag{20}$$

$$\int_\Omega G_\Omega(x, z) G_\Omega(z, y)\, dz = \tfrac{1}{3} xy(1-x)(1-y) - \tfrac{1}{6}(x-y)^2 \left((x \wedge y) - xy\right), \tag{21}$$

and

$$H_\Omega(x, y) = \tfrac{1}{3}\left(x \vee y\right) - \tfrac{1}{6}x^2 - \tfrac{1}{6}y^2. \tag{22}$$

The probabilistic interpretation of H_Ω is different from the one given for $m > 1$ since one dimensional Brownian motion has a positive probability of hitting any points of Ω. The exit time should be replaced by the quitting or last exit time $\gamma_{\{y\}}$ as defined by Chung in [3, page 209].

3 Proof of Theorem 1

The proof of Theorem 1 is based on some estimates for the Dirichlet heat kernel, which in turn imply precise estimates for the Green function.

Lemma 7. *Let Ω an open and connected set in $\mathbb{R}^m (m \geq 1)$. Then for $x, y \in \Omega$ and $t > 0$*

$$p(x, y; t) \geq p_\Omega(x, y; t) \geq p(x, y; t) - \frac{2m}{(4\pi t)^{m/2}} e^{-c^2(\delta(x) \vee \delta(y))^2/(4t)}, \tag{23}$$

where δ is the distance to the boundary

$$\delta(x) = \inf\left\{|x - y| : \ y \in \mathbb{R}^m \backslash \Omega\right\}, \tag{24}$$

and

$$c = \left(2\sqrt{2} - 2\right) m^{-1/2}. \tag{25}$$

Proof. The heat kernel estimates obtained in [1, Theorem 1] for $-\frac{1}{2}\triangle + \frac{\partial}{\partial t}$, yield (20), by scaling.

Lemma 8. *Let Ω be an open and connected set in $\mathbb{R}^m, m \geq 3$. Then for $x, y \in \Omega$*

$$G(x, y) \geq G_\Omega(x, y) \geq G(x, y) - \frac{m\,\Gamma(\frac{1}{2}m - 1)}{2\pi^{m/2}}\, c^{2-m}\, \left(\delta(x) \vee \delta(y)\right)^{2-m}. \tag{26}$$

Proof. Integrate inequality (23) with respect to t over $[0,\infty)$. $\square$

Lemma 9. *Let Ω be an open and connected set in $\mathbb{R}^m$, $m \geq 5$. Then for $x, y \in \Omega$*

$$\left| \int_0^\infty t\, p_\Omega(x,y;t)dt - \frac{\Gamma\left(\frac{1}{2}m-2\right)}{16\pi^{m/2}}\, |x-y|^{4-m} \right| \leq \frac{m\Gamma\left(\frac{1}{2}m-2\right)}{8\pi^{m/2}}\, c^{4-m}\ \left(\delta(x) \vee \delta(y)\right)^{4-m}. \tag{27}$$

Proof. Multiply inequality (23) by t and integrate the resulting inequality with respect to t over $[0,\infty)$. $\square$

Proof of Theorem 1. (i) The proof of Theorem 1 for $m \geq 5$ follows directly from (4), (9), Lemma 8 and Lemma 9.

(ii) The proof of Theorem 1 for $m = 4$ is more delicate. First note that by the semigroup property for heat kernels and Cauchy-Schwarz's inequality

$$\begin{aligned} p_\Omega(x,y;t) &= \int_\Omega dz\, p_\Omega(x,z;\tfrac{1}{2}t) p_\Omega(z,y;\tfrac{1}{2}t) \\ &\leq \left(\int_\Omega dz\, p_\Omega(x,z;\tfrac{1}{2}t)^2 \right)^{1/2} \left(\int_\Omega dz\, p_\Omega(z,y;\tfrac{1}{2}t)^2 \right)^{1/2} \\ &= \Big(p_\Omega(x,x;t)\ p_\Omega(y,y;t) \Big)^{1/2} . \end{aligned} \tag{28}$$

It is elementary that

$$p_\Omega(x,x;t) \leq e^{-t\lambda/2} p_\Omega(x,x;\tfrac{1}{2}t). \tag{29}$$

Let $T = \lambda^{-1}$. By (28), (29) and (2), (3) we have that for $x, y \in \Omega$

$$\int_T^\infty dt\, t\, p_\Omega(x,y;t) \leq \int_T^\infty dt\, t\, e^{-t\lambda/2} (2\pi t)^{-2} \leq 1. \tag{30}$$

Moreover for all $x, y \in \Omega$ with $|x-y|^2 \leq 4T$ we have that

$$\begin{aligned} \int_0^T dt\, t\, p_\Omega(x,y;t) &\leq \int_0^T dt\, t\, p(x,y;t) \\ &= \tfrac{1}{16\pi^2} \int_{|x-y|^2/(4T)}^\infty ds\, s^{-1}\, e^{-s} \\ &\leq \tfrac{1}{16\pi^2} \left(\int_{|x-y|^2/(4T)}^1 ds\, s^{-1} + \int_1^\infty ds\, e^{-s} \right) \\ &\leq \tfrac{1}{16\pi^2} \log\left(\frac{4T}{|x-y|^2} \right) + 1. \end{aligned} \tag{31}$$

By (30–31) we have for $x, y \in \Omega$ with $|x-y|^2 \leq 4T$

$$\int_0^\infty dt\, t\, p_\Omega(x,y;t) \leq \tfrac{1}{16\pi^2} \log\left(\frac{4T}{|x-y|^2}\right) + 2. \tag{32}$$

On the other hand we have by Lemma 7 that for $m = 4$ and $|x-y|^2 \leq 4T$

$$\begin{aligned}
\int_0^\infty dt\, t\, p_\Omega(x,y;t) &\geq \int_0^T dt\, t\, p_\Omega(x,y;t) \\
&\geq \tfrac{1}{16\pi^2} \int_{|x-y|^2/(4T)}^1 ds\, s^{-1} e^{-s} - \tfrac{1}{2\pi^2} \int_{c^2(\delta(x)\vee\delta(y))^2/(4T)}^\infty ds\, s^{-1} e^{-s} \\
&\geq \tfrac{1}{16\pi^2} \log\left(\frac{4T}{|x-y|^2}\right) - 1 - Tc^{-2}(\delta(x) \vee \delta(y))^{-2}.
\end{aligned} \tag{33}$$

Combining inequalities (32), (33), (26) with (9) and the expression for H_Ω we arrive at the conclusion of Theorem 1 (ii).

(iii) To prove Theorem 1 for $m = 3$ we first note that by (29) and (2)

$$\begin{aligned}
\int_0^\infty dt\, t\, p_\Omega(x,x;t) &\leq \int_0^\infty dt\, t\, e^{-t\lambda/2} p(x,x;\tfrac{1}{2}t) \\
&= \frac{1}{(2\pi)^{3/2}} \int_0^\infty dt\, t^{-1/2} e^{-t\lambda/2} \leq \lambda^{-1/2}.
\end{aligned} \tag{34}$$

This proves (12). To prove (13) we note that $y \to p_\Omega(x,y;t)$ is continuous and

$$\begin{aligned}
&\int_0^\infty dt\, t\, |p_\Omega(x,y;t) - p_\Omega(x,x;t)| \\
&\quad \leq \int_0^\infty dt\, t\, \Big(p_\Omega(x,x;t) p_\Omega(y,y;t)\Big)^{1/2} + \int_0^\infty dt\, t\, p_\Omega(x,x;t) \\
&\quad \leq 2\lambda^{-1/2},
\end{aligned} \tag{35}$$

by the estimate in (34). Hence by Lebesgue's dominated convergence theorem we have for $y \to x$

$$\int_0^\infty dt\, t\, p_\Omega(x,y;t) = \int_0^\infty dt\, t\, p_\Omega(x,x;t) + o(|x-y|). \tag{36}$$

The proof of (13) follows directly from (36) and (26).

(iv) Finally to prove Theorem 1 for $m = 2$ we first note that

$$\begin{aligned}
\int_0^\infty dt\, t\, p_\Omega(x,x;t) &\leq \int_0^\infty dt\, t\, e^{-t\lambda/2} p(x,x;\tfrac{1}{2}t) \\
&= \tfrac{1}{2\pi} \int_0^\infty dt\, e^{-t\lambda/2} \leq \lambda^{-1}.
\end{aligned} \tag{37}$$

This proves that (12) holds for $m = 2$. To prove (14) we note that

$$\begin{aligned}
&\int_0^\infty dt\, t\, |p_\Omega(x, y; t) - p_\Omega(x, x; t)| \\
&\quad \le \int_0^\infty dt\, t\, \Big(p_\Omega(x, x; t) p_\Omega(y, y; t)\Big)^{1/2} + \int_0^\infty dt\, t\, p_\Omega(x, x; t) \\
&\quad \le 2 \int_0^\infty dt\, t\, e^{-t\lambda/2} (2\pi t)^{-1} \le 2\lambda^{-1}. \qquad (38)
\end{aligned}$$

Hence by Lebesgue's dominated convergence theorem we have that (36) also holds for $m = 2$ and $y \to x$. It remains to find the asymptotic behaviour of $G_\Omega(x, y)$ as $y \to x$. By Lemma 7 we have for $|x - y|^2 \le 4T$

$$\begin{aligned}
G_\Omega(x, y) &= \int_0^\infty dt\, p_\Omega(x, y; t) \\
&\ge \int_0^T dt (4\pi t)^{-1} (e^{-|x-y|^2/(4t)} - 4e^{-c^2(\delta(x)\vee\delta(y))^2/(4t)}) \\
&\ge \tfrac{1}{4\pi} \log\left(\frac{4T}{|x - y|^2}\right) - 1 - 4T/(c^2(\delta(x) \vee \delta(y))^2). \qquad (39)
\end{aligned}$$

Secondly

$$\begin{aligned}
G_\Omega(x, y) &\le \int_0^T dt\, \tfrac{1}{4\pi t}\, e^{-|x-y|^2/(4t)} + \int_T^\infty dt\, \Big(p_\Omega(x, x; t) p_\Omega(y, y; t)\Big)^{1/2} \\
&\le \tfrac{1}{4\pi} \log\left(\frac{4T}{|x - y|^2}\right) + 3. \qquad (40)
\end{aligned}$$

This concludes the proof of Theorem 1 (iv) by (36), (39) and (40). □

4 Sketch of the proof of Theorem 4

The main idea in the proof of Theorem 4 is to approximate the domain by a half space. The cases $m = 2$ and $m = 3$ will be considered in Lemmas 11 and 10 respectively.

Lemma 10. *Let $\Omega_+ \subseteq \mathbb{R}^m$ be given by*

$$\Omega_+ = \{(x_1, \dots, x_m) : x_1 > 0\}. \qquad (41)$$

Then for $m \ge 3$ and $x_1 > 0$

$$\lim_{x_1 \to 0} H_{\Omega_+}((x_1, 0, \dots, 0), (y_1, 0, \dots, 0)) = \tfrac{1}{2m-4} y_1^2. \qquad (42)$$

Proof. By the reflection principle

$$p_{\Omega_+}((x_1, 0, \dots, 0), (y_1, 0, \dots, 0); t) = \frac{e^{-(y_1-x_1)^2/(4t)} - e^{-(y_1+x_1)^2/(4t)}}{(4\pi t)^{m/2}}. \qquad (43)$$

Hence

$$\lim_{x_1\to 0}\frac{\partial}{\partial x_1}\int_0^\infty dt\, t\, p_{\Omega_+}((x_1,0,\dots,0),(y_1,0\dots,0);t) = \frac{\Gamma(\frac{1}{2}m-1)}{4\pi^{m/2}} y_1^{3-m}. \quad (44)$$

Moreover, by (1) and (43)

$$\begin{aligned} &G_{\Omega_+}((x_1,0,\dots,0),(y_1,0,\dots,0)) \\ &\quad = \frac{\Gamma(\frac{1}{2}m-1)}{4\pi^{m/2}}\left(|y_1-x_1|^{2-m} - |y_1+x_1|^{2-m}\right). \end{aligned} \quad (45)$$

Hence

$$\lim_{x_1\to 0}\frac{\partial}{\partial x_1} G_{\Omega_+}((x_1,0,\dots,0),(y_1,0,\dots,0)) = \frac{(m-2)\Gamma(\frac{1}{2}m-1)}{2\pi^{m/2}} y_1^{1-m}, \quad (46)$$

and Lemma 10 follows by L' Hospital's rule with (44) and (46). □

Lemma 11. *Let $m=2$ and let Ω_+ be given by (41). Then for $y_1\to 0$ and $T>0$*

$$\lim_{x_1\to 0}\frac{\partial}{\partial x_1}\int_0^T dt\, t\, p_{\Omega_+}((x_1,0),(y_1,0);t) = \frac{1}{2\pi} y_1\left(\log\frac{1}{y_1}\right)(1+o(1)), \quad (47)$$

and

$$\lim_{x_1\to 0}\frac{\partial}{\partial x_1} G_{\Omega_+}((x_1,0),(y_1,0)) = \frac{1}{\pi} y_1^{-1}. \quad (48)$$

Proof. By (1) and (43)

$$\begin{aligned} G_{\Omega_+}((x_1,0),(y_1,0)) &= \int_0^\infty (4\pi t)^{-1}(e^{-(y_1-x_1)^2/(4t)} - e^{-(y_1+x_1)^2/(4t)})dt \\ &= \frac{1}{2\pi}\log\left(\frac{y_1+x_1}{y_1-x_1}\right), \end{aligned} \quad (49)$$

and (48) follows from (49).

To prove (47) we note that we may change the order of differentiation and limit with the integral. Hence the left hand side of (47) equals

$$\frac{1}{4\pi} y_1 \int_0^T dt\, t^{-1} e^{-y_1^2/(4t)} = \frac{1}{2\pi} y_1\left(\log\frac{1}{y_1}\right)(1+o(1)), \quad (50)$$

as $y_1\to 0$. □

The main idea in the proof of Theorem 4 is to replace $\partial\Omega$ by the plane tangent to $\partial\Omega$ at x_0. This is justified by the fact that the main contributions to the integrals in (1) and in (9) for y near x come from small t (see [2] for similar approximations). The formulae in Theorem 4 can be read-off from (42) for $m\ge 3$ and from (47) and (48) for $m=2$ respectively.

5 Proof of Proposition 6

In [4] it was shown that if $\mathbb{R}^2\backslash\Omega$ is non-polar and compact then

$$p_\Omega(x,y;t) = \frac{1}{\pi t(\log t)^2} u(x)u(y)(1+o(1)), \tag{51}$$

where $u:\Omega \to \mathbb{R}$ is the unique non-trivial harmonic function which is 0 on the regular points of $\partial\Omega$ and which satisfies $\lim_{|x|\to\infty} u(x)/\log|x| = 1$. The integral in (1) converges near 0 by (2) and (3). The case where $\mathbb{R}^2\backslash\Omega$ is non-polar and non-compact follows by monotonicity of the Dirichlet heat kernel. Conversely, if $\mathbb{R}^2\backslash\Omega$ is polar then $p_\Omega(x,y;t) = p(x,y;t)$ almost everywhere and (1) diverges. $\square$

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References

[1] M. van den Berg, *Bounds on Green's functions of second-order differential equations*, J. Math. Phys. **22**, 2452–2455 (1981).

[2] M. van den Berg, *On the asymptotics of the heat equation and bounds on traces associated with the Dirichlet Laplacian*, J. of Functional Analysis **71**, 279–293 (1987).

[3] Kai Lai Chung, *Lectures from Markov processes to Brownian motion.* Fundamental Principles of Mathematical Science 249. Springer-Verlag, New York-Berlin, 1982.

[4] P. Collet, S. Martínez, J.S. Martin, *Asymptotic behaviour of a Brownian motion on exterior domains*, Probability Theory and Related Fields **116**, 303 –316 (2000).

[5] M. Cranston, T.R. McConnell, *The lifetime of conditioned Brownian motion*, Z. Wahrsch. Verw. Gebiete **65**, 1–11 (1983).

[6] M. Cranston, *Lifetime of conditioned Brownian Motin in Lipschitz domains*, Z. Wahrsch. Verw. Gebiete **70**, 335 –340 (1985).

[7] A. Dall' Acqua, H.-C. Grunau, G.H. Sweers, *On a conditioned Brownian motion and a maximum principle on the disk*, J. d'Analyse Mathématiques **93**, 309–329 (2004).

[8] A. Dall' Acqua, *On the lifetime of a conditioned Brownian motion in the ball.* Preprint 2005.

[9] J. L. Doob, *Classical Potential Theory and its Probabilistic Counterpart*, Springer Verlag, Berlin 2001.

[10] R. Durrett, *Brownian Motion and Martingales in Analysis*, Wadsworth Mathematics Series, Belmont 1984.

[11] P.S. Griffin, T.R. McConnell, G. Verchota, *Conditioned Brownian motion in simply connected planar domains*, Ann. Inst. H. Poincaré Probab. Statist. **29**, 229–249 (1993).

[12] H.-Ch. Grunau, G. Sweers,. *Sharp estimates for iterated Green functions*. Proc. Roy. Soc. Edinburgh Sect. A **132**, 91–120 (2002).

[13] B. Kawohl, G. Sweers, *On anti-eigenvalues for elliptic systems and a question of McKenna and Walter*, Indiana University Math. J. **51**, 1023–1040 (2002).

[14] B. Kawohl, G. Sweers, *Among all two-dimensional convex domains the disk is not optimal for the lifetime of a conditioned Brownian motion*, J. d'Analyse Mathématiques **86**, 335–357 (2002).

[15] G. Sweers, *Positivity for a strongly Coupled Elliptic System by Green Function Estimates*, J. of Geometric Analysis **4**, 121–142 (1994).