Exercise Sheet 2 – Analysis II

(Homework solutions should be submitted at 08:00, before the class on 05.11.2018, N24 - H12)

(Please submit only the exercises 1, 2a, 4a).

1. Let $a > 0$ and $I := [0, 1 + a] \subset \mathbb{R}$. Let $f : I \to \mathbb{R}$ be the function defined by

$$f(x) := \begin{cases} ax & \text{for } 0 \leq x \leq 1, \\ a - (x - 1) & \text{for } 1 \leq x \leq a + 1. \end{cases}$$

For $k \in \mathbb{N}$, we divide the intervals $[0, 1]$ und $[1, a + 1]$ into $k$ sub-intervals by the partition:

$$\pi_k : 0 = x_0 < \frac{1}{k} = x_1 < \ldots < \frac{k-1}{k} = x_{k-1} < 1 = x_k < 1 + 1 \cdot \frac{a}{k} = x_{k+1} < \ldots < 1 + k \cdot \frac{a}{k} = x_{2k} = 1 + a.$$ 

Prove that $f$ is Riemann integrable on $I$ by calculating the upper and lower Riemann-Darboux sums $s(\pi_k, f)$ und $S(\pi_k, f)$. Determine the Riemann integral of $f$ on $I$.

2. Let $a > 1$. Calculate the following integrals by making use of Riemann sum $\sigma(\pi, f)$:

(a) $\int_1^a \frac{1}{x} dx$

(b) $\int_1^a \log x \, dx$.

Hint: Choose the partition $\pi : 1 = x_0 < x_1 < x_2 < \ldots < x_n = a$ with $x_k := a^{b/n}$ and choose $\xi_k$ in the $k$-th sub-interval as follows: $\xi_k = x_{k-1}$ für $k \in \mathbb{N}, 1 \leq k \leq n$.

3. Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be bounded, Riemann integrable on $[a, b]$ with $\int_a^b f(x) \, dx > 0$. [4*]

Prove that there exists an interval $[c, d] \subset (a, b)$, such that $f(x) > 0$ for all $x \in [c, d]$.

Hint: You can prove by contradiction. Consider the upper Riemann-Darboux integral of $f$ on $[a, b]$ in order to get $s(\pi, f) \leq 0$ for some partition $\pi$.

4. Let $a < b$ and $I = [a, b]$ be a compact interval. Prove the following statements:

(a) If $f : I \to \mathbb{R}$ is Lipschitz continuous or monotone or continuously differentiable, then $f$ is a function of bounded variation (a BV function). [15]

Hint: Consider any partition $\pi : a = x_0 < x_1 < \ldots < x_n = b$ of $I$ and prove that there is a constant $M \in \mathbb{R}$, such that $V(\pi, f) \leq M$ for every partition $\pi$.

(b) If $f$ is a BV function, then $f$ is bounded.

(c) The following function is continuous in $[0, 1]$, but it is not a BV function:

$$f(x) := \begin{cases} x \sin \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

- Please see the definition of a BV function on the next page!

- 4* in Exercise 3 means bonus points.
Definition 8.3.2 (Bounded Variation):

Let $a < b$ and $f : [a, b] \to \mathbb{R}$. Let $\pi : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $I$. Then, the following quantities

$$V(\pi, f) := \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

and

$$V_f = V_a^b(f) := \sup_{\pi} V(\pi, f)$$

are called the variation of $f$ with respect to $\pi$ and the total variation of $f$ on $I$, respectively. If $V_f < +\infty$ on $I$, then $f$ is said to be of bounded variation and we denote $f \in BV = BV(I)$. 

[Reference: https://www.uni-ulm.de/mawi/analysis/lehre/veranstaltungen/ws-20182019/analysis-2/]