Exercise Sheet 2 – Analysis II

(Homework solutions should be submitted at 08:00, before the class on 05.11.2018, N24 - H12)
*(Please submit only the exercises **1**, **2a**, **4a**).*

1. Let $a > 0$ and $I := [0, 1 + a] \subset \mathbb{R}$. Let $f : I \to \mathbb{R}$ be the function defined by [13]

$$f(x) := \begin{cases} ax & \text{for } 0 \le x \le 1, \\ a - (x-1) & \text{for } 1 \le x \le a+1. \end{cases}$$

For $k \in \mathbb{N}$, we devide the intervals $[0,1]$ und $[1, a+1]$ into k sub-intervals by the partition:

$$\pi_k : 0 = x_0 < \frac{1}{k} = x_1 < \ldots < \frac{k-1}{k} = x_{k-1} < 1 = x_k < 1+1\cdot\frac{a}{k} = x_{k+1} < \ldots < 1+k\cdot\frac{a}{k} = x_{2k} = 1+a.$$

Prove that f is Riemann integrable on I by calculating the upper and lower Riemann-Darboux sums $s(\pi_k, f)$ und $S(\pi_k, f)$. Determine the Riemann integral of f on I.

2. Let $a > 1$. Calculate the following intergrals by making use of Riemann sum $\sigma(\pi, f)$: [12+0]

(a) $\displaystyle\int_1^a \frac{1}{x} dx$ (b) $\displaystyle\int_1^a \log x \; dx.$

Hint: Choose the partition $\pi : 1 = x_0 < x_1 < x_2 < \ldots < x_n = a$ with $x_k := a^{k/n}$ and choose ξ_k in the k-th sub-interval as follows: $\xi_k = x_{k-1}$ für $k \in \mathbb{N}, 1 \le k \le n$.

3. Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be bounded, Riemann integrable on $[a, b]$ with $\int\limits_a^b f(x)dx > 0$. [4*]
Prove that there exists an interval $[c, d] \subset (a, b)$, such that $f(x) > 0$ for all $x \in [c, d]$.
Hint: You can prove by contradiction. Consider the upper Riemann-Darboux integral of f on $[a, b]$ in order to get $s(\pi, f) \le 0$ for some partition π.

4. Let $a < b$ and $I = [a, b]$ be a compact interval. Prove the following statements:

   (a) If $f : I \to \mathbb{R}$ is Lipschitz continuous or monotone or continuously differentiable, then f is a function of bounded variation (a BV function). [15]
   Hint: Consider any partition $\pi : a = x_0 < x_1 < \ldots < x_n = b$ of I and prove that there is a constant $M \in \mathbb{R}$, such that $V(\pi, f) \le M$ for every partition π.

   (b) If f is a BV function, then f is bounded.

   (c) The following function is continuous in $[0, 1]$, but it is not a BV function:

$$f(x) := \begin{cases} x \sin\frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

– Please see the definition of a BV function on the next page!
– 4 in Exercise 3 means bonus points.*

Definition 8.3.2 (Bounded Variation):

Let $a < b$ and $f : [a, b] \to \mathbb{R}$. Let $\pi : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of I. Then, the following quantities

$$V(\pi, f) := \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

and

$$V_f = V_a^b(f) := \sup_{\pi} V(\pi, f)$$

are called the *variation* of f with respect to π and the *total variation* of f on I, respectively.

If $V_f < +\infty$ on I, then f is said to be of *bounded variation* and we denote $f \in BV = BV(I)$.