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Total: 40

Exercise Sheet 3 – Analysis II

(Homework solutions will be handed in and discussed at 14:00-16:00, 08.11.2018, N24 - H12

*(Please submit only the exercises **1-4**).*

1. Let $a, b \in \mathbb{R}^+$. Calculate the surface area of the ellipse $E := \left\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$. [8]

2. Study the convergence of the following improper integrals: [2x6]

   (a) $\int\limits_0^1 \frac{x}{\sqrt{1-x}}$ (b) $\int\limits_0^\infty \frac{x^2+3}{e^x} dx$

3. Study the convergence of the series: [7]

$$\sum_{k=2}^{\infty} \frac{1}{k \log^{\mu} k}, \ \mu > 0.$$

4. Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of functions on $[0,1]$ given by [5+5+3]

$$f_k(x) := \frac{k^{\mu} x}{(1+k^2x^2)^2} \text{ for } \mu \in \mathbb{R}.$$

   Find μ such that:

   (a) $(f_k)_{k \in \mathbb{N}}$ converges pointwise on $[0,1]$,
   (b) $(f_k)_{k \in \mathbb{N}}$ converges uniformly on $[0,1]$,
   (c) the equality $\int_0^1 \lim\limits_{k \to \infty} f_k(x)dx = \lim\limits_{k \to \infty} \int_0^1 f_k(x)dx$ holds.

   Hint: (a) Determine the limit function f for $\mu = 4$ and $\mu < 4$.

   (b) Find μ such that $\lim\limits_{k \to \infty} \left(\sup\limits_{x \in [0,1]} |f_k(x) - f(x)| \right) = 0.$

5. Let the functions $f, g : [a,b] \to \mathbb{R}, \ a < b$ be bounded and Riemann integrable on $[a,b]$. Prove that:

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx.$$

   Hint: For all $\lambda \in \mathbb{R}$, $0 \leq \int_a^b (f(x) - \lambda g(x))^2 dx$. Consider the case $B := \int_a^b g^2(x)dx = 0$ and $B \neq 0$.