Exercise Sheet 3 – Analysis II

(Homework solutions will be handed in and discussed at 14:00-16:00, 08.11.2018, N24 - H12)

(Please submit only the exercises 1-4).

1. Let \( a, b \in \mathbb{R}^+ \). Calculate the surface area of the ellipse \( E := \{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \} \). [8]

2. Study the convergence of the following improper integrals:
   (a) \( \int_0^1 \frac{x}{\sqrt{1-x}} \, dx \)
   (b) \( \int_0^\infty \frac{x^2 + 3}{e^x} \, dx \) [2x6]

3. Study the convergence of the series:
   \[ \sum_{k=2}^{\infty} \frac{1}{k \log^\mu k}, \quad \mu > 0. \] [7]

4. Let \((f_k)_{k \in \mathbb{N}}\) be a sequence of functions on \([0,1]\) given by
   \[ f_k(x) := \frac{k^\mu x}{(1 + k^2 x^2)^2} \text{ for } \mu \in \mathbb{R}. \] [5+5+3]

   Find \( \mu \) such that:
   (a) \((f_k)_{k \in \mathbb{N}}\) converges pointwise on \([0,1]\),
   (b) \((f_k)_{k \in \mathbb{N}}\) converges uniformly on \([0,1]\),
   (c) the equality \( \lim_{k \to \infty} \int_0^1 f_k(x) \, dx = \lim_{k \to \infty} \int_0^1 f_k(x) \, dx \) holds.

   Hint: (a) Determine the limit function \( f \) for \( \mu = 4 \) and \( \mu < 4 \).
   (b) Find \( \mu \) such that \( \lim_{k \to \infty} \sup_{x \in [0,1]} |f_k(x) - f(x)| = 0 \).

5. Let the functions \( f, g : [a, b] \to \mathbb{R}, \quad a < b \) be bounded and Riemann integrable on \([a, b]\).

   Prove that:
   \[ \left( \int_a^b f(x)g(x) \, dx \right)^2 \leq \int_a^b f^2(x) \, dx \cdot \int_a^b g^2(x) \, dx. \]

   Hint: For all \( \lambda \in \mathbb{R}, \quad 0 \leq \int_a^b (f(x) - \lambda g(x))^2 \, dx \). Consider the case \( B := \int_a^b g^2(x) \, dx = 0 \) and \( B \neq 0 \).

https://www.uni-ulm.de/mawi/analysis/lehre/veranstaltungen/ws-20182019/analysis-2/