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Total point: 40

Exercise Sheet 1 – Analysis III

(Homework solutions will be handed in and discussed at 10:00, 29.10.18, O27-H20)

0. Please pay attention to the following points!

   (a) *Sign in in Moodle (moodle.uni-ulm.de) for this lecture.*

   (b) *Submit the solutions individually before the exercise class (10:00-10:15).*

   (c) *To obtain 50 % of the total points in the homework assignments is a precondition for participating in the final exam.*

1. Let ϕ be a function given by [8]

$$\phi : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \phi(x) := \begin{cases} e^{-\frac{1}{x}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

   Show that $\phi \in \mathcal{C}^\infty(\mathbb{R})$. *Hint*: Prove by induction that for $x > 0$ and $k \geq 0$, the k^{th} derivative of ϕ is given by the formula $\phi^{(k)}(x) = P_k(\frac{1}{x})\ e^{-\frac{1}{x}}$ where P_k is some polynomial.

2. Suppose $f : \mathbb{R}^k \to \mathbb{R}^n$ $(k < n)$ is a C^r function $(r \geq 1)$ such that $\det\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j=1}^k \neq 0$. Let $\Pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection $\Pi(x, y) = x$, where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$. [5+5]

   (a) Prove that there exist open subsets U and V of $\mathbb{R}^k$ and a C^r function $g : V \to U$ such that $a \in U$ and $(g \circ \Pi)(f(x)) = x$ for all $x \in U$. ($G = g \circ \Pi$ is called a "local left inverse" of f).

   Hint: Use the inverse function theorem.

   (b) Give a "local left inverse" at $x = 0$ of the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) := (\cos x, \sin x)$.

3. Let $\alpha : [a, b] \to \mathbb{R}^n$, $\alpha(t) = (x_1(t), ..., x_n(t))$, be a regular parametrized C^1-curve (i.e. a continuously differentiable curve). The arc length of α is defined by: [3×4]

$$s(t) = \int_a^t |\alpha'(\tau)|\, d\tau$$

   where $|\alpha'(\tau)| = \sqrt{(x_1'(\tau))^2 + ... + (x_n'(\tau))^2}$.

   (a) Show that $s : [a, b] \to [0, \mathcal{L}(\alpha)]$ is a C^1-diffeomorphism.

   (b) Let γ be the reparametrization of α by arc length (i.e. $\gamma = \alpha \circ s^{-1}$). Prove that $|\gamma'(t)| = 1\ \forall t \in [0, \mathcal{L}(\alpha)]$ (γ is called a unit speed curve).

   (c) Is the path $\alpha : [0, 2] \to \mathbb{R}^3$ with $\alpha(t) = \left(t\sqrt{2}, e^t, 1 - e^{-t}\right)$ regular? If so, determine the arc length $s(t)$.

4. Consider the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) := \left(xy, \dfrac{2 - 2y^2}{1 + x^2 + y^2}\right)$ and two regular parametrized differentiable curves $\gamma_1 : [-1, 1] \to \mathbb{R}^2$ and $\gamma_2 : [0, \pi] \to \mathbb{R}^2$ defined by $\gamma_1(t) = (0, t)$, $\gamma_2(t) = (\sin t, \cos t)$. Calculate $\int_{\gamma_1} F(u) \cdot d\overrightarrow{u}$ and $\int_{\gamma_2} F(u) \cdot d\overrightarrow{u}$. [5+5]