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Total point: 40+6*

Exercise Sheet 2 – Analysis III

(Homework solutions will be handed in and discussed at 10:00-12:00, 12.11.18, O27-H20)

1. Consider the vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by [3x5]

$$f(x,y) := (f_1(x,y), f_2(x,y)) = \left(e^{xy} + xye^{xy},\ x^2e^{xy} - 2y\right)$$

   (a) Prove that this vector field admits a potential.

   (b) Determine this potential F.
   Hint: From the relation $\frac{\partial F}{\partial x}(x,y) = f_1(x,y)$, we can determine the function $h : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $F(x,y) = h(x,y) + g(y)$.

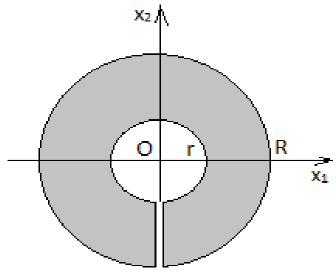
   (c) Calculate the line integral $\int_\gamma f(u) \cdot d\overrightarrow{u}$ where the curve $\gamma : [0,1] \to \mathbb{R}^2$ given by $\gamma(t) := \left(t, 1-t^2\right)$.

2. (a*) Prove that every convex subset of $\mathbb{R}^n$ is simply connected. [3*]

   (b) Let (X_1, d_1) and (X_2, d_2) be metric spaces and $g : X_1 \to X_2$ a homeomorphism, i.e. g is a continuous bijection whose inverse is also continuous. Prove that if $A \subseteq X_1$ is simply connected, then $g(A)$ is simply connected. [5]

   [5]

   (c) Apply (a) and (b) to prove that: for $0 < r < R$, the set $\left\{x \in \mathbb{R}^2 : r < |x| < R\right\} \setminus \left\{(0, x_2) \in \mathbb{R}^2 : x_2 \leq 0\right\}$ is simply connected.



3. For $f, h \in C(\mathbb{R}^n)$ and $g \in C_0(\mathbb{R}^n)$, i.e. g is a continuous function with compact support. Remind that the convolution of f with g is defined by:

$$f * g : \mathbb{R}^n \to \mathbb{R} \text{ mit } \quad (f * g)(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy.$$

   Show the following properties:

   (a) $f * g$ is continuous. [5]

   (b) $f * g = g * f$ and $(f+h) * g = f * g + h * g$. [5]

   (c) if $f \in C^k(\mathbb{R}^n)$, $0 \leq k \leq \infty$, then we have $f * g \in C^k(\mathbb{R}^n)$ and for every $|\alpha| \leq k$, [5]

$$D^\alpha(\varphi_\varepsilon * f) \longrightarrow D^\alpha f \quad \text{uniformly on every compact subset of } \mathbb{R}^n \text{ as } \varepsilon \to 0,$$

   where the family of functions $\{\varphi_\varepsilon,\ \varepsilon > 0\}$ is a mollifier, i.e. $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$ with $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfying: $\varphi \geq 0$ on $\mathbb{R}^n$, $\text{supp}(\varphi) \subset \overline{B_1(0)}$ and $\int_{\mathbb{R}^n} \varphi dx = 1$.

   (d) Is there a suitable condition on $f \in C(\mathbb{R}^n)$ s.t. $\varphi_\varepsilon * f \longrightarrow f$ uniformly on $\mathbb{R}^n$? [3*]