Exercise Sheet 2 – Analysis III

(Homework solutions will be handed in and discussed at 10:00-12:00, 12.11.18, O27-H20)

1. Consider the vector field \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by

\[
( f_1(x, y), f_2(x, y)) = (e^{xy} + xye^{xy}, x^2 e^{xy} - 2y)
\]

(a) Prove that this vector field admits a potential.

(b) Determine this potential \( F \).

Hint: From the relation \( \frac{\partial F}{\partial x}(x, y) = f_1(x, y) \), we can determine the function \( h : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) such that \( F(x, y) = h(x, y) + g(y) \).

(c) Calculate the line integral \( \int_{\gamma} f(u) \cdot d\vec{u} \) where the curve \( \gamma : [0, 1] \rightarrow \mathbb{R}^2 \) given by \( \gamma(t) := (t, 1-t^2) \).

2. (a*) Prove that every convex subset of \( \mathbb{R}^n \) is simply connected. [3*]

(b) Let \((X_1, d_1)\) and \((X_2, d_2)\) be metric spaces and \( g : X_1 \rightarrow X_2 \) a homeomorphism, i.e. \( g \) is a continuous bijection whose inverse is also continuous. Prove that if \( A \subseteq X_1 \) is simply connected, then \( g(A) \) is simply connected. [5]

(c) Apply (a) and (b) to prove that:

for \( 0 < r < R \), the set \( \{ x \in \mathbb{R}^2 : r < |x| < R \} \) \( \backslash \{ (0, x_2) \in \mathbb{R}^2 : x_2 \leq 0 \} \) is simply connected.

3. For \( f, h \in C(\mathbb{R}^n) \) and \( g \in C_0(\mathbb{R}^n) \), i.e. \( g \) is a continuous function with compact support. Remind that the convolution of \( f \) with \( g \) is defined by:

\[
(f * g)(x) := \int_{\mathbb{R}^n} f(x - y) g(y) dy.
\]

Show the following properties:

(a) \( f * g \) is continuous. [5]

(b) \( f * g = g * f \) and \( (f + h) * g = f * g + h * g \). [5]

(c) if \( f \in C^k(\mathbb{R}^n) \), \( 0 \leq k \leq \infty \), then we have \( f * g \in C^k(\mathbb{R}^n) \) and for every \( |\alpha| \leq k \), \( D^\alpha(\varphi_\varepsilon * f) \rightarrow D^\alpha f \) uniformly on every compact subset of \( \mathbb{R}^n \) as \( \varepsilon \rightarrow 0 \),

where the family of functions \( \{ \varphi_\varepsilon, \varepsilon > 0 \} \) is a mollifier, i.e. \( \varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon) \) with \( \varphi \in C^\infty_0(\mathbb{R}^n) \) satisfying: \( \varphi \geq 0 \) on \( \mathbb{R}^n \), \( \text{supp}(\varphi) \subseteq \overline{B_1}(0) \) and \( \int_{\mathbb{R}^n} \varphi dx = 1 \).

(d) Is there a suitable condition on \( f \in C(\mathbb{R}^n) \) s.t. \( \varphi_\varepsilon * f \rightarrow f \) uniformly on \( \mathbb{R}^n \)? [3*]