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WS 2018/2019
Total points: 40

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Exercise Sheet 3 – Analysis III

((Homework solutions will be handed in and discussed at 10:00, 26.11.18, O27-H20)

1. (a) (*Polar coordinate tranformation*) Let $U := B_R(0) = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\right\}$, for $R > 0$ or $R = +\infty$, and let $f \in C^0(U)$ with $\iint_U |f(x,y)|\, dxdy < +\infty$. Use the transformation rule (Theorem A.4.2 in the appendix) to prove that [5]

$$\iint_U f(x,y)\, dxdy = \int_{-\pi}^{\pi} \int_0^R f(r\cos\varphi, r\sin\varphi)\, r\ drd\varphi.$$

   (b) Use the formula in Part (a) to prove that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$, where the function φ is given by [5]

$$\varphi(x) := \frac{1}{\left(\sqrt{2\pi}\right)^n} e^{-\frac{|x|^2}{2}}.$$

2. Let $U := B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$, for $R > 0$ or $R = +\infty$, and let $f \in C^0(U)$ with $\int_U |f(x)|\, dx < +\infty$. Write $x = (x_1, ..., x_n) \in \mathbb{R}^n$ as follows: $x = r\xi$ where $r = |x|$ and $\xi = \frac{x}{|x|} = (\xi_1, ..., \xi_{n-1}, \xi_n)$ with $\xi_n = \pm\sqrt{1 - \sum_{j=1}^{n-1} \xi_j^2}$. Prove that [10]

$$\int_U f(x)dx = \int_0^R r^{n-1} \int_{\mathbb{S}^{n-1}} f(r\xi)dS_\xi dr,$$

where $\mathbb{S}^{n-1} := \partial B_1(0) = \{x \in \mathbb{R}^n : |x| = 1\}$ and the integral of a function $g \in C^0\left(\mathbb{S}^{n-1}\right)$ on the sphere $\mathbb{S}^{n-1}$ is defined by :

$$\int_{\mathbb{S}^{n-1}} g(\xi)\, dS_\xi = \int_{|\xi'|<1} \frac{g\left(\xi', \sqrt{1-|\xi'|^2}\right) + g\left(\xi', -\sqrt{1-|\xi'|^2}\right)}{\sqrt{1-|\xi'|^2}} d\xi',$$

where $\xi' = (\xi_1, .., \xi_{n-1}) \in \mathbb{R}^{n-1}$ (the general definition of the surface integral will be given later).

Hint: Apply the transformation rule by considering

$$I = \left\{(\xi', r) \in \mathbb{R}^n \,\middle|\, 0 < r < R,\ \xi' = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1},\ \left|\xi'\right|^2 = \sum_{j=1}^{n-1} \xi_j^2 < 1\right\},$$

and maps $T^\pm = x^\pm : I \to U_R^\pm := \{x = (x_1, \ldots, x_n) \in U_R |\ x_n \gtrless 0\}$ defined by

$$\begin{cases} x_j^\pm(\xi', r) := r\xi_j, & \text{for } j = 1, \ldots, n-1, \\ x_n^\pm(\xi', r) := \pm\, r\sqrt{1 - |\xi'|^2}. \end{cases}$$

3. Prove that [5×4]
   (a) if $M \subset \mathbb{R}^n$ is open, then M is an n-dimensional C^∞- submanifold of $\mathbb{R}^n$.
   (b) $M = \left\{\left(3x, -x^2\right),\ x \in \mathbb{R}\right\}$ is a 1-dimensional C^∞- submanifold of $\mathbb{R}^2$.
   (c) $M = \left\{(x,y) \in \mathbb{R}^2 : xy = 0\right\}$ is not an 1-dimensional submanifold of $\mathbb{R}^2$.
   (d) if $M = M_1 \times M_2$ where M_i is a k_i-dimensional submanifold of $\mathbb{R}^{n_i}$, for $i = 1, 2$, then M is $(k_1 + k_2)$-dimensional submanifold of $\mathbb{R}^n$ with $n = n_1 + n_2$.
   (e) The set $SL(n, \mathbb{R}) := \left\{A \in M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2} : \det A = 1\right\}$ is a $(n^2 - 1)$-dimensional C^∞-submanifold of $\mathbb{R}^{n^2}$.