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WS 18/19
Total point: 40

Exercise Sheet 4 – Analysis III

(Homework solutions will be handed in and discussed at 10:00, 10.12.18, O27-H20)

1. Let us denote $E_m \in \mathbb{R}^n$ the m-dimensional plane: [10]

$$E_m := \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_{m+1} = ... = x_n = 0\}.$$

Prove that: A non-empty subset $M \subset \mathbb{R}^n$ is an m-dimensional submanifold of class C^l if and only if for every $x_0 \in M$ there exists an open neighborhood $U \subset \mathbb{R}^n$ of x_0, an open set $V \subset \mathbb{R}^n$ and a C^l-diffeomorphism $F : U \to V$ such that $F(M \cap U) = E_m \cap V$.

2. Let $M = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0 \text{ and } x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4\right\}$. [6+6]

   (a) Prove that M is a 2-dimensional submanifold of $\mathbb{R}^4$.

   (b) Prove that there exists a smooth function $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(1, -1) = (1, -1)$ and $(g_1(x_2, x_3), x_2, x_3, g_2(x_2, x_3)) \subset M$ for (x_2, x_3) sufficiently close to the point $(1, -1)$. Calculate $Dg(1, -1)$. *Hint*: Use the implicit function theorem.

3. Let $\mathbb{S}^n$ be a unit sphere in $\mathbb{R}^{n+1}$. [4+8+6]

   (a) Prove that $\mathbb{S}^n$ is an n-dimensional C^∞-submanifold of $\mathbb{R}^{n+1}$.

   (b) Let N, S be the north and resp. south poles of S^n, i.e. $N = (0, ..., 0, 1)$, $S = (0, ..., 0, -1)$. Consider the map π_N, called the stereographic projection from the north pole N, defined by

$$\pi_N : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n,$$

   which maps the point $P \in \mathbb{S}^n \setminus \{N\}$ to the intersection of the line NP with the hyperplane $\left\{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\right\} \cong \mathbb{R}^n$. The stereographic projection π_S from S is defined similarly. Determine $\pi_N(x)$ for $x \in \mathbb{S}^n \setminus \{N\}$ and $\pi_S(x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

   (c) Show that π_N, π_S are bijective and the inverse maps π_N^{-1}, π_S^{-1} are C^∞-smooth charts (parameterisations) of the submanifold $\mathbb{S}^n$.