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Total points: 40+6*

Exercise Sheet 8 – Analysis III

(Homework solutions will be handed in and discussed at 10:00, 04.02.19, O27-H20)

1. Investigate the convergence of the Fourier series for the following functions. Determine their (pointwise or uniform) limits, if they exist. [6+6]
   (a) The function f given in Exercise 2 on Sheet 7.
   (b) The function $g(x) := \sin(2x) + \cos(3x)$.

2. The sequence space of square-summable sequences is defined as follows: [6x2+6*]

$$\ell^2 := \left\{ \alpha : \mathbb{N}_0 \to \mathbb{C}, \alpha = (\alpha_n)_{n\in\mathbb{N}_0} \;\middle|\; \sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \right\}.$$

   Moreover, on ℓ^2, define the following map:

$$\langle\cdot,\cdot\rangle : \ell^2 \times \ell^2 \to \mathbb{C}, \quad \langle\alpha,\beta\rangle := \sum_{n=0}^{\infty} \alpha_n \overline{\beta_n}$$

   for $\alpha = (\alpha_n)_{n\in\mathbb{N}_0}$ and $\beta = (\beta_n)_{n\in\mathbb{N}_0}$.
   (a) Give a non-zero element of ℓ^2.
   (b) Prove that $\langle\cdot,\cdot\rangle$ is a scalar product of the space ℓ^2.
   (c) Prove that ℓ^2 with the induced norm is a Hilbert space.

3. The Riemann zeta function is the function defined by [8+8]

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for } s > 1.$$

   (a) Let f be a 2π-periodic function defined by $f(x) := x - \pi, \quad x \in [0, 2\pi)$. Prove that the Fourier coefficients of f are given as follows:

$$c_0(f) = 0; \quad c_n(f) = \frac{i}{n}, \quad n \in \mathbb{Z}\backslash\{0\},$$

   and then use Parseval's identity from Theorem 2.5.4 to determine $\zeta(2)$.
   (b) Calculate $\zeta(4)$ by considering the 2π-periodic function g defined by $g(x) := (x-\pi)^2$, $x \in [0, 2\pi)$. (Solution: $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$).