A 53 Sei $f \in L^1(\Omega)$

Z $u = T^* f \in L^{p'}(\Omega)$ wohldefiniert und

$\forall \varphi \in C^2(\overline{\Omega}) : \varphi|_{\partial\Omega} = 0$

$\int u \Delta \varphi \, dx = \int f \varphi \, dx$

Bew $p > \frac{n}{2}$

$T : L^p(\Omega) \longrightarrow C_0(\overline{\Omega})$

$T : T_1 \circ T_2$

$T_2 : L^p(\Omega) \longrightarrow W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

$T_2(f) =$ eindeutige schw. Lösung von $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$

$T_1 : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow C_0(\overline{\Omega})$ Einbettung

$T : L^p(\Omega) \longrightarrow C_0(\overline{\Omega})$

$T^* : C_0(\overline{\Omega})^* \longrightarrow (L^p(\Omega))^*$

Für die Wohldefiniertheit braucht man

1) $L^1(\Omega) \subset C_0(\overline{\Omega})^*$

2) $(L^p(\Omega))^* = L^{p'}(\Omega)$

zu 2) bekannt aus FA.

zu 1 Sei $f \in L^1(\Omega)$. Setze

$$T_f : C_0(\overline{\Omega}) \longrightarrow \mathbb{R}$$

$$T_f(\varphi) := \int_\Omega f \varphi \, dx \quad \in \mathbb{R}$$

Dann gilt $T_f \in C_0^{'}(\overline{\Omega})$ da

$\forall \varphi \in C_0(\overline{\Omega})$ gilt $|T_f(\varphi)| \leq \|f\|_{L^2} \|\varphi\|_{L^\infty} \leq \|f\|_{L^2} \|\varphi\|_{C^0(\overline{\Omega})}$

$\Longrightarrow$ Beh.

Nun sei $u = T^* f \mathrel{\hat{=}} T^*(T_f)$

Dann gilt

$$\int u \, \Delta\varphi \, dx = \int \underbrace{T^*(T_f)}_{\in L^{p'}} \cdot \underbrace{(\Delta\varphi)}_{\in L^\infty \subset L^p} dx$$

$$\underset{L^{p'} \cong (L^p)'}{=\!=\!=} \; T^*(T_f)(\Delta\varphi)$$

$$= T_f(T(\Delta\varphi))$$

$$= T_f(T_1 \circ T_2(\Delta\varphi))$$

$$= T_f(\varphi) = \int f \varphi \, dx \Longrightarrow \text{Beh}$$

A54

$u \in C^2(\bar{\Omega}; \mathbb{R}^3)$

$$\begin{cases} \Delta u = 2H\ (\partial_x u \times \partial_y u) & \text{in } \Omega \\ u = u_0 & \text{auf } \partial\Omega \end{cases}$$

$\|u\|_\infty < \frac{1}{|H|}$

z.z. $\Delta |u|^2 \geq 2(1 - |H||u|)\, |Du|^2 \ (\geq 0)$

[Dann folgt $\|u\|_\infty \leq \|u_0\|_\infty$!].

$$\Delta |u|^2 = \sum_{i=1}^{2} \partial_i^2 |u|^2$$

$$= \sum_{i=1}^{2} \partial_i [2\, \partial_i u \cdot u]$$

$$= \sum_{i=1}^{2} 2\left[(\partial_i u)^2 + u\, \partial_i^2 u\right]$$

$$= 2|Du|^2 + 2u\, \Delta u$$

$$= 2|Du|^2 + 2\, u\, (2H(\partial_x u \times \partial_y u))$$

$$\geq 2|Du|^2 - 4|H|\,|u|\,|\partial_x u|\,|\partial_y u|$$

$$\geq 2|Du|^2 - 4|H|\,|u| \left(\tfrac{1}{2}(|\partial_x u|^2 + |\partial_y u|^2)\right)$$

$$\geq 2|Du|^2 (1 - |H||u|)$$

A55

$\Omega \subset \mathbb{R}^n$, $\varphi, u \in L^\infty(\Omega) \cap H^1(\Omega)$

Z $u, \varphi \in L^\infty(\Omega) \cap H^1(\Omega)$ und

$\nabla(u\varphi) = \varphi \nabla u + u \nabla \varphi$

Schritt 1 $u\varphi \in L^\infty(\Omega)$ klar (Produkt von zwei $L^\infty(\Omega)$-Funktionen.

Schritt 2 $u\varphi$ ist schwach diffbar und $\nabla(u\varphi) = \underbrace{u\partial_i\varphi + \varphi\partial_i u}_{\in L^2}$

Beachte A52(c) $\Rightarrow \exists (u_n) \subset L^\infty(\Omega) \cap C^\infty(\bar{\Omega}) : \|u_n\|_{L^\infty} \leq \|u\|_{L^\infty}$ und $u_n \longrightarrow u$ in $H^1(\Omega)$ und punktweise f.ü.

Sei nun $\eta \in C_0^\infty(\Omega)$

$$\int_\Omega u\varphi\, \partial_i\eta \, dx = \lim_{n\to\infty} \int u_n \varphi\, \partial_i \eta$$

$$= \lim_{n\to\infty} \int \varphi\, \partial_i(\eta u_n) - \underbrace{\int \underbrace{\varphi\, \eta}_{\in L^\infty} \underbrace{\partial_i u_n}_{\to \partial_i u \text{ in } L^2}}_{\to \int \varphi \eta\, \partial_i u}$$

$$= \lim_{n\to\infty} - \int \underbrace{\partial_i \varphi\, \eta\, u_n}_{\substack{\to \partial_i\varphi\, \eta\, u \text{ pktw} \\ |\cdot| \leq |\partial_i\varphi| |\eta| \|u\|_\infty \to L^2 \text{ Majorante}}} - \int \varphi\, \eta\, \partial_i u$$

$$\overset{\text{Lebesgue}}{=} -\int \partial_i\varphi\, \eta\, u - \int \varphi\, \eta\, \partial_i u = -\int (u\partial_i\varphi + \varphi\partial_i u)\, \eta$$

$\Rightarrow$ Bew.

A56

$$\partial_z f = \frac{\partial_x f - i\partial_y f}{2} \quad , \quad \partial_{\bar z} f = \frac{\partial_x f + i\partial_y f}{2}$$

(a) Sei $f \in Hol(\mathbb{C})$

$$\partial_{\bar z} f = \frac{1}{2}\left(\partial_x f + i\partial_y f\right)$$

$$= \frac{1}{2}\left(\underbrace{\partial_x Re f + i\partial_y \cancel{Im f}}_{=0} + \overbrace{i\partial_x Re f - \partial_y Im f}^{=0}\right)$$

$\overset{\text{C.Riemann}}{\underset{\text{DGL's}}{=}}$ 0.

(b) $\Delta f = \partial_x^2 f + \partial_y^2 f$

$$\partial_z \partial_{\bar z} f = (\partial_x - i\partial_y)(\partial_x + i\partial_y) f$$
$$= (\partial_x^2 + \partial_y^2) f$$

(c) $\partial_z (f\circ g) = \cancel{\partial_x \mp i}\ \frac{1}{2}(\partial_x - i\partial_y)(f\circ g)$

$= \frac{1}{2}(\partial_x - i\partial_y)\left[(Re\, f\circ g) + i(Im\, f)\circ g\right]$

$= \Big[\partial_{\tilde x}(Re f)\ \partial_x Re g + \partial_{\tilde y}(Re f)\ \partial_x(Im g)$

$- i\,\partial_{\tilde x} Re f\ \partial_y Re g - i\,\partial_{\tilde y}(Re f)\ \partial_y Im g$

$+ i\,\partial_{\tilde x} Im f\ \partial_x Re g + i\,\partial_{\tilde y} Im f\ \partial_x Im g$

$+ \partial_{\tilde x}(Im f)\ \partial_y Re g + \partial_{\tilde y}(Im f)\ \partial_y(Im g)\Big]\cdot\frac{1}{2}$

<u>Now</u>

$$\partial_w f \, \partial_z g + \partial_{\bar w} f \, \partial_z \bar g$$

$$= \frac{1}{4}(\partial_{\tilde x} f - i\partial_{\tilde y} f)(\partial_x g - i\partial_y g)$$

$$+\frac{1}{4}(\partial_{\tilde x} f + i\partial_{\tilde y} f)(\partial_x \bar g - i\partial_y \bar g)$$

$$= \frac{1}{4}\Big[\partial_{\tilde x} f(\partial_x g + \partial_x \bar g) + i\partial_{\tilde y} f\,(\overbrace{\partial_x \bar g - \partial_x g}^{-2i\partial_x \mathrm{Im}\, g})$$

$$+ \partial_{\tilde x} f(-i\partial_y g - i\partial_y \bar g) + i\partial_{\tilde y} f\,\underbrace{(+i\partial_y g - i\partial_y \bar g)}_{i(2i\mathrm{Im}\, g) = -2\mathrm{Im}\, g}\Big]$$

$$= \frac{1}{4}\Big[2\partial_{\tilde x} f \, \partial_x \mathrm{Re}\, g + 2\partial_{\tilde y} f \, \partial_{\cancel{y}x} \mathrm{Im}\, g$$

$$-2i\partial_{\tilde x} f \, \partial_y \mathrm{Re}\, g - 2i \, \partial_{\tilde y} f \, \partial_y \mathrm{Im}\, g\Big]$$

$$= \frac{1}{2}\Big[(\partial_{\tilde x}\mathrm{Re}\, f + i\partial_{\tilde x}\mathrm{Im}\, f)\,\partial_x \mathrm{Re}\, g$$

$$+(\partial_{\tilde y}\mathrm{Re}\, f + i\partial_{\tilde y}\mathrm{Im}\, f)\,\partial_x \mathrm{Im}\, g$$

$$- i\,(\partial_{\tilde x}\mathrm{Re}\, f + i\partial_{\tilde x}\mathrm{Im}\, f)\,\partial_y \mathrm{Re}\, g$$

$$- i\,(\partial_{\tilde y}\mathrm{Re}\, f + i\partial_{\tilde y}\mathrm{Im}\, f)\,\partial_y \mathrm{Im}\, g\Big]$$

$$= \frac{1}{2}\Big[\partial_{\tilde x}\mathrm{Re}\, f \, \partial_x \mathrm{Re}\, g + \partial_{\tilde y}(\mathrm{Re}\, f)\,\partial_y \mathrm{Im}\, g$$

$$- i\,\partial_{\tilde x}\mathrm{Re}\, f\,\partial_y \mathrm{Re}\, g - i\,\partial_{\tilde y}\mathrm{Re}\, f\,\partial_y \mathrm{Im}\, g$$

$$+ i\,\partial_{\tilde x}\mathrm{Im}\, f\,\partial_x \mathrm{Re}\, g + i\,\partial_{\tilde y}(\mathrm{Im}\, f)\,\partial_x(\mathrm{Im}\, g)$$

$$+ \partial_{\tilde x}\mathrm{Im}(f)\,\partial_y \mathrm{Re}\, g + \partial_{\tilde y}\mathrm{Im}\, f\,\partial_y \mathrm{Im}\, g\Big]$$

$\boxed{A\,57}$

$$w(r,\theta) = w(r\cos\theta, r\sin\theta)$$

$$\partial_r w = \partial_x w \cos\theta + \partial_y w \sin\theta$$

$$\partial_\theta w = -\partial_x w \, r\sin\theta + \partial_y w \, r\cos\theta$$

$$\Rightarrow |\partial_r w|^2 + \frac{1}{r^2} (\partial_\theta w)^2$$

$$= (\cos\theta\, \partial_x w + \sin\theta\, \partial_y w)^2$$

$$+ (-\sin\theta\, \partial_x w + \cos\theta\, \partial_y w)^2$$

$$\underset{\cos^2\theta + \sin^2\theta = 1}{=} (\partial_x w)^2 + (\partial_y w)^2$$

A58)

$\underline{n=2}$ $\overset{?}{\text{z}}$ $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ $\forall p$

Vergleich der Sobolev-Exponenten

$$1-\frac{2}{2} \qquad 0-\frac{2}{p}$$

$$\overset{\shortparallel}{0} \quad > \quad \overset{\shortparallel}{-\frac{2}{p}}$$

$\Longrightarrow$ $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ $\forall p$.

$\underline{\text{Nun}}$ $H_0^1(\Omega) \not\subset L^\infty(\Omega)$ denn

sei $f(x) := \log\left(\log \frac{1}{|x|}\right)$, $x \in \Omega := B_{\frac{1}{2}}(0)$

1) $f \in L^\infty(\Omega)$ denn sei $N \in \Omega$ eine Nullmenge

$\Rightarrow$ 0 ist kein innerer Punkt von N

$\Longrightarrow$ $\exists\, (x_n) \in$ Nullfolge : $x_n \in \Omega \setminus N$ $\forall n \in \mathbb{N}$

$$\Rightarrow \sup_{x \in \Omega \setminus N} |f(x)| \geq \sup_{n \in \mathbb{N}} |f(x_n)| = \sup_{n \in \mathbb{N}} \log\left(\log \frac{1}{|x_n|}\right) = \infty.$$

Da N beliebige Nullmenge $\Longrightarrow$ $\|f\|_{L^\infty} = \inf\limits_{N \text{ Nullmenge}} \sup\limits_{x \in \Omega \setminus N} |f(x)| = \infty$

$\Rightarrow f \notin L^\infty(\Omega)$

ABER $f \in H_0^1(\Omega)$

Bew

1) $f \in L^2(\Omega)$ da

$$\int_\Omega |f(x)|^2 \, dx = \int_\Omega \log\left(\log \frac{1}{|x|}\right) dx$$

$$= \int_0^{1/2} \log\left(\log\left(\frac{1}{r}\right)\right) r \, dr \; 2\pi$$

$$= \int_2^\infty \log\log(z) \frac{1}{z^3} \, dz$$

Nun $\log\log z \underset{\substack{\text{Bernoulli} \\ e^x \geq x+1}}{\leq} \log(z-1) \leq z-2$

$$\Rightarrow \int_\Omega |f(x)|^2 \, dx \leq \int_2^\infty \frac{z-2}{z^3} \, dz < \infty$$

$$\Longrightarrow f \in L^2(\Omega)$$

2) f schwach diffbar da

$\forall \varphi \in C_0^\infty(\Omega)$

da $f \in L^2$, Lebesgue

$$\int f \partial_j \varphi \, dx = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon < \frac{1}{2}}} \int_{B_{\frac{1}{2}}(0) \setminus B_\varepsilon(0)} f \, \partial_j \varphi \, dx$$

$$= \lim_{\varepsilon \to 0} \Big(\underbrace{- \int_{\substack{\cancel{\partial B_{\frac{1}{2}}(0)} \\ \partial B_\varepsilon(0)}} \partial_j \varphi \, f}_{} - \int_{B_{\frac{1}{2}}(0) \setminus B_\varepsilon(0)} \varphi \, \partial_j f \, dx \Big)$$

$$\leq 2\pi\varepsilon \, \|\partial_j \varphi\|_\infty \underbrace{\sup_{x \in \partial B_\varepsilon(0)} |f(x)|}_{= \log \log \frac{1}{\varepsilon}}$$

$$\leq C \cdot 2\pi \varepsilon \log \log \frac{1}{\varepsilon} \longrightarrow 0 \quad (\varepsilon \to 0)$$

$$= \lim_{\varepsilon \to 0} - \int_{B_{\frac{1}{2}}(0) \setminus B_\varepsilon(0)} \varphi \, \partial_j f \, dx \qquad (*)$$

Wir zeigen noch

$\partial_j f \in L^2(\Omega)$. Dann gilt:

1) Der GW in (*) existiert und stimmt mit $\int_{B_{\frac{1}{2}}(0)} \partial_j f \, \varphi \, dx$ überein

$\Longrightarrow$ f schwach diffbar und die schwache Ableitung ist $\partial_j f$.

$\Longrightarrow f \in H_0^1(\Omega)$ da $\partial_j f \in L^2(\Omega)$

Nun $\partial_j f = -\frac{1}{\log \frac{1}{|x|}} \frac{x}{|x|^3}$

$$\Longrightarrow \int_\Omega |\partial_j f|^2 \, dx \overset{\log \frac{1}{|x|} = -\log|x|}{\leq} \int_\Omega \frac{1}{\log^2 |x|} \frac{1}{|x|^2} \, dx$$

$$\cancel{= \int_0^{\frac{1}{2}} \frac{1}{\log^2 |x|} \frac{1}{}}$$

$$= 2\pi \int_0^{\frac{1}{2}} \frac{1}{\log^2 r} \frac{1}{r^2} r \, dr = 2\pi \int_0^{\frac{1}{2}} \frac{1}{r} \frac{1}{\log^2 r} \, dr$$

$$\underset{s = \log r}{=} \int_{-\infty}^{-\log 2} \frac{1}{s^2} \, ds < \infty$$

$\Longrightarrow$ Beh

Aufgabe 59

(a) $2\sum_{j=1}^{2} \partial_j \left(e^{-f/2} \partial_j |z|^2 e^{f/2} \right)$

$= 2\sum_{j=1}^{2} -\frac{(\partial_j f)}{2} e^{-f/2} \partial_j \left(|z|^2 e^{f/2} \right)$

$+ 2\sum_{j=1}^{2} e^{-f/2} \partial_j^2 \left(|z|^2 e^{f/2} \right)$

$= -\sum_{j=1}^{2} (\partial_j f) e^{-f/2} \partial_j |z|^2 e^{f/2}$

$+ 2\sum_{j=1}^{2} e^{-f/2} \Big((\partial_j^2 |z|^2) e^{f/2}$

$+ 2 \partial_j |z|^2 \frac{1}{2}(\partial_j f) e^{f/2}$

$+ |z|^2 \left(\frac{(\partial_j^2 f)}{2} e^{f/2} + \frac{(\partial_j f)^2}{4} e^{f/2} \right) \Big)$

$= -\sum_{j=1}^{2} (\partial_j f) e^{-f/2} \left((\partial_j |z|^2) e^{f/2} + \frac{1}{2}|z|^2 \partial_j f \, e^{f/2} \right)$

$+ 2\sum_{j=1}^{2} \partial_j^2 |z|^2 + 2\sum_{j=1}^{2} \partial_j (|z|^2) \partial_j (f)$

$+ |z|^2 \Delta f + \frac{1}{2} \sum_{j=1}^{2} |z|^2 (\partial_j f)^2$

$= -\nabla f \cdot \nabla |z|^2 \cancel{+} \frac{1}{2} |z|^2 \sum_{j=1}^{2} (\partial_j f)^2$

$+ 2 \Delta |z|^2 + 2 \nabla |z|^2 \cdot \nabla f$

$+ |z|^2 \Delta f + \frac{1}{2} |z|^2 \sum_{j=1}^{2} (\partial_j f)^2$

$= 2 \Delta |z|^2 + \nabla f \nabla |z|^2 + |z|^2 \Delta f$

(b) $\boxed{(7)}$:

$$2H\left(\partial_x(u_1-u_2)\times\partial_y\left(\frac{u_1+u_2}{2}\right)+\partial_x\left(\frac{u_1+u_2}{2}\right)\times\partial_y(u_1-u_2)\right)$$

$$\begin{aligned}
= \cancel{2}H\Big(&\partial_x u_1\times\partial_y u_1-\partial_x u_2\times\partial_y u_2\\
&+\partial_x u_1\times\partial_y u_2-\partial_x u_2\times\partial_y u_1\\
&+\partial_x u_1\times\partial_y u_1-\partial_x u_2\times\partial_y u_2\\
&+\partial_x u_2\times\partial_y u_1-\partial_x u_1\times\partial_y u_2\Big)
\end{aligned}$$

$$=2H\left(\partial_x u_1\times\partial_y u_1-\partial_x u_2\times\partial_y u_2\right)$$

$$=\cancel{2}\,\Delta u_1-\Delta u_2=\Delta z$$

Gl (8)

$$\begin{aligned}
z\cdot\Delta z=2H\Big(&z\cdot\partial_1 z\times\partial_2\frac{u_1+u_2}{2}\\
&+z\cdot\partial_1\left(\frac{u_1+u_2}{2}\right)\times\partial_2 z\Big)
\end{aligned}$$

$$\begin{aligned}
=2H\Big(&\frac{\partial}{\partial x_2}\left(\frac{u_1+u_2}{2}\right)\cdot\cancel{z\times\partial_1}(\partial_1 z\times z)\\
&+\frac{\partial}{\partial x_1}\left(\frac{u_1+u_2}{2}\right)\cdot(\partial_2 z\times z)\Big)
\end{aligned}$$

a

$$\cancel{\Rightarrow |z\cdot\Delta z|\le 2|H|\left(\left|\nabla\left(\frac{u_1+u_2}{2}\right)\right|\left(|\partial_1 z\times z|^2+|\partial_2 z\times z|^2\right)\right)^{\frac{1}{2}}}$$

$$\cancel{\le 2|H|\,|\nabla(u_1+u_2)|\left(|\partial_1 z\times z|^2+|\partial_2 z\times z|^2\right)^{\frac{1}{2}}}$$

$$\Rightarrow \quad |z\cdot\Delta z| \le |H| \left(\left|\frac{\partial}{\partial x_2}(u_1+u_2)\right| \, |\partial_1 z\times z| + \left|\frac{\partial}{\partial x_1}(u_1+u_2)\right| \, |\partial_2 z\times z| \right)$$

$ab+cd \le \sqrt{a^2+c^2}\,\sqrt{b^2+d^2}$

$$\le |H| \left(\left|\frac{\partial}{\partial x_1}(u_1+u_2)\right|^2 + \left|\frac{\partial}{\partial x_2}(u_1+u_2)\right|^2 \right)^{\frac12} \left(|\partial_1 z\times z|^2 + |\partial_2 z\times z|^2 \right)^{\frac12}$$

$$\le |H| \, |\nabla(u_1+u_2)| \left(|\partial_1 z\times z|^2 + |\partial_2 z\times z|^2 \right)^{\frac12}$$

$$\le |H| \, |\nabla(u_1+u_2)| \left(|\partial_1 z|^2|z|^2 + |\partial_2 z|^2|z|^2 - (\partial_1 z\cdot z)^2 - (\partial_2 z\cdot z)^2 \right)^{\frac12}$$

$$\le |H| \, |\nabla(u_1+u_2)| \left(|\nabla z|^2|z|^2 - \frac14 |\nabla|z|^2|^2 \right)^{\frac12}$$

(g(9))

$$2\Delta|z|^2 = 4(z\cdot\Delta z) + 4|\nabla z|^2$$

$$\ge -4|H| \, |\nabla(u_1+u_2)| \, |z| \, \frac{\left(|\nabla z|^2|z|^2 - \frac14 |\nabla|z|^2|^2 \right)^{\frac12}}{|z|} + 4|\nabla z|^2$$

$$\ge -|H|^2 |\nabla(u_1+u_2)|^2 |z|^2 - \frac{4}{|z|^2} \left(|\nabla z|^2|z|^2 - \frac14 |\nabla|z|^2|^2 \right) + 4|\nabla z|^2$$

$$\ge -|H|^2 |\nabla(u_1+u_2)|^2 |z|^2 + \frac{|\nabla|z|^2|}{|z|^2}$$

$$\ge -2|H|^2 \left(|\nabla u_1|^2 + |\nabla u_2|^2 \right) |z|^2 + \frac{|\nabla|z|^2|}{|z|^2}$$

(c) (gl. 10)

$$2\sum_{j=1}^{2} \partial_j\left(e^{-f/2}\, \partial_j\left(|z|^2 e^{f/2}\right)\right)$$

$$\geq -2|H|^2 \left(|Du_1|^2 + |Du_2|^2\right)|z|^2$$

$$+ \frac{|D|z|^2|^2}{|z|^2} + |z|\underbrace{Df}_{a} \cdot \underbrace{\frac{D|z|^2}{|z|}}_{b}) + |z|^2 \Delta f$$

$a \cdot b \leq \frac{a^2}{4} + b^2$

$$\geq -2|H|^2 \left(|Du_1|^2 + |Du_2|^2\right)|z|^2$$

$$- \frac{1}{4}|z|^2 |Df|^2 + |z|^2 \Delta f$$

(d) $$= \sum_{i=1}^{2} -2|H|^2 |Du_i|^2 \; |z|^2 - \frac{1}{4}|z|^2 |Df|^2 + |z|^2 \Delta f$$

$$\boxed{f = \sum_{i=1}^{2} \varphi(|u_i|^2)}$$

$$\Rightarrow Df = \sum_{i=1}^{2} \varphi'(|u_i|^2) \; D|u_i|^2$$

$$\Rightarrow |Df|^2 \leq 2 \; \varphi'(|u_i|^2)^2 \; |D|u_i|^2|^2$$

$$\Delta f = \sum_{i=1}^{2} \varphi''(|u_i|^2) \; |D|u_i|^2|^2 + \varphi'(|u_i|^2) \; \Delta|u_i|^2$$

$$\geq \sum_{i=1}^{2} \varphi''(|u_i|^2) \; |D|u_i|^2|^2 + 2\varphi'(|u_i|^2)\left(1 - |H||u_i|\right)|Du_i|^2$$

$$\Rightarrow -\frac{1}{4}|z|^2 |Df|^2 + |z|^2 \Delta f$$

$$\geq \sum_{i=1}^{2} \left(\varphi''(|u_i|) - \frac{1}{2}\varphi'(|u_i|^2)^2\right) |D|u_i|^2|^2 \; |z|^2$$

$$+ 2\varphi'(|u_i|^2)\left(1 - |H||u_i|\right)|Du_i|^2 |z|^2$$

$$\Rightarrow L(z) \geq \sum_{i=1}^{2} 2\left(-|H|^2 + \varphi'(|u_i|^2)\right)\left(1 - |H||u_i|\right)|Du_i|^2 |z|^2$$

$$+\sum_{i=1}^{2}\left(\varphi''(|u_i|^2)-\frac{1}{2}\varphi'(|u_i|^2)^2\right)|D|u_i|^2|\,|z|^2$$

$$\varphi(x)=-2\log\left(\frac{1}{|H|^2}-x\right)$$

$$\varphi'(x)=\frac{+2}{\frac{1}{|H|^2}-x}\qquad\varphi''(x)=\frac{2}{\left(\frac{1}{|H|^2}-x\right)^2}$$

<u>Beachte</u> $\quad\varphi''-\frac{1}{2}(\varphi')^2=0.$

$$\Longrightarrow L(z)\geq\sum_{i=1}^{2}2\left(-|H|^2+(1-|H|\,|u_i|)\frac{2}{\frac{1}{|H|^2}-|u_i|^2}\right)|Du_i|^2|z|^2$$

$$=2\sum_{i=1}^{2}\left(\frac{-|H|^2\left(\frac{1}{|H|^2}-|u_i|^2\right)+2-2|H|\,|u_i|}{\frac{1}{|H|^2}-|u_i|^2}\right)|Du_i|^2|z|^2$$

$$=2\sum_{i=1}^{2}\left(\frac{-1+|H|^2|u_i|^2+2-2|H|\,|u_i|}{\frac{1}{|H|^2}-|u_i|^2}\right)|Du_i|^2|z|^2$$

$$=2\sum_{i=1}^{2}\cancel{|H|\,|u_i|}\frac{(1-|H|\,|u_i|)^2}{\left(\frac{1}{|H|^2}-|u_i|^2\right)}|Du_i|^2|z|^2$$

$$\geq 0$$