

Advanced Topics in the Calculus of Variations

Blatt 7

A391

Erinnerung $S(p, \Omega) = \inf_{v \in H_0^1(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 dx}{\left(\int_{\Omega} |v|^{p+1} dx \right)^{\frac{2}{p+1}}}$

(a) Aus dem Beweis von Lemma 1.2 sieht man

$$S(p, \Omega) = r^{n-2 - \frac{2n}{p+1}} S(p, \mathbb{R}^n), \quad \forall r > 0, \forall p \in \left[1, \frac{n+2}{n-2}\right]$$

Falls nun $p = \frac{n+2}{n-2}$ so gilt $p+1 = \frac{2n}{n-2}$

$$\Rightarrow S(p, \Omega) = S(p, \mathbb{R}^n) \Rightarrow \forall r > 0: S\left(\frac{n+2}{n-2}, \Omega\right) = S\left(\frac{n+2}{n-2}, \mathbb{R}^n\right)$$

Nun

$$S\left(\frac{n+2}{n-2}, \mathbb{R}^n\right) = S\left(\frac{n+2}{n-2}, \frac{2}{p} B_f(0)\right) = S\left(\frac{n+2}{n-2}, B_f(0)\right)$$

(b) Sei $v \in H_0^1(\Omega) \setminus \{0\}$, $p = \frac{n+2}{n-2}$

$$\Rightarrow \exists (v_k)_{k=1}^{\infty} \subset C_0^{\infty}(\Omega) : v_k \xrightarrow{H_0^1(\Omega)} v$$

$$\begin{aligned} C_0^{\infty}(\Omega) &\subset C_0^{\infty}(\mathbb{R}^n) \\ \Rightarrow H_0^1(\Omega) &\hookrightarrow L^{\frac{2n}{n-2}}(\Omega) \end{aligned} \quad \exists (v_k)_{k=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}^n) : \begin{cases} \nabla v_k \longrightarrow \nabla v & \text{in } L^2(\Omega) \\ v_k \longrightarrow v & \text{in } L^{\frac{2n}{n-2}}(\Omega) \end{cases}$$

Beachte: $\exists k_0 \in \mathbb{N} : k \geq k_0 \Rightarrow v_k \neq 0$

Nun

$$\frac{\int_{\Omega} |\nabla v|^2 dx}{\left(\int_{\Omega} |v|^{p+1} dx \right)^{\frac{2}{p+1}}} = \lim_{k \rightarrow \infty} \frac{\int_{\Omega} |\nabla v_k|^2}{\left(\int_{\Omega} |v_k|^{p+1} dx \right)^{\frac{2}{p+1}}}$$

$$= \lim_{k \rightarrow \infty} \frac{\int_{\mathbb{R}^n} |\nabla v_k|^2 dx}{\left(\int_{\mathbb{R}^n} |v_k|^{p+1} dx \right)^{\frac{2}{p+1}}} \geq \inf_{v \in C_0^{\infty}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla v|^2 dx}{\left(\int_{\mathbb{R}^n} |v|^{p+1} dx \right)^{\frac{2}{p+1}}} = S\left(\frac{n+2}{n-2}, \mathbb{R}^n\right)$$

$$\Rightarrow \inf_{k \in H_c^{1,2}(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 dx}{\left(\int_{\Omega} |v|^{p+2} dx \right)^{\frac{2}{p+2}}} \geq S$$

$$\Rightarrow S\left(\frac{n+2}{n-2}, \Omega\right) \geq S$$

(c) „ $\geq$ “ folgt aus Aufgabe 39b

„ $\leq$ “ Wähle $w \in D^{1,2}(\mathbb{R}^n) \setminus \{0\}$

$$\frac{\int_{\Omega} |\nabla w|^2}{\left(\int_{\Omega} |w|^{p+2} \right)^{\frac{2}{p+2}}} \leq S + \varepsilon$$

$$\text{Da } D^{1,2}(\mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}^n)}^{H^{1,2}}$$

gibt es $(\varphi_k) \subset C_0^\infty(\mathbb{R}^n)$: $\varphi_k \xrightarrow{D^{1,2}} w \Rightarrow \varphi_k \xrightarrow{L^{p+2}} w$

$$\lim_{k \rightarrow \infty} \left| \frac{\int_{\mathbb{R}^n} |\nabla w|^2 dx}{\left(\int_{\mathbb{R}^n} |w|^{p+2} dx \right)^{\frac{2}{p+2}}} - \frac{\int_{\mathbb{R}^n} |\nabla \varphi_k|^2}{\left(\int_{\mathbb{R}^n} |\varphi_k|^{p+2} \right)^{\frac{2}{p+2}}} \right| = 0$$

Wähle nun $k_0 \in \mathbb{N}$ sodass

$$\left| \frac{\int_{\mathbb{R}^n} |\nabla w|^2}{\left(\int_{\mathbb{R}^n} |w|^{p+2} \right)^{\frac{2}{p+2}}} - \frac{\int_{\mathbb{R}^n} |\nabla \varphi_{k_0}|^2}{\left(\int_{\mathbb{R}^n} |\varphi_{k_0}|^{p+2} \right)^{\frac{2}{p+2}}} \right| < \varepsilon$$

$$\Rightarrow \frac{\int_{\Omega} |\nabla \varphi_{k_0}|^2}{\left(\int_{\Omega} |\varphi_{k_0}|^{p+2} \right)^{\frac{2}{p+2}}} < S + 2\varepsilon$$

Nun gibt es $\tilde{R}_\varepsilon > 0$ sodass

$$\text{supp}(\varphi_{k_0}) \subset B_{\tilde{R}_\varepsilon}(0).$$

$$\Rightarrow S\left(\frac{n+2}{n-2}, B_{\tilde{R}_\varepsilon}(0)\right) \leq \frac{\int |\nabla \varphi_{k_0}|^2}{\left(\int |\varphi_{k_0}|^{p+2}\right)^{\frac{2}{p+2}}} \leq S + 2\varepsilon$$

Skalierungsinvarianz

$$\Rightarrow \forall R > 0 \quad S\left(\frac{n+2}{n-2}, B_R(0)\right) \leq S + 2\varepsilon$$

$$\xrightarrow{\varepsilon \rightarrow 0} \forall R > 0: S\left(\frac{n+2}{n-2}, B_R(0)\right) = S$$

(d) Wir wissen bereits

$$S\left(\frac{n+2}{n-2}, B_R(0)\right) = S \quad \forall R > 0$$

Nun sei $\Omega \subset \mathbb{R}^n$, $\Omega \neq \emptyset$ Ω offen

$$\Rightarrow \exists x_0 \in \mathbb{R}^n, \varepsilon > 0 : B_\varepsilon(x_0) \subset \Omega$$

$$\Rightarrow S \underset{\text{A39b}}{\leq} S\left(\frac{n+2}{n-2}, \Omega\right) \underset{\substack{\text{Lemma} \\ 1.1}}{\leq} S\left(\frac{n+2}{n-2}, B_\varepsilon(x_0)\right)$$

$$\xRightarrow{\text{Translationsinvarianz}} S\left(\frac{n+2}{n-2}, B_\varepsilon(0)\right) = S$$

A401

(a) (i) $\mu_k \equiv \mu$ erfüllt Fall 1 'Kompaktheit'

$\exists \exists (x_k)_{k=1}^{\infty} \subset \mathbb{R}^n : \forall \varepsilon > 0 \exists R > 0 :$

$$\int_{B_R(x_k)} d\mu_k \geq 1 - \varepsilon \quad \forall k \in \mathbb{N}.$$

Bew Wähle $x_k = 0 \quad \forall k \in \mathbb{N}.$

Sei $\varepsilon > 0.$

Bem $\exists R > 0 : \int_{B_R(0)} d\mu \geq 1 - \varepsilon$, denn

$$\lim_{\substack{j \rightarrow \infty \\ j \in \mathbb{N}}} \int_{B_j(0)} d\mu = \lim_{j \rightarrow \infty} \int_{\mathbb{R}^n} \chi_{B_j(0)} d\mu \quad \begin{array}{c} \text{monotone} \\ \text{Konvergenz} \end{array}$$

$$= \int_{\mathbb{R}^n} 1 d\mu = 1$$

$\Rightarrow R$ wie in Bem kann gefunden werden.

(ii) $\mu_k \equiv \delta_{k e_1}$ erfüllt Fall 1 'Kompaktheit'

$\exists \exists (x_k)_{k=1}^{\infty} \subset \mathbb{R}^n : \forall \varepsilon > 0 \exists R > 0 :$

$$\int_{B_R(x_k)} d\mu_k \geq 1 - \varepsilon$$

Bew: Wähle $x_k = k e_1$. Sei $\varepsilon > 0$, Wähle $R := 1$

$$\int_{B_1(x_k)} d\mu_k = \int_{B_1(k e_1)} d\delta_{k e_1} = 1 \geq 1 - \varepsilon \quad \checkmark$$

(iii)

$$\mu_k = \frac{1}{k} \sum_{\ell=1}^k \delta_{\ell e_1} \quad \text{erfüllt Fat 2 'Verschwinden'}$$

$$\geq \forall R > 0$$

$$\lim_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^n} \int_{B_R(x)} d\mu_k = 0$$

Bew Sei $x \in \mathbb{R}^n$, $R > 0$

$$\Rightarrow \# \{ B_R(x) \cap \{ \ell e_1 \mid \ell \in \mathbb{N} \} \} \leq \lceil 2R \rceil$$

da diam $B_R(x) \leq 2R$, weswegen nur $2R$
Zahlen der Gestalt $\{ \ell e_1 \mid \ell \in \mathbb{N} \}$ in $B_R(x)$ liegen
können.

$$\Rightarrow \int_{B_R(x)} d\mu_k = \frac{1}{k} \sum_{\ell=1}^k \mathbb{1}_{\ell e_1 \in B_R(x)} \leq \frac{\lceil 2R \rceil}{k}$$

$$\Rightarrow \lim_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^n} \int_{B_R(x)} d\mu_k$$

$$\leq \lim_{k \rightarrow \infty} \frac{\lceil 2R \rceil}{k} = 0$$

$\Rightarrow$ Bew

(IV)

Seien $(\mu_k)_{k=1}^{\infty}$ eine Folge die Fall 1 (Kompaktheit) erfüllt und $(\nu_k)_{k=1}^{\infty}$ eine Folge die Fall 2 (Verschwinden) erfüllt

B.w.

$\forall \lambda \in (0,1)$ erfüllt $\sigma_k = \lambda \mu_k + (1-\lambda) \nu_k$

Fall 3 (Dichotomie).

Bew:

$\exists \forall \varepsilon > 0 \exists R > 0 \exists (\mu_k)_{k=1}^{\infty}$:

$\forall R' > R \exists \mu_k^{(1)}, \mu_k^{(2)}:$

$$\sigma_k \geq \mu_k^{(1)} + \mu_k^{(2)}$$

$$\mu_k^{(1)} \Big|_{B_R(x_k)^c} = 0$$

$$\mu_k^{(2)} \Big|_{B_{R'}(x_k)} = 0$$

$$\limsup_{k \rightarrow \infty} \left(\left| \lambda - \int_{\mathbb{R}^n} d\mu_k^{(1)} \right| + \left| (1-\lambda) - \int_{\mathbb{R}^n} d\mu_k^{(2)} \right| \right) \leq \varepsilon$$

Set also $\varepsilon > 0$.

$\Rightarrow \exists (\mu_k)_{k=1}^{\infty} \exists R > 0:$

$$\int_{B_R(x_k)} d\mu_k \geq 1 - \varepsilon$$

Wähle für $R' > R$ und $A \in \mathcal{B}(\mathbb{R}^n)$

$$\mu_k^{(1)}(A) := \lambda \mu_k(A \cap B_R(x_k))$$

$$\mu_k^{(2)}(A) := (1-\lambda) \nu_k(A \cap B_{R'}(x_k)^c)$$

Nun $\sigma_k(A) = \lambda \mu_k(A) + (1-\lambda) \nu_k(A) \geq \mu_k^{(1)}(A) + \mu_k^{(2)}(A)$

$$\left| \lambda - \int_{\mathbb{R}^n} d\mu_k^{(1)} \right| = \lambda \left| 1 - \int_{B_R(x_k)} d\mu_k \right| \leq \lambda \varepsilon$$

$$\left| 1-\lambda - \int_{\mathbb{R}^n} d\mu_k^{(2)} \right| = (1-\lambda) \left| 1 - \int_{B_{R'}(x_k)^c} d\nu_k \right|$$

$$= (1-\lambda) \int_{B_{R'}(x_k)} d\mu_k \leq (1-\lambda) \sup_{x \in \mathbb{R}^n} \int_{B_{R'}(x)} d\mu_k$$

$$\Rightarrow \limsup_{k \rightarrow \infty} \left(\underbrace{\left| \lambda - \int_{\mathbb{R}^n} d\mu_k^{(1)} \right|}_{\leq \lambda \varepsilon} + \left| 1-\lambda - \int_{\mathbb{R}^n} d\mu_k^{(2)} \right| \right)$$

$$\leq \limsup_{k \rightarrow \infty} \lambda \varepsilon + (1-\lambda) \underbrace{\sup_{x \in \mathbb{R}^n} \int_{B_{R'}(x)} d\mu_k}_{\rightarrow 0 \text{ (} k \rightarrow \infty \text{)}}$$

$$\leq \lambda \varepsilon < \varepsilon \quad \text{p.d.}$$

(V) Verschwinden:

$$\lim_{k \rightarrow \infty} \sup_{x \in \mathbb{R}} \int_{B_R(x)} \frac{z}{k\pi} \frac{1}{1 + \frac{y^2}{k^2}} dy$$

$$= \lim_{k \rightarrow \infty} \sup_{x \in \mathbb{R}} \int_{x-R}^{x+R} \frac{z}{k\pi} \frac{1}{1 + \frac{y^2}{k^2}} dy$$

$$= \lim_{k \rightarrow \infty} \sup_{x \in \mathbb{R}} \int_{\frac{x-R}{k}}^{\frac{x+R}{k}} \frac{z}{\pi} \frac{1}{1+z^2} dz$$

$$= \lim_{k \rightarrow \infty} \sup_{x \in \mathbb{R}} \left[\arctan\left(\frac{x+R}{k}\right) - \arctan\left(\frac{x-R}{k}\right) \right]$$

MWS: arctan Lipschitz mit Lipschitz Konstante 1

$$\leq \lim_{k \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \frac{x+R}{k} - \frac{x-R}{k} \right|$$

$$\leq \lim_{k \rightarrow \infty} \frac{2R}{k} = 0$$

(vi) Kompaktheit

Es genügt $\exists (x_k)_{k=1}^{\infty} \subset \mathbb{R}^n$

$\forall \varepsilon > 0 \exists R > 0$

$$\liminf_{k \rightarrow \infty} \int_{B_R(x_k)} d\mu_k \geq 1 - \varepsilon$$

Nun

$x_k = 0 \forall k$, Sei $\varepsilon > 0$, Wähle $R = 1$

$$\liminf_{k \rightarrow \infty} \int_{B_R(x_k)} d\mu_k = \lim_{k \rightarrow \infty} \int_{-R}^R \frac{2}{\pi} \frac{k}{1+k^2 x^2} dx$$

$$= \lim_{k \rightarrow \infty} \frac{2}{\pi} (\arctan(kR) - \arctan(-kR))$$

$$= \lim_{k \rightarrow \infty} \frac{4}{\pi} \arctan(k\pi) = 2$$

oops. Wir haben $\mu_k = \frac{1}{\pi} \frac{k}{1+k^2 x^2}$ untersucht
wollen!

sonst μ_k ist kein W-Maß.)

A41

(a) $(V, \|\cdot\|)$ gleichmäßig konvex

$\Rightarrow$ klar,

$\Leftarrow$ Sei $x_k \xrightarrow{k \rightarrow \infty} x$ und $\|x_k\| \xrightarrow{k \rightarrow \infty} \|x\|$.

OBdA $x \neq 0$ sonst folgt die Beh aus 'ö.'.

Nun gibt es als $k_0 \in \mathbb{N}$: $x_k \neq 0 \quad \forall k \geq k_0$,

da $\|x_k\| \rightarrow \|x\| > 0$.

Wir zeigen nun

$$\left\| \frac{1}{2} \left(\frac{x_k}{\|x_k\|} + \frac{x}{\|x\|} \right) \right\|_V \rightarrow 1$$

1) Wegen der Δ -Ungleichung gilt

$$\limsup_{k \rightarrow \infty} \left\| \frac{1}{2} \left(\frac{x_k}{\|x_k\|} + \frac{x}{\|x\|} \right) \right\| \leq 1$$

2) Wegen ~~dem~~ dem Satz von Hahn Banach existiert $v^* \in V^*$ mit $\|v^*\| = 1$ und $v^*(x) = \|x\|$

$$\Rightarrow \liminf_{k \rightarrow \infty} \left\| \frac{1}{2} \left(\frac{x_k}{\|x_k\|} + \frac{x}{\|x\|} \right) \right\|_V$$

$$\geq \liminf_{k \rightarrow \infty} v^* \left(\frac{1}{2} \left(\frac{x_k}{\|x_k\|} + \frac{x}{\|x\|} \right) \right)$$

$$= \liminf_{k \rightarrow \infty} \frac{v^*(x_k)}{2\|x_k\|} + \frac{v^*(x)}{2\|x\|}$$

$$\underline{\underline{x_k \rightarrow x}} \quad \frac{v^*(x)}{\|x\|} = \frac{\|x\|}{\|x\|} = 1$$

$$\Rightarrow \left\| \frac{1}{2} \left(\frac{x_k}{\|x_k\|} + \frac{x}{\|x\|} \right) \right\| \rightarrow 1$$

gln
 $\xrightarrow{\text{Konvergenz}}$

$$\left\| \frac{x_k}{\|x_k\|} - \frac{x}{\|x\|} \right\| \rightarrow 0$$

$\|x_k\|$ beschr.
 $\xrightarrow{\quad}$

$$\left\| x_k - \frac{\|x_k\|}{\|x\|} x \right\| \rightarrow 0$$

Now

$$\|x_k - x\| = \underbrace{\left\| x_k - \frac{\|x_k\|}{\|x\|} x \right\|}_{\rightarrow 0} + \underbrace{\left(\frac{\|x_k\|}{\|x\|} - 1 \right)}_{\rightarrow 0 \text{ nach Vorr}} \|x\|$$

$\xrightarrow{k \rightarrow \infty} 0 \Rightarrow \text{Beh.}$

(b) Sei $f \in L^1(\mathbb{R}^n) : \int f(x) dx = 1, f \geq 0, \text{supp}(f) \subset B_1(0).$

Sei $f_k(x) := \frac{1}{k^n} f\left(\frac{x}{k}\right), g_k(x) := f(x).$

Wir zeigen $\|f_k\|_{L^1} = \|g_k\|_{L^1} = 1$

$\left\| \frac{1}{2} (f_k + g_k) \right\|_{L^1} \rightarrow 1$ aber

$\|f_k - g_k\|_{L^1} \not\rightarrow 0,$

$\|f_k\| = \int \frac{1}{k^n} f\left(\frac{x}{k}\right) dx = \int f(y) dy = 1$

$\|g_k\| = \int f(x) dx = 1$

$\left\| \frac{1}{2} (f_k + g_k) \right\| = \frac{1}{2} \int |f_k + g_k| \stackrel{f_k, g_k \geq 0}{=} \frac{1}{2} \int (f_k + g_k) = 1$

Nun

$\|f_k - g_k\|_{L^1} = \int_{B_1(0)} |f_k(x) - g_k(x)| dx$

$= \int_{B_1(0)} \left| \frac{1}{k^n} f\left(\frac{x}{k}\right) - f(x) \right| dx$

$\geq \int_{(B_{\frac{1}{k}}(0))^c} |f(x)| dx \xrightarrow{k \rightarrow \infty} \int_{\mathbb{R}^n} |f(x)| dx = 1$

$\Rightarrow \|f_k - g_k\|_{L^1} \not\rightarrow 0,$

A 41 c)

Wir gehen von wie im Hinweis

Zwischen beh: $(V_i, \|\cdot\|_i)$ glm konvex $\forall i=1,2,3, \dots, m$

Dann $(V_1 \times \dots \times V_m, \|(\cdot)_i\|_{i=1, \dots, m} := \left(\sum_{i=1}^m \|v_i\|_i^p \right)^{\frac{1}{p}})$ glm konvex

Mit dieser gilt

$(L^p(\mathbb{R}^n), \|\cdot\|_{L^p})$ glm konvex

$\Rightarrow ((L^p(\mathbb{R}^n))^m, \|v\|_{(L^p)^m} := \left(\sum_{i=1}^m \|v_i\|_{L^p}^p \right)^{\frac{1}{p}})$ glm konvex.

Sei nun $(v_k)_{k=1}^\infty, (w_k)_{k=1}^\infty \subset D^{m,p}(\mathbb{R}^n)$ mit

$\|v_k\|_{D^{m,p}} = \|w_k\|_{D^{m,p}} = 1 \quad \forall k$ und $\left\| \frac{1}{2}(v_k + w_k) \right\|_{D^{m,p}(\mathbb{R}^n)} \xrightarrow{k \rightarrow \infty} 1$
 $\sum \|v_k - w_k\|_{D^{m,p}} \rightarrow 0, (k \rightarrow \infty)$

Nun: $\|v_k\|_{D^{m,p}} \stackrel{(1) \text{ auf Blatt 7}}{=} \left\| (D^\alpha v_k)_{|\alpha|=m} \right\|_{(L^p)^m} = 1$

$\|w_k\|_{D^{m,p}} \stackrel{(1) \text{ auf Blatt 7}}{=} \left\| (D^\alpha w_k)_{|\alpha|=m} \right\|_{(L^p)^m} = 1$

$\left\| \frac{1}{2}(v_k + w_k) \right\|_{D^{m,p}} = \left\| \frac{1}{2} \left[(D^\alpha v_k)_{|\alpha|=m} + (D^\alpha w_k)_{|\alpha|=m} \right] \right\|_{(L^p)^m}$
 $\xrightarrow{k \rightarrow \infty} 1$

$\Rightarrow \left\| (D^\alpha v_k)_{|\alpha|=m} - (D^\alpha w_k)_{|\alpha|=m} \right\|_{(L^p)^m} = 0$

$\Rightarrow \|v_k - w_k\|_{D^{m,p}} \rightarrow 0. \Rightarrow \text{Beh}$

(Noch $\Leftarrow$: Zwischen beh!)

Seien $V_1, \dots, V_m$ gleichmäßig konvex

$\Rightarrow (V_1 \times \dots \times V_m, \|\cdot\|_V)$ gleichmäßig konvex, wobei

$$\|v\|_V := \|(v^1, \dots, v^m)\|_V := \left(\sum_{i=1}^m \|v^i\|_{V_i}^p \right)^{\frac{1}{p}}.$$

Seien $\|v_k\|_V, \|w_k\|_V \leq 1$ und $\| \frac{1}{2}(v_k + w_k) \|_V \rightarrow 1$

$v_k = (v_k^1, \dots, v_k^m), w_k = (w_k^1, \dots, w_k^m)$ dann gilt

$$1 \leftarrow \lim_{k \rightarrow \infty} \left(\sum_{i=1}^m \left\| \frac{v_k^i + w_k^i}{2} \right\|_{V_i}^p \right)^{\frac{1}{p}} \stackrel{\Delta_{\text{ungl}}}{\leq} \left(\sum_{i=1}^m \left(\frac{\|v_k^i\|_{V_i} + \|w_k^i\|_{V_i}}{2} \right)^p \right)^{\frac{1}{p}} \stackrel{(\frac{1}{2})}{\leq}$$

Nun gilt $A_k := (\|v_k^1\|_{V_1}, \dots, \|v_k^m\|_{V_m}), B_k := (\|w_k^1\|_{V_1}, \dots, \|w_k^m\|_{V_m})$

sind Vektoren in $\mathbb{R}^m$ mit $\|A_k\|_p, \|B_k\|_p \leq 1$ wobei

$$\|x\|_p := \left(\sum_{i=1}^m |x_i|^p \right)^{\frac{1}{p}}. \text{ Außerdem gilt durch } (\frac{1}{2})$$

$$\text{dass } \left\| \frac{1}{2}(A_k + B_k) \right\|_p \rightarrow 1$$

$\Rightarrow (\mathbb{R}^m, \|\cdot\|_p)$ gl.konvex

$$\|B_k - A_k\|_p \rightarrow 0$$

$$\Rightarrow \left(\sum_{i=1}^m \left| \|v_k^i\|_{V_i} - \|w_k^i\|_{V_i} \right|^p \right)^{\frac{1}{p}} \rightarrow 0 \quad (k \rightarrow \infty)$$

$$\Rightarrow \forall i=1, \dots, m : \|v_k^i\|_{V_i} - \|w_k^i\|_{V_i} \rightarrow 0 \quad (k \rightarrow \infty)$$

Wegen dem Satz von Bolzano Weierstraß (da $\|v_k^i\|_{V_i} \leq \|v_k\|_V \leq 1$)
gibt nach Auswahl einer Teilfolge $\|v_k^i\|_{V_i} \rightarrow h_i, \forall i=1, \dots, m$

und durch (da die Differenz gegen 0 strebt)

$$\|w_k^i\|_{V_i} \rightarrow h_i, \forall i=1, \dots, m.$$

Beachte

$$\left(\sum_{i=1}^m \lambda_i^p \right)^{\frac{1}{p}} = \lim_{k \rightarrow \infty} \left(\sum_{i=1}^m \|w_k^i\|_i^p \right)^{\frac{1}{p}} \leq 1.$$

~~Nun sei $i \in \{1, \dots, m\}$~~

gewisse

Nach Auswahl einer Teilfolge gilt auch für $\tilde{\lambda}_1, \dots, \tilde{\lambda}_m \in \mathbb{R}$

$$\forall i=1, \dots, m: \left\| \frac{v_k^i + w_k^i}{2} \right\|_i \rightarrow \tilde{\lambda}_i. \text{ Beachte: Wegen den}$$

Δ Ungleichung gilt $\tilde{\lambda}_i \leq \lambda_i$. Aber

$$\begin{aligned} \left(\sum_{i=1}^m \tilde{\lambda}_i^p \right)^{\frac{1}{p}} &= \lim_{k \rightarrow \infty} \left(\sum_{i=1}^m \left\| \frac{v_k^i + w_k^i}{2} \right\|_i^p \right)^{\frac{1}{p}} \\ &= 1 \geq \left(\sum_{i=1}^m \lambda_i^p \right)^{\frac{1}{p}} \left[\geq \left(\sum_{i=1}^m \tilde{\lambda}_i^p \right)^{\frac{1}{p}} \right] \\ &\quad \text{da } \lambda_i \geq \tilde{\lambda}_i \forall i \end{aligned}$$

$$\Rightarrow \tilde{\lambda}_i = \lambda_i, \forall i=1, \dots, m.$$

OBdA $\lambda_i \neq 0 \forall i$ (sonst gilt $\|v_k^i - w_k^i\|_i \leq \|v_k^i\|_i + \|w_k^i\|_i \xrightarrow{(k \rightarrow \infty)} 0$)

Wir wissen: $\forall \varepsilon > 0 \exists \delta > 0 : \forall i=1, \dots, m: \|v_k^i\|_i \leq 1, \|w_k^i\|_i \leq 1$

$$\left\| \frac{v_k^i + w_k^i}{2} \right\|_i \geq 1 - \delta \Rightarrow \|v_k^i - w_k^i\|_i < \varepsilon.$$

Sei nun $\varepsilon > 0$ fix. Wähle δ wie oben

Nun gilt für $\theta > 0$ beliebig:

$$\exists k_0 \in \mathbb{N} : \forall k \geq k_0 \quad \left\| \frac{v_k^i}{\lambda_i + \theta} \right\| \leq 1, \quad \left\| \frac{w_k^i}{\lambda_i + \theta} \right\| \leq 1$$

Beachte

~~$\lim_{k \rightarrow \infty} \left\| \frac{v_k^i + w_k^i}{2} \right\|$~~

$$\begin{aligned} \lim_{k \rightarrow \infty} \left\| \frac{1}{2} \left(\frac{v_k^i}{\lambda_i + \theta} + \frac{w_k^i}{\lambda_i + \theta} \right) \right\| \\ = \frac{1}{2(\lambda_i + \theta)} \lim_{k \rightarrow \infty} \|v_k^i + w_k^i\| \end{aligned}$$

$$= \frac{\tilde{\lambda}_i}{\lambda_i + \theta} = \frac{\lambda_i}{\lambda_i + \theta} \geq 1 - \frac{\delta}{2} \quad \forall i=1, \dots, m$$

$$\text{für } \theta \leq \min_{i=1, \dots, m} \left(\lambda_i \frac{\delta}{1-\delta} \right) \Rightarrow \left\{ \begin{array}{l} \exists k_1 \in \mathbb{N} : k_1 \geq k_0 \\ \frac{1}{2} \left\| \frac{v_k^i}{\lambda_i + \theta} + \frac{w_k^i}{\lambda_i + \theta} \right\| \geq 1 - \delta \\ \forall k \geq k_1 \end{array} \right.$$

$$\Rightarrow \cancel{\|v^i - w^i\|_i} \quad \left\| \frac{v_k^i}{\lambda_i + \theta} - \frac{w_k^i}{\lambda_i + \theta} \right\|_i < \varepsilon \quad \forall i=1, \dots, m$$

$$\forall k \geq k_1$$

$$\Rightarrow \|v_k - w_k\|_V = \left(\lambda_i + \theta \right) \left\| \frac{v_k}{\lambda_i} \right\|$$

$$= \left(\sum_{i=1}^m \|v_k^i - w_k^i\|_i^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^m (\lambda_i + \theta)^p \left\| \frac{v_k^i}{\lambda_i + \theta} - \frac{w_k^i}{\lambda_i + \theta} \right\|_i^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^m (\lambda_i + \theta)^p \varepsilon^p \right)^{\frac{1}{p}} = \varepsilon \left(\sum_{i=1}^m (\lambda_i + \theta)^p \right)^{\frac{1}{p}} \quad \forall k \geq k_1$$

$$\Rightarrow \limsup_{k \rightarrow \infty} \|v_k - w_k\|_V \leq C \varepsilon$$

$$\xrightarrow{\varepsilon \rightarrow 0} \|v_k - w_k\|_V \rightarrow 0 \quad \text{qed}$$

A42)

$$(a) \quad f_\delta(x) = \begin{cases} \frac{1}{|x|} & |x| > \delta \\ \frac{1}{\delta} & |x| \leq \delta \end{cases}$$

$$\Rightarrow f_\delta = \min \left\{ \frac{1}{\delta}, \frac{1}{|x|} \right\}$$

PDE
 $\Rightarrow f_\delta \in W^1$
| $f, g \in W^1$
| $\Rightarrow \min(f, g) \in W^1$

$$\nabla f_\delta = \begin{cases} \nabla \frac{1}{\delta} & \frac{1}{\delta} < \frac{1}{|x|} \\ \nabla \frac{1}{|x|} & \frac{1}{\delta} \geq \frac{1}{|x|} \end{cases}$$
$$= \begin{cases} 0 & |x| < \delta \\ -\frac{x}{|x|^3} & |x| \geq \delta \end{cases}$$

$$\Rightarrow \int_{\mathbb{R}^3} |\nabla f_\delta|^2 dx = \int_{\mathbb{R}^3 \setminus B_\delta(0)} \left| -\frac{x}{|x|^3} \right|^2 dx$$

$$= \int_{\mathbb{R}^3 \setminus B_\delta(0)} \frac{1}{|x|^4} dx = \int_\delta^\infty \int_{\partial B_s(0)} \frac{1}{|x|^4} dS(x) ds$$

$$= \int_\delta^\infty \frac{1}{s^4} \frac{|\partial B_s(0)|}{4\pi s^2} ds$$

$$= 4\pi \int_\delta^\infty \frac{1}{s^2} ds = \frac{4\pi}{\delta} < \infty$$

$$\Rightarrow f_\delta \in D^{1,2}(\mathbb{R}^3)$$

Aber $f_\delta \notin W^{1,2}(\mathbb{R}^n)$ da

$$\int |f_\delta|^2 dx = \int_{B_\delta(0)} \frac{1}{\delta^2} dx + \int_{\mathbb{R}^3 \setminus B_\delta(0)} \frac{1}{|x|^2} dx$$

$$= \frac{\frac{4}{3}\pi\delta^3}{\delta^2} + \int_\delta^\infty \int_{\partial B_\delta(0)} \frac{1}{|x|^2} dS(x) ds$$

$$= \frac{4}{3}\pi\delta + \int_\delta^\infty \frac{1}{s^2} 4\pi s^2 ds$$

$$= \frac{4}{3}\pi\delta + 4\pi \int_\delta^\infty 1 ds = \infty,$$

$$\Rightarrow f_\delta \notin W^{1,2}(\mathbb{R}^n)$$

Nun

$$\lim_{\delta \rightarrow 0} f_\delta = \frac{1}{|x|} \quad x \in \mathbb{R}^3 \setminus \{0\}$$

$\frac{1}{4\pi} \frac{1}{|x|} \triangleq$ Fundamentallösung der Poisson Gleichung d.h.

$$\Delta u = f \quad \text{in } \mathbb{R}^n, \quad f \in C_c^\infty(\mathbb{R}^n)$$

$$\Rightarrow u(x) = \int_{\mathbb{R}^3} \frac{1}{4\pi |x-y|} f(y) dy \quad \forall x$$

$$= \frac{1}{4\pi |x|} * f$$

$\hookrightarrow$ Hierdurch ist es wichtig zu wissen in welchem Raum $\frac{1}{4\pi |x|}$ lebt, damit man die

Royalavität der Faltung unter sehen kann!

A42(b) (Bonus)

$$(i) \quad -\Delta g = \delta_0$$

$$\Rightarrow -\widehat{\Delta g} = \widehat{\delta_0} = 1$$

$$\Rightarrow |k|^2 \hat{g}(k) = 1$$

$$\Rightarrow \hat{g}(k) = \frac{1}{|k|^2}$$

$$(ii) \quad \int_{\mathbb{R}^n} f(x) g(x) dx = \int_{\mathbb{R}^n} \hat{f}(k) \overline{\hat{g}(k)} dk$$

$$= \int_{\mathbb{R}^n} |k| \hat{f}(k) \frac{1}{|k|} \overline{\hat{g}(k)} dk$$

$$\leq \left(\int_{\mathbb{R}^n} |k|^2 |\hat{f}(k)|^2 dk \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \frac{1}{|k|^2} |\hat{g}(k)|^2 dk \right)^{\frac{1}{2}}$$

$$= \left(\int_{\mathbb{R}^n} |ik \hat{f}(k)|^2 dk \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \hat{g}(k) \frac{1}{|k|^2} \overline{\hat{g}(k)} dk \right)^{\frac{1}{2}}$$

$$= \left(\int_{\mathbb{R}^n} |\widehat{\nabla f}(k)|^2 dk \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \hat{g}(k) \widehat{g^* g}(k) dk \right)^{\frac{1}{2}}$$

$$\stackrel{\text{Plancherel}}{=} \|\nabla f\|_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} g(x) (g^* g)(x) dx$$

$$= \|\nabla f\|_{L^2(\mathbb{R}^n)} \iint_{\mathbb{R}^n \times \mathbb{R}^n} g(x) g(y) \frac{1}{c_n |x-y|^{n-2}} dx dy$$

$$(iii) f \in C_0^\infty(\mathbb{R}^n)$$

$$\|f\|_{L^{p+1}(\mathbb{R}^n)} = \sup_{\substack{g \in (L^{p+1})' \\ \|g\|_{(L^{p+1})'} \leq 1}} \int f(x) g(x) dx$$

$$\frac{1}{p+1} + \frac{1}{q} = 1$$

$$\Rightarrow \frac{1}{q} = \frac{p}{p+1}$$

$$= \frac{\frac{n+2}{n-2}}{\frac{2n}{n-2}} = \frac{n+2}{2n}$$

$$\Rightarrow q = \frac{2n}{n+2}$$

$$\sup_{g \in C_0^\infty(\mathbb{R}^n)}$$

$$\|g\|_{\frac{2n}{n+2}} \leq 1$$

$$\int f(x) g(x) dx$$

(ii)



$$\sup_{g \in C_0^\infty(\mathbb{R}^n)}$$

$$\|g\|_{\frac{2n}{n+2}} \leq 1$$

$$C \|f\|_{L^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} g(x) g(y) \frac{1}{|x-y|^{n-2}} dx dy$$

Wir verwenden nun die Hardy Littlewood Sobolev Ungleichung:

$$(HLS) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) h(y) \frac{1}{|x-y|^{n-2}} dx dy \leq C \|f\|_{L^p} \|h\|_{L^r}$$

$f \in L^p \wedge h \in L^r$

$$\frac{1}{p} + \frac{1}{r} + \frac{1}{n} = 2$$

$$\text{Nun } p=r = \frac{2n}{n+2} \quad \text{und } n = n-2$$

$$\frac{1}{p} + \frac{1}{r} + \frac{1}{n} = 2 \left(\frac{n+2}{2n} \right) + \frac{n-2}{n} = 2 \quad \checkmark$$

[siehe, Lieb, Loss Thm 4.3]

$$\leq \sup_{\|g\|_{\frac{2n}{n+2}} = 1} \tilde{C} \|\nabla f\|_{L^2} \|g\|_{\frac{2n}{n+2}}^2 \leq C \|\nabla f\|_{L^2}$$

$$\Rightarrow \|f\|_{L^{p+2}} \leq C \|\nabla f\|_{L^2} \quad \forall f \in C_0^\infty(\mathbb{R}^n)$$

Sei nun $f \in D^{2,2}(\mathbb{R}^n)$ Wähle $(f_n) \subset C_0^\infty(\mathbb{R}^n)$

$$f_n \rightarrow f \text{ in } D^{2,2}(\mathbb{R}^n)$$

~~$$\Rightarrow f_n \rightarrow f \text{ in } L^{p+2}(\mathbb{R}^n)$$~~

~~$$\Rightarrow f_n \rightarrow f \text{ in } L^{p+2}(\Omega) \quad \forall \Omega \subset \mathbb{R}^n$$~~

$$\Rightarrow f_n \rightarrow f \text{ p.k.w. für (n.A.E.T.)}$$

● Fatou's Lemma:

$$\left(\int |f|^{p+2} \right)^{\frac{1}{p+2}} \leq \liminf_{n \rightarrow \infty} \left(\int |f_n|^{p+2} \right)^{\frac{1}{p+2}}$$

$$\leq \liminf_{n \rightarrow \infty} C \left(\int |\nabla f_n|^2 \right)^{\frac{1}{2}}$$

$\rightarrow \int |\nabla f|^2$

$$= C \int |\nabla f|^2 dx \quad \Rightarrow \text{Bsp für alle } f \in D^{2,2}(\mathbb{R}^n)$$